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ON THE EIGENFREQUENCIES OF LONGITUDINALLY VIBRATING RODS CARRYING A TIP MASS AND SPRING–MASS IN-SPAN

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The present paper is concerned with the determination of the frequency equation and sensitivity of the eigenfrequencies of a fixed–free longitudinally vibrating rod carrying a tip mass to which a spring–mass system is attached in-span. First, the exact frequency equation is established, and then an approximate formula is given for the fundamental frequency, based on Dunkerley's procedure. Moreover, using the normal mode method, a second approximate but very accurate frequency equation is established with the help of which a sensitivity formula is derived later. Frequency equations of some simpler systems are also obtained from the general expressions by using limiting processes. These novel equations can be very useful for a design engineer who is interested in the eigencharacteristics of similar systems, and their sensitivity.

1. INTRODUCTION

Examination of the existing literature shows that the solution of the frequency equations governing the bending vibrations of Bernoulli–Euler beams restrained in various manners and carrying point or heavy masses has attracted the interest of many investigators [1–6]. Beside the efforts towards the determination of the eigenfrequencies, some researchers have dealt with the determination of the sensitivity of the eigenfrequencies with respect to size and shape design variables [7–9]. In connection with the subject of investigating the general properties of the eigenfrequencies of combined systems, there are a number of studies [10–22] dealing with the problem of free and forced transverse vibrations of beams and plates carrying elastically mounted concentrated masses, using analytical and numerical approaches.

Unlike the investigations on bending vibrations of Bernoulli–Euler beams, there are not so many publications on longitudinally vibrating elastic bars. In reference [23], Laura *et al*. investigated a system consisting of a spring and longitudinally vibrating rod with a mass attached at the other end, moving axially with constant velocity. In reference [24], Cutchins investigated the effect of an arbitrarily located mass on the longitudinal vibrations of a bar. Kohoutek derived the frequency equation of longitudinally vibrating rods with a tip mass or a spring [25]. Ebrahimi dealt with the longitudinal vibrations of fixed–fixed bars with a lumped mass [26]. Kukla *et al*. [27] published an interesting study on the problem of the natural longitudinal vibrations of two rods coupled by many translational springs where the Green function method was employed. Motivated by this study, Mermertaş and Gürgöze [28] investigated longitudinal vibrations of rods coupled by a double spring–mass system. More recently, the present author presented in reference [29] alternative formulations of the frequency equation of a similar system.

The problem to be handled in the present paper represents to some extent the counterpart of the study in reference [22] where the lateral vibrations were investigated. In this paper, longitudinal vibrations are analyzed. The present study is concerned with the determination of the eigenfrequencies and their sensitivity of a fixed–free longitudinally vibrating rod carrying a tip mass to which a spring–mass system is attached in-span. After establishing the exact frequency equation, an approximate formula is given for the fundamental frequency, based on Dunkerley's method. The attribute "exact" will be used henceforth in the sense that the frequency equation is obtained by means of a boundary value problem formulation. Moreover, using the normal mode method, a second approximate but very accurate frequency equation is established with the aid of which a sensitivity formula is derived later. Then, from the general expressions of the frequency equations, the corresponding equations of some simpler systems are obtained as special cases. These equations are then numerically solved for various combinations of physical parameters. The eigenfrequency sensitivity of the original system is calculated with respect to the location of the attachment point of the spring–mass system.

The rod spring–mass assembly has a variety of practical applications such as in modelling of mechanism linkages, automatic control systems and piping connections.

2. EXACT FREQUENCY EQUATION

The system to be dealt with in the present study is shown in Figure 1. It is essentially an axially vibrating fixed–free rod of axial rigidity EA and mass per unit length m carrying a tip mass M (primary system) to which a secondary spring–mass system is attached in-span. The first aim of the present paper is to derive the exact frequency equation of the combined system described above in order to determine it's eigenfrequencies. It is well known that the longitudinal vibrations of a uniform elastic rod are governed by the partial differential equation [30]

$$EA\,\frac{\partial^2 u(x,\,t)}{\partial x^2} = m\,\frac{\partial^2 u(x,\,t)}{\partial t^2}\,, \tag{1}$$

where $u(x,\,t)$ denotes the axial displacement at point x and time t. The axial displacements in the regions to the left and right of the in-span attachment of the spring–mass will be

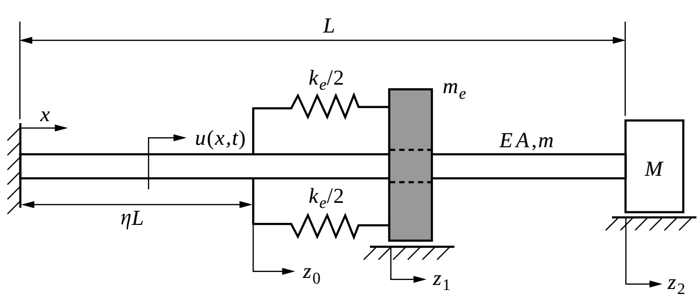

Figure 1. Longitudinally vibrating fixed–free rod with a tip mass and a spring–mass attached in-span.

denoted hereafter as $u_1(x, t)$ and $u_2(x, t)$ where both are subject to the differential equation (1). The corresponding boundary and matching conditions are:

$$u_1(0, t) = 0, \qquad u_1(\eta L, t) = u_2(\eta L, t), \qquad EAu_2'(L, t) + M\ddot{u}_2(L, t) = 0,$$
$$EAu_1'(\eta L, t) - EAu_2'(\eta L, t) + m_e\ddot{z}_1(t) = 0,$$
$$m_e\ddot{z}_1(t) + k_e[z_1(t) - z_0(t)] = 0. \tag{2}$$

Here, $z_0(t)$ means the axial displacement of the attachment point of the spring–mass system and $z_1(t)$ denotes the axial displacement of the mass m_e. Dots and primes denote partial derivatives with respect to time t and x. If harmonical solutions are assumed for both regions and for the displacement of m_e:

$$u_i(x, t) = U_i(x)\cos\omega t \quad (i = 1, 2), \qquad z_1(t) = Z_1\cos\omega t. \tag{3}$$

$U_i(x)$, Z_1 and ω being the amplitudes and the system eigenfrequency to be determined, the following ordinary differential equations are obtained from equation (1)

$$U_i''(x) + \beta^2 U_i(x) = 0 \quad (i = 1, 2), \tag{4}$$

where

$$\beta^2 = \frac{m\omega^2}{EA}. \tag{5}$$

The general solutions of the ordinary differential equations (4) are:

$$U_1(x) = C_1\sin\beta x + C_2\cos\beta x, \qquad U_2(x) = C_3\sin\beta x + C_4\cos\beta x, \tag{6}$$

where C_i $(i = 1, \ldots, 4)$ are arbitrary integration constants to be evaluated from the boundary and matching conditions for $U_1(x)$ and $U_2(x)$.

If these conditions are considered with equations (6), a set of five homogeneous equations for the unknowns C_1–C_5 are obtained, where $C_5 = Z_1$ is introduced. In order to obtain non-vanishing solutions for C_1–C_5, the corresponding determinant of coefficients has to be equated to zero. It can be shown after same algebraic manipulations, that this determinantal equation results in the following simple form

$$(\cos\bar{\beta} - \beta_M\bar{\beta}\sin\bar{\beta})\left[-1 + \frac{\alpha_{me}}{\alpha_{ke}}\bar{\beta}^2 + \frac{\alpha_{me}}{2}\bar{\beta}\sin 2\eta\bar{\beta}\right]$$
$$+\alpha_{me}\bar{\beta}\sin^2\eta\bar{\beta}(\sin\bar{\beta} + \beta_M\bar{\beta}\cos\bar{\beta}) = 0, \tag{7}$$

where the following non-dimensional parameters are introduced:

$$\bar{\beta} = \beta L, \qquad \alpha_{me} = \frac{m_e}{mL}, \qquad \alpha_{ke} = \frac{k_e}{EA/L}, \qquad \beta_M = \frac{M}{mL}. \tag{8}$$

The roots of the transcendental equation above give the dimensionless frequency parameters $\bar{\beta}$ and therefore by considering equation (5) the eigenfrequencies of the system in Figure 1. Having obtained the frequency equation of the general system in Figure 1, the frequency equations of simpler systems can be easily obtained as limiting cases. This point will be handled in detail later.

3. AN APPROXIMATE FREQUENCY EQUATION

Interest lies not only in obtaining the eigenfrequencies of the system but also in the investigation of the behaviour of the eigenfrequencies with respect to small chances of the location of the spring–mass attachment, which can occur in a real system. To this end, the exact frequency equation (7) should be differentiated partially with respect to η. Instead, in order to make direct use of the expressions derived in reference [22], an approximate frequency equation will be derived first, then a sensitivity formula will be given. In order to derive an approximate frequency equation, it is necessary to first formulate the equation of motion of the system in Figure 1 in another way. To this end, the approach in reference [17] will be adopted here, which was also used in references [20, 22].

The kinetic and potential energies of the system are

$$T = \tfrac{1}{2}m \int_0^L \dot{u}^2(x, t)\,\mathrm{d}x + \tfrac{1}{2}m_e \dot{z}_1^2 + \tfrac{1}{2}M\dot{z}_2^2, \qquad V = \tfrac{1}{2}EA \int_0^L u'^2(x, t)\,\mathrm{d}x + \tfrac{1}{2}k_e(z_1 - z_0)^2, \tag{9, 10}$$

where $u(x, t)$ now represents the axial displacements over the entire rod and $z_1(t)$ denotes the displacements of the appended mass m_e. Finally, $z_2(t)$ is the axial displacement of the rod end section. The displacement of the rod at point x is assumed to be expressible in the form of a finite series

$$u(x, t) = \sum_{i=1}^{n} U_i(x)\eta_i(t), \tag{11}$$

where

$$U_i(x) = \sqrt{\frac{2}{mL}} \sin (2i - 1) \frac{\pi}{2}\frac{x}{L} \quad (i = 1, \ldots, n) \tag{12}$$

are the orthonormalized eigenfunctions of a fixed–free uniform rod and $\eta_i(t)$ $i = (1, \ldots, n)$ denote generalized co-ordinates to be determined [30]. If the assumed series solution (11) is substituted into the energy equations (9) and (10), the Lagrangian is obtained as

$$L = T - V = \frac{1}{2}\sum_{i=1}^{n} \dot{\eta}_i^2 + \tfrac{1}{2}m_e\dot{z}_1^2 + \tfrac{1}{2}M\dot{z}_2^2 - \frac{1}{2}\sum_{i=1}^{n} \omega_i^2\eta_i^2 - \tfrac{1}{2}k_e(z_1 - z_0)^2, \tag{13}$$

where the following relations are used which are due to the special normalization of the eigenfunctions $U_i(x)$

$$\int_0^L mU_i(x)U_j(x)\,\mathrm{d}x = \delta_{ij}, \qquad \int_0^L EAU_i'(x)U_j'(x)\,\mathrm{d}x = \delta_{ij}\omega_i^2. \tag{14}$$

Here, δ_{ij} represents the Kronecker delta and ω_i is the ith eigenfrequency of the fixed–free uniform rod. The equations of motion of the system in Figure 1 can be established by means of the Lagrange's equations in connection with Lagrange's multipliers, which when considered for a system with n degrees of freedom where v redundant co-ordinates are used, are as follows [31]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{\ell=1}^{v} \lambda_\ell \frac{\partial f_\ell}{\partial q_k} \quad (k = 1, \ldots, n + v). \tag{15}$$

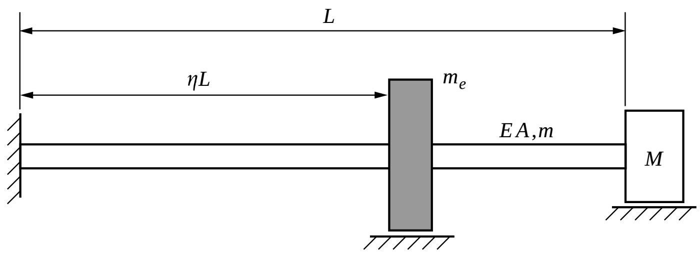

Figure 2. Longitudinally vibrating rod with a tip mass and an in-span mass.

Here, λ_ℓ denotes the ℓth Lagrangian multiplier, L is the Lagrangian and there are v constraint equations of the form $f_\ell(t; q_1, \ldots, q_{n+v}) = 0$ $(\ell = 1, \ldots, v)$. In the present case, there are two constraint equations

$$f_1 = \sum_{k=1}^{n} U_k(L)\eta_k(t) - z_2(t) = 0, \qquad f_2 = \sum_{k=1}^{n} U_k(\eta L)\eta_k(t) - z_0(t) = 0. \tag{16}$$

Recognizing that the Lagrangian L in equation (13) and the constraint equations in (16) are formally the same as given in reference [22], the frequency equation of the mechanical system in Figure 1 can be written directly as

$$\left[\sum_{k=1}^{n} \frac{U_k^2(L)}{\omega_k^2 - \omega^2} - \frac{1}{M\omega^2}\right] \cdot \left[\sum_{k=1}^{n} \frac{U_k^2(\eta L)}{\omega_k^2 - \omega^2} + \frac{\omega^2 - \omega_e^2}{m_e \omega_e^2 \omega^2}\right] - \left[\sum_{k=1}^{n} \frac{U_k(L)U_k(\eta L)}{\omega_k^2 - \omega^2}\right]^2 = 0, \tag{17}$$

where according to $\omega_e^2 = k_e/m_e$, ω_e represents the eigenfrequency of the attached spring–mass system.

For further investigations, it is more suitable to rewrite the frequency equation (17) in terms of non-dimensional quantities as

$$\left[\sum_{k=1}^{n} \frac{a_k^2}{\lambda_k - \Omega^2} - \frac{1}{\beta_M \Omega^2}\right] \cdot \left[\sum_{k=1}^{n} \frac{b_k^2}{\lambda_k - \Omega^2} + \frac{1}{\alpha_{ke}} - \frac{1}{\alpha_{me}\Omega^2}\right] - \left[\sum_{k=1}^{n} \frac{a_k b_k}{\lambda_k - \Omega^2}\right]^2 = 0, \tag{18}$$

where additional to those given in equations (5) and (8), the following abbreviations are used

$$\lambda_k = \left[(2k-1)\frac{\pi}{2}\right]^2, \qquad a_k = \sqrt{2}(-1)^{k+1}, \qquad b_k(\eta) = \sqrt{2}\sin(2k-1)\frac{\pi}{2}\eta,$$

$$\omega_0^2 = EA/mL^2, \qquad \Omega = \omega/\omega_0 \equiv \bar{\beta}. \tag{19}$$

It is reasonable to expect that the dimensionless eigenfrequencies Ω obtained from equation (18) converge to those of the exact frequency equation (7) if n goes to infinity. It is worth noting that $\bar{\beta}$ and Ω in the exact and approximate frequency equations represent the same dimensionless eigenfrequency parameter.

Having obtained both exact and approximate frequency equations of the general system in Figure 1, frequency equations of many special systems can be obtained from equations (7) and (18) as limit cases.

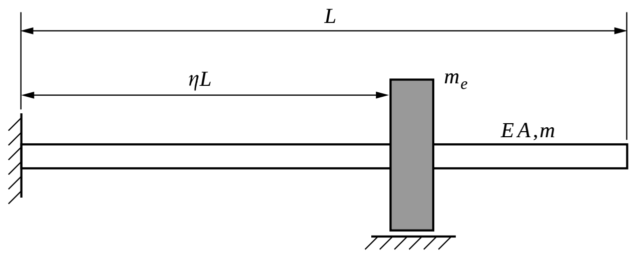

Figure 3. Longitudinally vibrating rod carrying a mass attached in-span.

Case 1

If α_{ke} goes to infinity, the frequency equations of the system in Figure 2 are obtained:

$$(\cos\overline{\beta} - \beta_M\overline{\beta}\sin\overline{\beta})\cdot\left(-1+\frac{\alpha_{me}}{2}\overline{\beta}\sin 2\eta\overline{\beta}\right)+\alpha_{me}\overline{\beta}\sin^2\eta\overline{\beta}\cdot(\sin\overline{\beta}+\beta_M\overline{\beta}\cos\overline{\beta})=0, \tag{20a}$$

$$\left[\sum_{k=1}^{n}\frac{a_k^2}{\lambda_k-\Omega^2}-\frac{1}{\beta_M\Omega^2}\right]\cdot\left[\sum_{k=1}^{n}\frac{b_k^2}{\lambda_k-\Omega^2}-\frac{1}{\alpha_{me}\Omega^2}\right]\cdot\left[\sum_{k=1}^{n}\frac{a_k b_k}{\lambda_k-\Omega^2}\right]^2=0. \tag{20b}$$

Case 2

For α_{ke} going to infinity and β_M to zero, the frequency equations of the system in Figure 3 are obtained:

$$\alpha_{me}\overline{\beta}\sin^2\eta\overline{\beta}\cdot(\tan\overline{\beta}+\cot\eta\overline{\beta})-1=0, \qquad \sum_{k=1}^{n}\frac{b_k^2}{\lambda_k-\Omega^2}-\frac{1}{\alpha_{me}\Omega^2}=0. \tag{21a, b}$$

The frequency equation (21a) was derived by Cutchins in reference [24].

Case 3

Taking $\beta_M = 0$ yields the frequency equation of the system in Figure 4:

$$\cos\overline{\beta}\cdot\left[-1+\frac{\alpha_{me}}{a_{ke}}\overline{\beta}^2+\frac{\alpha_{me}}{2}\overline{\beta}\sin 2\eta\overline{\beta}\right]+\alpha_{me}\overline{\beta}\sin^2\eta\overline{\beta}\sin\overline{\beta}=0, \tag{22a}$$

$$\sum_{k=1}^{n}\frac{b_k^2}{\lambda_k-\Omega^2}+\frac{1}{\alpha_{ke}}-\frac{1}{\alpha_{me}\Omega^2}=0. \tag{22b}$$

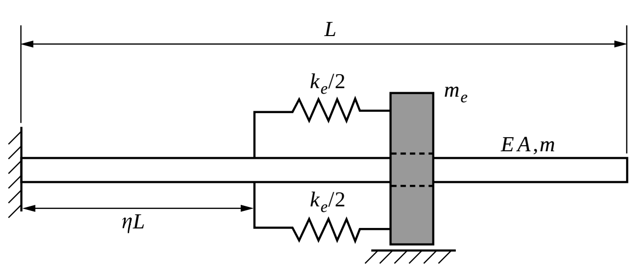

Figure 4. Longitudinally vibrating rod carrying a spring–mass system attached in-span.

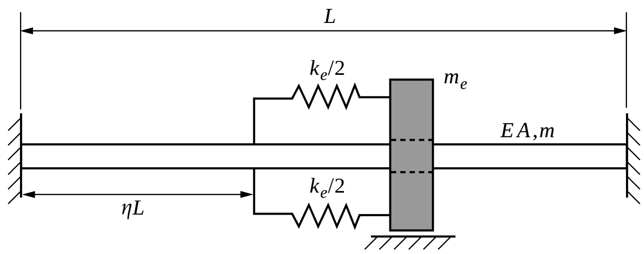

Figure 5. Longitudinally vibrating fixed–fixed rod carrying a spring–mass system attached in-span.

Case 4

If β_M goes to infinity, the frequency equations of the system in Figure 5 are obtained:

$$\alpha_{me}\bar{\beta}^2 \sin^2 \eta\bar{\beta} \cdot \cos\bar{\beta} - \bar{\beta}\sin\bar{\beta} \cdot \left[-1 + \frac{\alpha_{me}}{\alpha_{ke}}\bar{\beta}^2 + \frac{\alpha_{me}}{2}\bar{\beta}\sin 2\eta\bar{\beta} \right] = 0, \tag{23a}$$

$$\left(\sum_{k=1}^{n} \frac{a_k^2}{\lambda_k - \Omega^2} \right) \cdot \left[\sum_{k=1}^{n} \frac{b_k^2}{\lambda_k - \Omega^2} + \frac{1}{\alpha_{ke}} - \frac{1}{\alpha_{me}\Omega^2} \right] - \left[\sum_{k=1}^{n} \frac{a_k b_k}{\lambda_k - \Omega^2} \right]^2 = 0. \tag{23b}$$

Case 5

Finally, if α_{me} goes to infinity, the corresponding equations of the system in Figure 6 are obtained:

$$(\cos\bar{\beta} - \beta_M\bar{\beta}\sin\bar{\beta})\left(\frac{\bar{\beta}^2}{\alpha_{ke}} + \frac{\bar{\beta}}{2}\sin 2\eta\bar{\beta} \right) + \bar{\beta}\sin^2 \eta\bar{\beta} \cdot (\sin\bar{\beta} + \beta_M\bar{\beta}\cos\bar{\beta}) = 0, \tag{24a}$$

$$\left[\sum_{k=1}^{n} \frac{a_k^2}{\lambda_k - \Omega^2} - \frac{1}{\beta_M\Omega^2} \right] \cdot \left[\sum_{k=1}^{n} \frac{b_k^2}{\lambda_k - \Omega^2} + \frac{1}{\alpha_{ke}} \right] - \left[\sum_{k=1}^{n} \frac{a_k b_k}{\lambda_k - \Omega^2} \right]^2 = 0. \tag{24b}$$

4. DUNKERLEY-BASED FORMULA FOR THE FUNDAMENTAL FREQUENCY

It is intended now to derive a formula for the approximate determination of the fundamental eigenfrequency of the system in Figure 1 by means of Dunkerley's procedure.

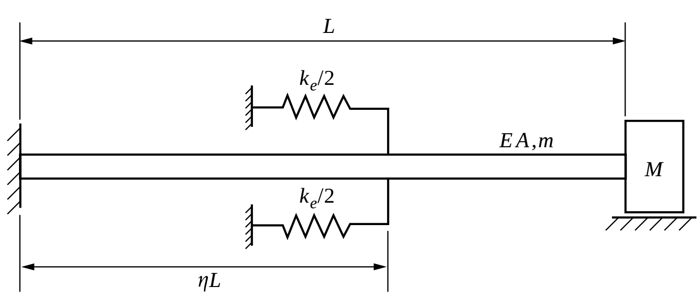

Figure 6. Longitudinally vibrating rod with a tip mass restrained by springs in-span.

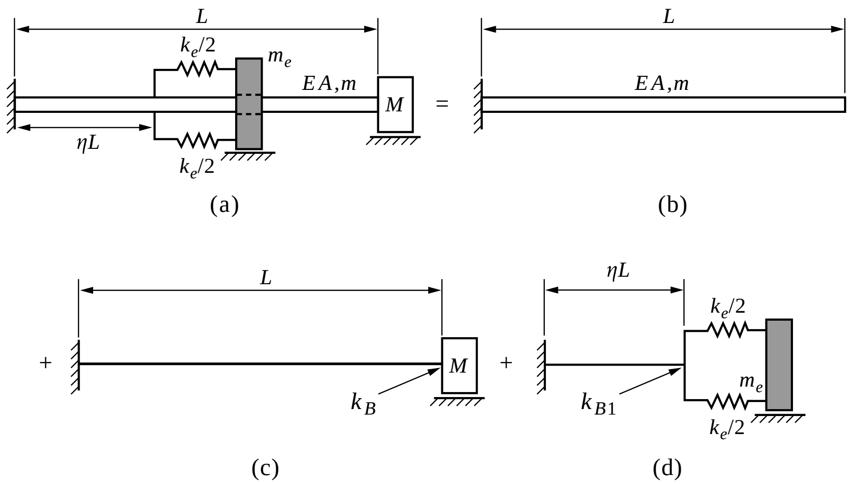

Figure 7. Partial systems for the application of Dunkerley's method.

The system can be thought of as the "sum" of the three partial systems shown in Figure 7. According to Dunkerley's method the following relation can be written [32]

$$\frac{1}{\omega_1^2} \approx \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}, \tag{25}$$

where ω_1 denotes the fundamental eigenfrequency of the original system reshown in Figure 7(a). In formula (25), ω_{11} represents the fundamental eigenfrequency of the bare fixed–free rod, as shown if Figure 7(b):

$$\omega_{11}^2 = \lambda_1(EA/mL^2) = \frac{\pi^2}{4}\omega_0^2. \tag{26}$$

ω_{22} represents the eigenfrequency of the massless fixed–free rod carrying a tip mass M, as shown in Figure 7(c). The equivalent stiffness of this rod at the tip is $k_B = AE/L$ such that

$$\omega_{22}^2 = AE/ML. \tag{27}$$

Finally, ω_{33} represents the eigenfrequency of the system shown in Figure 7(d) which is made up of a massless fixed–free rod of length ηL to the tip of which the spring–mass system k_e,m_e is appended. Recognizing that the equivalent stiffness which acts on the mass m_e is $k_e k_{B1}/(k_e + k_{B1})$ (springs connected in series), the square of the eigenfrequency is simply

$$\omega_{33}^2 = k_e k_{B1}/(m_e(k_e + k_{B1})), \tag{28}$$

where $k_{B1} = AE/\eta L$.

If now the expressions for ω_{11}^2, ω_{22}^2 and ω_{33}^2 from equations (26)–(28) are put into Dunkerley's formula (25), the following approximate formula is obtained for the fundamental eigenfrequency of the system in Figure 1.

$$\omega_1 = \left[\frac{1}{4/\pi^2 + \beta_M + \alpha_{me}(\eta + 1/\alpha_{ke})}\right]^{1/2}\sqrt{AE/mL^2} \equiv \overline{\beta}_1\omega_0 = \Omega_1\omega_0. \tag{29}$$

5. DERIVATION OF A SENSITIVITY FORMULA

A formula will now be derived for the sensitivity of the eigenfrequencies of the combined system with respect to small changes in the location of the in-span spring–mass attachment around its nominal position, i.e., the rate of change of the eigenfrequencies with respect to the location parameter η of the attachment point of the spring–mass. Instead of differentiating the exact frequency equation (7) partially with respect to η, one can directly use the sensitivity formula given in reference [22] as it is. It was obtained by differentiating partially the approximate frequency equation (18) with respect to η:

$$\Omega^{2\prime} = \frac{\partial \Omega^2(\eta)}{\partial \eta}$$

$$= 2\,\frac{S_1S_2 - S_3S_4 + S_4/\beta_M\Omega^2}{S_5S_6 + S_3S_7 - 2S_1S_8 + \dfrac{S_3}{\alpha_{me}\Omega^4} + \dfrac{S_6}{\beta_M\Omega^4} + S_5\left(\dfrac{1}{\alpha_{ke}} - \dfrac{1}{\alpha_{me}\Omega^2}\right) - \dfrac{S_7}{\beta_M\Omega^2} - \dfrac{2}{\alpha_{me}\beta_M\Omega^6} + \dfrac{1}{\alpha_{ke}\beta_M\Omega^4}}, \tag{30}$$

where the following abbreviations are used with the definitions previously given by equations (8) and (19):

$$S_1 = \sum_{k=1}^{n} \frac{a_kb_k}{\lambda_k - \Omega^2}, \qquad S_2 = \sum_{k=1}^{n} \frac{a_kb_k'}{\lambda_k - \Omega^2}, \qquad S_3 = \sum_{k=1}^{n} \frac{a_k^2}{\lambda_k - \Omega^2},$$

$$S_4 = \sum_{k=1}^{n} \frac{b_kb_k'}{\lambda_k - \Omega^2}, \qquad S_5 = \sum_{k=1}^{n} \frac{a_k^2}{(\lambda_k - \Omega^2)^2}, \qquad S_6 = \sum_{k=1}^{n} \frac{b_k^2}{\lambda_k - \Omega^2},$$

$$S_7 = \sum_{k=1}^{n} \frac{b_k^2}{(\lambda_k - \Omega^2)^2}, \qquad S_8 = \sum_{k=1}^{n} \frac{a_kb_k}{(\lambda_k - \Omega^2)^2},$$

$$b_k' = \frac{\partial b_k(\eta)}{\partial \eta} = \frac{\sqrt{2}}{2}(2k-1)\pi \cos[(2k-1)\pi\eta/2]. \tag{31}$$

In the expressions above, primes denote partial derivatives with respect to the location parameter η. It is now possible to give an approximate formula for the modified value Ω_{mod} of a non-dimensionalized eigenfrequency, if the location of the spring–mass attachment is changed by an amount $\Delta\eta$

$$\Omega_{mod} \approx \Omega(\eta) + \frac{1}{2\Omega(\eta)}\frac{\partial \Omega^2(\eta)}{\partial \eta}\Delta\eta. \tag{32}$$

In a similar manner, other sensitivities can also be obtained, for example sensitivities with respect to k_e, m_e or other parameters.

6. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluation of the formulae established in the preceding sections. The numerical solutions of the exact and approximate frequency equations were carried out by using MATLAB.

The first two dimensionless eigenfrequency parameters of the system in Figure 1 are given in Table 1 for various values of the stiffness and mass parameters α_{ke} and α_{me} of the appended spring–mass system, where $\eta = 0{\cdot}4$ and $\beta_M = 0{\cdot}5$ are taken. The values in the first rows are fundamental frequency parameters obtained from the Dunkerley-based expression (29), whereas those of the second row are values obtained from the exact

TABLE 1

The first two dimensionless eigenfrequency parameters of the system in Figure 1 for various values of stiffness and mass parameters, where $\eta = 0{\cdot}4$ and $\beta_M = 0{\cdot}5$

	α_{ke}							
α_{me}	0·5	1	1·5	2	2·5	3	5	10
0·5	0·689199	0·789267	0·833733	0·858983	0·875281	0·886676	0·910867	0·930370
	0·850233	0·953756	0·977593	0·987214	0·992338	0·995507	1·001306	1·005227
	0·850389	0·953982	0·977840	0·987470	0·992599	0·995772	1·001577	1·005501
	1·208986	1·453911	1·657715	1·810792	1·927295	2·017846	2·234698	2·427454
	1·209499	1·454794	1·659020	1·812489	1·929334	2·020174	2·237779	2·431220
1	0·550042	0·658624	0·712118	0·744264	0·765776	0·781198	0·815062	0·843564
	0·630291	0·779607	0·841108	0·870964	0·887830	0·898483	0·918143	0·931348
	0·630413	0·779818	0·841352	0·871220	0·888092	0·898748	0·918414	0·931621
	1·154610	1·263386	1·374885	1·472540	1·554239	1·622153	1·802633	1·990980
	1·155084	1·264103	1·375909	1·473869	1·555850	1·624016	1·805241	1·994447
1·5	0·471128	0·576842	0·631788	0·665885	0·689199	0·706174	0·744264	0·777255
	0·519665	0·659127	0·728699	0·767886	0·792152	0·808358	0·840067	0·862543
	0·519769	0·659326	0·728950	0·768163	0·792441	0·808654	0·840374	0·862853
	1·143873	1·221809	1·299376	1·369719	1·430989	1·483660	1·631244	1·797499
	1·144334	1·222462	1·300269	1·370866	1·432377	1·485271	1·633546	1·800674
2	0·418660	0·519504	0·573670	0·607986	0·631788	0·649303	0·689199	0·724469
	0·451894	0·578955	0·647194	0·688452	0·715455	0·734237	0·772969	0·802119
	0·451986	0·579140	0·647439	0·688733	0·715757	0·734553	0·773308	0·802468
	1·139407	1·205462	1·268717	1·325881	1·376147	1·419905	1·545732	1·693698
	1·139863	1·206085	1·269545	1·326927	1·377406	1·421363	1·547828	1·696644
2·5	0·380548	0·476445	0·529112	0·562964	0·586686	0·604274	0·644788	0·681158
	0·405099	0·521750	0·586750	0·627555	0·655134	0·674821	0·716974	0·750286
	0·405182	0·521922	0·586984	0·627830	0·655437	0·675143	0·717331	0·750664
	1·136969	1·196893	1·252669	1·302605	1·346509	1·384867	1·496461	1·630887
	1·137422	1·197499	1·253461	1·303592	1·347688	1·386228	1·498415	1·633660
3	0·351250	0·442578	0·493547	0·526660	0·550042	0·567478	0·607986	0·644788
	0·370331	0·478515	0·540117	0·579670	0·606948	0·626751	0·670288	0·705998
	0·370407	0·478675	0·540340	0·579938	0·607246	0·627072	0·670655	0·706396
	1·135435	1·191647	1·242908	1·288395	1·328279	1·363138	1·465056	1·589516
	1·135886	1·192243	1·243676	1·289343	1·329405	1·364433	1·466909	1·592157
5	0·278366	0·355665	0·400365	0·430121	0·451511	0·467680	0·506027	0·541905
	0·287627	0·373826	0·424762	0·458762	0·483065	0·501269	0·543445	0·580953
	0·287687	0·373955	0·424949	0·458994	0·483331	0·501562	0·543803	0·581369
	1·132567	1·182165	1·225480	1·263048	1·295625	1·323969	1·406950	1·510037
	1·133013	1·182741	1·226204	1·263924	1·296650	1·325136	1·408593	1·512379
10	0·200380	0·259018	0·293966	0·317736	0·335101	0·348396	0·380548	0·411509
	0·203764	0·265859	0·303404	0·329072	0·347835	0·362177	0·396627	0·429190
	0·203807	0·265953	0·303543	0·329248	0·348041	0·362408	0·396925	0·429538
	1·130576	1·175847	1·214083	1·246579	1·274412	1·298448	1·368326	1·455199
	1·131020	1·176410	1·214778	1·247406	1·275368	1·299525	1·369811	1·457296

TABLE 2

Fundamental eigenfrequency parameters of the system in Figure 6 for various values of the stiffness and mass parameters, where $\eta = 0{\cdot}4$

	α_{ke}					
β_M	0·5	1	1·5	2	2·5	3
0·5	1·128709	1·170129	1·203959	1·232088	1·255829	1·276123
	1·129151	1·170680	1·204626	1·232870	1·256721	1·277118
1	0·897444	0·926799	0·950594	0·970265	0·986794	1·000874
	0·898004	0·927459	0·951354	0·971120	0·987738	1·001901
1·5	0·766084	0·789787	0·808947	0·824756	0·838021	0·849308
	0·766656	0·790447	0·809695	0·825586	0·838926	0·850283
2	0·679134	0·699476	0·715898	0·729434	0·740784	0·750438
	0·679690	0·700112	0·716612	0·730221	0·741639	0·751354
2·5	0·616244	0·634315	0·648892	0·660902	0·670968	0·679527
	0·616778	0·634923	0·649571	0·661647	0·671774	0·680389
3	0·568066	0·584477	0·597710	0·608608	0·617740	0·625504
	0·568578	0·585058	0·598355	0·609314	0·618503	0·626318

frequency equation (7). The values of the third rows are obtained from the numerical solution of the approximate frequency equation (18), where $n = 100$ is taken. Finally, the values of the fourth and fifth rows are second eigenfrequency parameters obtained from the numerical solution of the equations (7) and (18), respectively. As can be seen from Table 1, the Dunkerley-based values of the first rows yield lower bounds. The accuracy of these values improves for increasing values of the appended mass. It is worth noting that the agreement

TABLE 3

Fundamental eigenfrequency parameters of the system in Figure 5 for various values of the stiffness and tip mass parameters, where $\eta = 0{\cdot}4$

	α_{ke}							
α_{me}	0·5	1	1·5	2	2·5	3	3·5	4
1	0·666825	0·892081	1·037077	1·140592	1·218585	1·279482	1·328307	1·368282
	0·667029	0·892579	1·037869	1·141654	1·219885	1·329997	1·329997	1·370130
1·5	0·544831	0·730063	0·850506	0·937523	1·003904	1·056369	1·098921	1·134135
	0·544997	0·730465	0·851148	0·938836	1·004967	1·057609	1·100318	1·135670
2	0·471995	0·632960	0·738124	0·814527	0·873151	0·919751	0·957754	0·989364
	0·472138	0·633307	0·738678	0·815273	0·874072	0·920829	0·958971	0·990706
2·5	0·422249	0·566510	0·661023	0·729909	0·782943	0·825241	0·859845	0·888715
	0·422376	0·566820	0·661517	0·730576	0·783768	0·826207	0·860938	0·889922
3	0·385509	0·517376	0·603925	0·667139	0·715914	0·754898	0·786858	0·813575
	0·385625	0·517658	0·604375	0·667748	0·716667	0·755782	0·787859	0·814681
3·5	0·356946	0·479145	0·559451	0·618193	0·663586	0·699924	0·729757	0·754731
	0·357053	0·479406	0·559868	0·618756	0·664284	0·700743	0·730685	0·755758
4	0·333915	0·448302	0·523546	0·578645	0·621272	0·655433	0·683510	0·707038
	0·334016	0·448546	0·523935	0·579171	0·621924	0·656200	0·683430	0·708001

TABLE 4

Comparison of the dimensionless eigenfrequencies of the system in Figure 1 if the location of the spring–mass attachment is changed slightly around the nominal value $\eta = 0{\cdot}4$, where $\beta_M = \alpha_{ke} = \alpha_{me} = 0{\cdot}5$

η	From equation (18)	From equation (32)
0·350	0·866253682	0·865666746
0·355	0·864615603	0·864138926
0·360	0·862991259	0·862611105
0·365	0·861378232	0·861083284
0·370	0·859774078	0·859555463
0·375	0·858181126	0·858027642
0·380	0·856601637	0·856499821
0·385	0·855033178	0·854972000
0·390	0·853473297	0·853444179
0·395	0·851924326	0·851916358
0·400	**0·850389**	**0·850389**
0·405	0·848863480	0·848860716
0·410	0·847346696	0·847332894
0·415	0·845840523	0·845805073
0·420	0·844347249	0·844277252
0·425	0·842864408	0·842749431
0·430	0·841389526	0·841221610
0·435	0·839924954	0·839693789
0·440	0·838473004	0·838165968
0·445	0·837031184	0·836638147
0·450	0·835597008	0·835110326

of the values from the second and third rows is very good, which in turn means that the approximate frequency equation (18) gives very accurate results, especially for the fundamental eigenfrequency parameters.

As a second numerical application, the fundamental frequency parameters of the system in Figure 6 are collected in Table 2 for various values of the tip mass parameter β_M and the stiffness parameter α_{ke}, where $\eta = 0{\cdot}4$ is taken. The first figures are exact values obtained from equation (24a), whereas those of the second rows are numerical results from equation (24b), for $n = 100$.

As another numerical application, the fundamental frequency parameters of the system shown in Figure 5 are given in Table 3 for various values of α_{ke} and α_{me}, where $\eta = 0{\cdot}4$ is taken. The first figures are exact values calculated from equation (23a), whereas those of the second rows are results from equation (23b), for $n = 100$. Comparison of the exact and approximate eigenfrequency parameters indicate close agreement.

Table 4 gives an indication on the accuracy of the sensitivity-related equation (32) in connection with equations (30) and (31), where the tip mass ratio is taken as $\beta_M = 0{\cdot}5$. As an application, frequency parameters from the numerical solution of the frequency equation (18) are given together with approximate values obtained from equation (32), assuming that slight changes occur around the nominal in-span spring–mass attachment $\eta = 0{\cdot}4$, where stiffness and mass parameters are taken as $\alpha_{ke} = 0{\cdot}5$, $\alpha_{me} = 0{\cdot}5$ and $n = 100$. The comparison of the values in the second and third columns indicate clearly that equation (32) gives very accurate approximations to the eigenfrequency parameters of the modified system without having to resolve frequency equation (18) for the parameters of the modified system.

7. CONCLUSIONS

The present study deals with the determination of the eigenfrequencies and their sensitivity of a fixed–free longitudinally vibrating rod carrying a tip mass (primary system) to which a spring–mass (secondary system) is attached in-span. After establishing the exact frequency equation, a formula is given for the approximate determination of the fundamental frequency, based on Dunkerley's procedure. Using the normal mode method, a second approximate but very accurate frequency equation is established with the aid of which a sensitivity formula is later derived. From the general expressions of the frequency equations, the frequency equations of some simpler systems are obtained as special cases. These frequency equations are then numerically solved for various combinations of physical parameters. To the authors best knowledge, the frequency equation derived here and its special cases are novel. These equations shall be considered valuable for the design engineers who are working in the dynamics of such systems.

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