Journal of Sound and Vibration (1998) **216**(2), 315–320
Article No. sv981686

COMMENTS ON "A NOTE ON A FINITE ELEMENT FOR VIBRATING THIN, ORTHOTROPIC RECTANGULAR PLATES"

Serge Abrate
Department of Technology, Southern Illinois University, Carbondale, IL 62901-6603, *U.S.A*

(*Received* 26 *January* 1998)

Rossi [1] used a classical finite element formulation to analyse the free vibrations of rectangular orthotropic plates. In order to show the accuracy of the finite element model, three examples are presented: (1) a simply-supported square isotropic plate, (2) a simply-supported square orthotropic plate; and (3) a clamped square orthotropic plate. The accuracy of the results is established by comparison with the results from the exact closed-form solution for simply supported plates and by comparison with results from previous investigators for the clamped plate example. Convergence of the finite element solution was demonstrated by giving the first four natural frequencies for 8, increasingly refined, uniform meshes. This discussion will address the convergence of the finite element solution and point out two typographical errors in the numerical results given in reference [1].

Strang and Fix [2] studied the eigenvalue and eigenfunction errors in finite element approximations. They proved the following convergence theorem: *If S^k is a finite element space of degree $k-1$, then there is a constant δ such that the approximate eigenvalues are bounded for small h by*

$$\lambda_i \leqslant \lambda_i^h \leqslant \lambda_i + 2\delta h^{2(k-m)}\lambda^{k/m_i}, \tag{1}$$

where $\lambda_i = \omega_i^2$ is the exact eigenvalue, $\lambda_i^h = (\omega_i^h)^2$ is the finite element approximation for a uniform mesh with element size h,and m is the highest degree of the derivatives in the strain energy density functional. Let us define the error for the ith eigenvalue as

$$E_i = \frac{\lambda_i^h}{\lambda_i} - 1. \tag{2}$$

For the bi-cubic element used in reference [1], $k = 4$ and $m = 2$. Therefore, inequalities (1) imply that the error E_i is proportional to h^4 or, in other words, that E_i is inversely proportional to the square of the number of elements. Plotting the error E_i vs the number of elements using a logarithmic scale for both axes should result in a series of straight lines with a slope of -2. The convergence theorem also indicates that the error increases with λ_i.

For each example, reference [1] gives results obtained using 25, 100, 225, 400, and 625 element meshes for the entire plate but also 225, 400, 625 element meshes for a quarter of the plate. Because of symmetry, these last three models are equivalent to using 900, 1600, and 2500 elements for the entire plate. Figures 1–3 show that for all three examples, the error for each eigenvalue is inversely proportional to the square of the number of elements

0022–460X/98/370315 + 06 $30.00/0

in the model. The hypotenuse of the triangles drawn in dotted lines in these figures has a slope equal to -2 to emphasize that point.

Deviations from perfectly straight lines on these plots are attributed to the fact that the results given in Tables 1–3 of reference [1] do not always carry enough significant figures to calculate the error accurately. It can be shown that, if the error E is to be known within $\pm\varepsilon$, the natural frequency must be known within $\pm\alpha$, where

$$\alpha = \tfrac{1}{2}\omega E\varepsilon. \tag{3}$$

For example, for the first mode of the clamped square orthotropic plate predicted a 625 element mesh, reference [1] gives a non-dimensional frequency of 29·9792 in Table 3. In that case, to have the error estimated within $\pm 1\%$ requires that the frequency be known within $\pm 3.3 \times 10^{-6}$. Therefore, the frequency should be given with six decimal places instead of four. Discrepancies observed in Figures 1–3 are attributable to the fact that not enough significant figures were available to calculate the error.

Displacement-based finite element approximations overestimate the natural frequencies and converge uniformly to the exact solution from above as indicated by the convergence

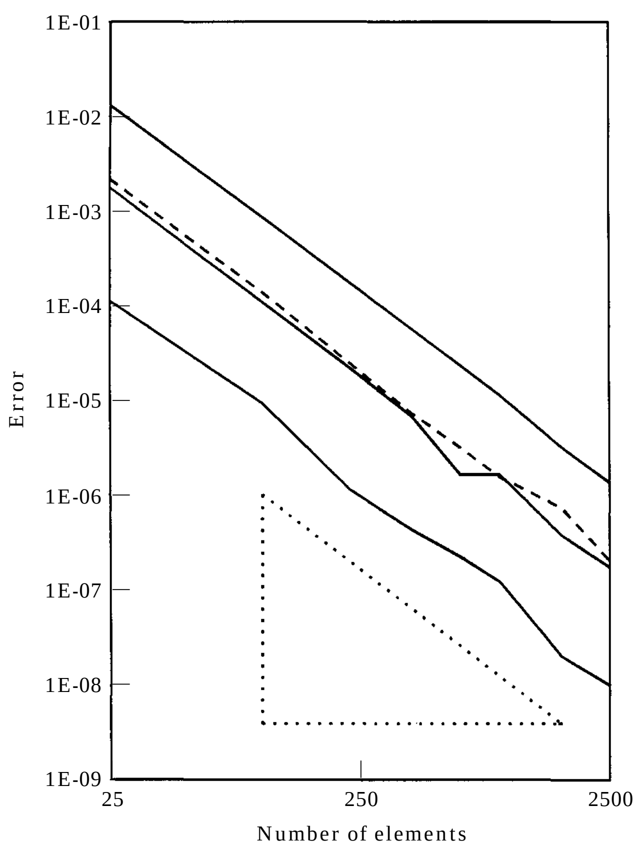

Figure 1. Error in eigenvalues as a function of the number of elements for the simply supported, square, isotropic plate in reference [1].

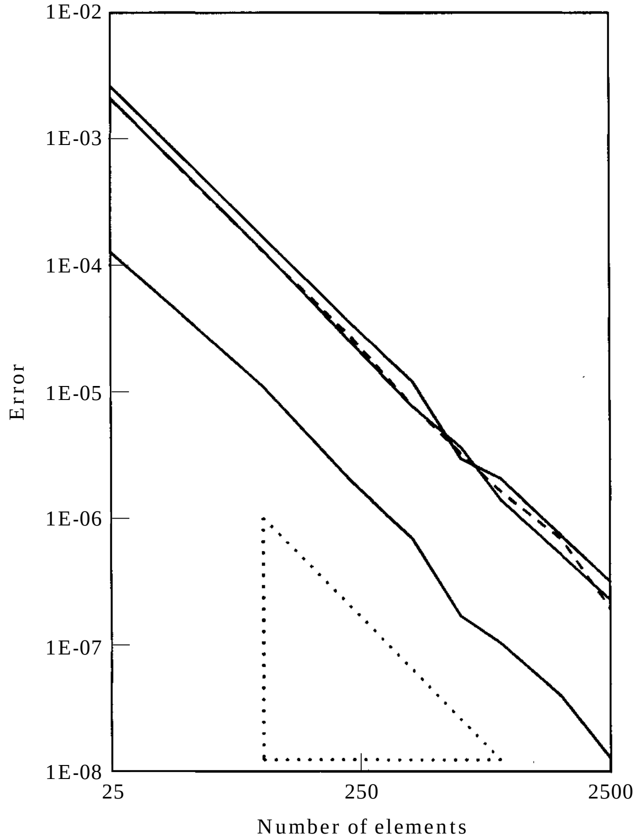

Figure 2. Error in eigenvalues as a function of the number of elements for the simply supported, square, orthotropic plate in reference [1].

theorem. In reference [1], the second non-dimensional frequency for the square isotropic plate is predicted to be 49·3514, 49·3587, and 49·3482 for uniform meshes with 100, 225, and 400 elements respectively. The value of 49·3587 is greatly in error and is inconsistent, since the frequency for that mesh is expected be less than 49·3514. This data point was omitted in Figure 1 but, from the general trend of the error for that frequency, we suspect that the actual finite element must have been 49·3487. The exact solution for the first natural frequency for the simply supported orthotropic plate given in Table 2 of reference [1] should be 15·6052147 instead of 16·605214.

The purpose of that discussion was to examine the convergence rate of the eigenvalues for the finite element approximation using the element developed in reference [1]. With the error defined as in equation (2), the error is inversely proportional to the square of the number of elements. The author is to be congratulated for the quality of his work and for providing extensive numerical results that have made these observations possible.

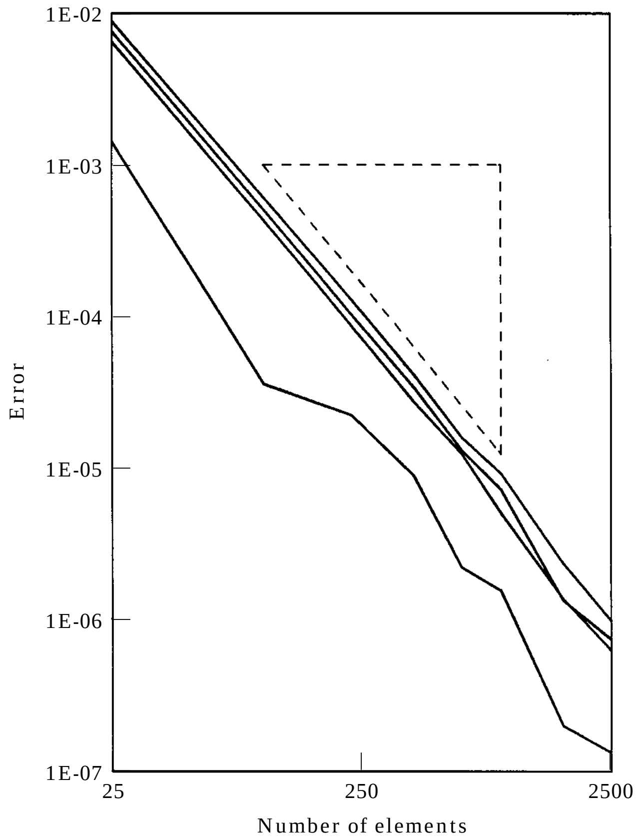

Figure 3. Error in eigenvalues as a function of the number of elements for the clamped, square, orthotropic plate in reference [1].

REFERENCES

1. R. E. Rossi 1997 *Journal of Sound and Vibration* **208**(5), 864–865. A note on a finite element for vibrating thin, orthotropic rectangular plates.
2. G. Strang and G. J. Fix 1973 *An Analysis of the Finite Element Method*. Englewood Cliffs, NJ: Prentice–Hall; 230.

AUTHOR'S REPLY

R. E. Rossi

Department of Engineering, Universidad Nacional del Sur, 8000 *Bahía Blanca, Argentina*

(*Received* 14 *April* 1998)

The author is indebted to Professor Serge Abrate for his valuable comments and constructive criticism [1] which have enhanced considerably the value of his contribution.

At the same time the author apologizes for two typographical errors in Tables 1 and 2 pointed out by Professor Abrate and another one in Table 3.

TABLE 1

The frequency coefficients of a simply supported, square, isotropic plate

Number of elements	Ω_1	$\Omega_2 = \Omega_3$	Ω_4	$\Omega_5 = \Omega_6$
25	19·7402843	49·4014425	79·0264868	99·3402231
100	19·7392757	49·3514311	78·9611372	98·7389575
225	19·7392220	49·3486979	78·9576819	98·7046187
400	19·7392130	49·3482361	78·9571028	98·6987680
625	19·7392105	49·3481098	78·9569447	98·6971618
225*	19·7392096	49·3480643	78·9568880	98·6965836
400*	19·7392091	49·3480354	78·9568519	98·6962149
625*	19·7392089	49·3480275	78·9568420	98·6961140
Exact solution	19·7392088	49·3480220	78·9568352	98·6960440

* Results obtained using one-quarter of the plate.

TABLE 2

The frequency coefficients of a simply supported, square, orthotropic plate ($D_2/D_1 = 0{\cdot}5$; $D_3/D_1 = 0{\cdot}5$; $\nu_2 = 0{\cdot}3$)

Number of elements	Ω_1	Ω_2	Ω_3	Ω_4
25	15·6062290	35·6225523	44·7449250	62·4856077
100	15·6052781	35·5877378	44·6902833	62·4249160
225	15·6052273	35·5858352	44·6872787	62·4216601
400	15·6052187	35·5855138	44·6867704	62·4211124
625	15·6052164	35·5854258	44·6866312	62·4209628
225*	15·6052155	35·5853942	44·6865812	62·4209091
400*	15·6052150	35·5853741	44·6865493	62·4208750
625*	15·6052148	35·5853686	44·6865406	62·4208655
Exact solution	15·6052147	35·5853647	44·6865345	62·4208590

* Results obtained using one-quarter of the plate.

TABLE 3

The frequency coefficients of a clamped, square, orthotropic plate ($D_2/D_1 = 0{\cdot}5$; $D_3/D_1 = 0{\cdot}5$; $\nu_2 = 0{\cdot}3$)

Number of elements	Ω_1	Ω_2	Ω_3	Ω_4
25	30·0006322	54·5135497	68·0546078	88·5513045
100	29·9806946	54·3484336	67·8147929	88·1860084
225	29·9794826	54·3390446	67·8011100	88·1646577
400	29·9792685	54·3374237	67·7987567	88·1608931
625	29·9792086	54·3369750	67·7981068	88·1598383
225*	29·9791869	54·3368127	67·7978719	88·1594539
400*	29·9791729	54·3367088	67·7977217	88·1592064
625*	29·9791691	54·3366803	67·7976805	88·1591382
Reference [2]	29·979167	54·336663	67·797655	88·159097

* Results obtained using one-quarter of the plate.

Table 1: The third value of the column $\Omega_2 = \Omega_3$ should be 49·3487 (instead of 49·3587).

Table 2: The last value of the column corresponding to Ω_1 is 15·6052147 (instead of 16·6052147).

Table 3: The second value of the column corresponding to Ω_1 is 29·9807 (instead of 29·9797).

Also the author is providing again Tables 1, 2 and 3 with additional significant figures (seven decimal places) in order that the error analysis suggested by Professor Abrate may be performed, if so desired.

REFERENCES

1. S. Abrate 1998 *Journal of Sound and Vibration* **216,** 315–318. Comments on "A note on a finite element for vibrating thin, orthotropic rectangular plates".
2. T. Sakata and K. Hosokawa 1988 *Journal of Sound and Vibration* **125**, 429–439. Vibrations of clamped orthotropic rectangular plates.