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RECURRENCE SCHEME FOR THE GENERATION OF TWO-DIMENSIONAL BOUNDARY CHARACTERISTIC ORTHOGONAL POLYNOMIALS TO STUDY VIBRATION OF PLATES

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1. INTRODUCTION

The Rayleigh–Ritz method is a very powerful technique that can be used to predict the natural frequencies and mode shapes of vibrating structures. The method requires assumption of deflection shapes that satisfy at least the geometrical boundary conditions of the vibrating structure in order to evaluate its maximum kinetic and potential energies. Vibration of one-dimensional structures or structures whose deflection can be assumed in the form of product of one-dimensional functions, such as rectangular plates, were studied using one-dimensional boundary characteristic orthogonal polynomials proposed by Bhat [1]. The method has been successfully used to study plate vibration with complicating effects by Dickinson and Di Blasio [2], Kim and Dickinson [3], Liew *et al*. [4, 5], etc. Many studies on the vibration of non-rectangular plates assuming various deflection shapes in the Rayleigh–Ritz method are reported by Leissa [6]. Bhat [7] proposed two-dimensional boundary characteristic orthogonal polynomials to study non-rectangular plates in the Rayleigh–Ritz method. Singh and Chakraverty [8–10] generated the two-dimensional boundary characteristic orthogonal polynomials in a systematic fashion to study the vibration of elliptic plates with clamped, simply-supported and free boundaries. They also generated the two-dimensional boundary characteristic orthogonal polynomials over other domains in order to study plates of various geometries [11].

In constructing one-dimensional boundary characteristic orthogonal polynomials one can use the well-known three-term recurrence relation, as given in Chihara [12]. However, no such recurrence relation was employed in constructing the two-dimensional boundary characteristic orthogonal polynomials. They were generated by orthogonalizing with all the previously generated orthogonal polynomials.

The present paper provides an efficient numerical scheme for generating the two-dimensional boundary characteristic orthogonal polynomials using recurrence relations suggested in references [13–15] for the study of vibration of plates of various geometries. The three-term recurrence relation to generate multi-dimensional orthogonal polynomials, presented by Kowlaski [13, 14], did not consider these polynomials to satisfy any conditions at the boundary of the domain. In the present study, however, two-dimensional orthogonal polynomials are constructed so as to satisfy the essential boundary conditions of the vibrating structure. Such two-dimensional orthogonal polynomials constructed using the recurrence relations are compared with those constructed by orthogonalizing with all the previous polynomials, in order to verify the validity of the recurrence relations.

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2. RECURRENCE RELATIONS FOR MULTI-DIMENSIONAL ORTHOGONAL POLYNOMIALS

For one-dimensional orthogonal polynomials $\{\phi_k(x)\}$, we have the three-term recurrence relation as

$$\phi_{k+1}(x) = (d_k x + e_k)\phi_k(x) + p_k \phi_{k-1}(x), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where the coefficients d_k, e_k, p_k, $k = 0, 1, 2, \ldots,$ can be found using the orthogonality property.

A similar three-term recurrence relation among orthogonal polynomials in n-variables was reported by Kowalski [13, 14]. In order to develop such a recurrence scheme that can be numerically implemented for two variables, some preliminaries are discussed as follows.

Let

$$\prod\nolimits_n^\infty$$

be a vector space of all polynomials with real coefficients in n variables and let

$$\prod\nolimits_n^k$$

be its subspace of polynomials whose total degree in n variables is not larger than k, then

$$\dim \prod_n^k = \sum_{i=0}^{k} r_n^i = \binom{n+k}{k},$$

where

$$r_n^k = \binom{n+k-1}{k}$$

is the number of monomials in the basis whose degree is equal to k.

If a basis in

$$\prod\nolimits_n^\infty$$

be denoted by $\{\phi_i^k\}_{k=0,i=1}^{\infty,r_n^k}$, and each polynomial is of the degree indicated by its superscript, then we define

$$\overline{\phi_k}(x) = [\phi_1^k(x), \phi_2^k(x), \ldots, \phi_{r_n^k}^k(x)]^T, \tag{2}$$

and

$$\overline{x\phi_k(x)} = [x_1\overline{\phi_k}(x)^T | x_2\overline{\phi_k}(x)^T| \cdots |x_n\overline{\phi_k}(x)^T]^T, \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$, $k = 0, 1, \ldots.$

A recurrence formula relating $\overline{\phi_{k-1}}$, $\overline{\phi_k}$ and $\overline{\phi_{k+1}}$ can be written as

$$\begin{aligned} \overline{x\phi_k} &= A_k\overline{\phi_{k+1}} + B_k\overline{\phi_k} + C_k\overline{\phi_{k-1}}, \\ \overline{\phi_{k+1}} &= D_k\overline{x\phi_k} + E_k\overline{\phi_k} + G_k\overline{\phi_{k-1}}, \qquad k = 0, 1, \ldots, \end{aligned} \tag{4}$$

where A_k, B_k, C_k, D_k, E_k and G_k are matrices with

$$A_k : nr_n^k \times r_n^{k+1},\ B_k : nr_n^k \times r_n^k,\ C_k : nr_n^k \times r_n^{k-1},$$

$$D_k : r_n^{k+1} \times nr_n^k,\ E_k : r_n^{k+1} \times r_n^k,\ G_k : r_n^{k+1} \times r_n^{k-1},$$

and $\overline{\phi_{-1}} = 0$, $C_0 = G_0 = 0$.

Further, if relations (4) hold then

$$\left.\begin{aligned} D_k A_k &= I, \\ E_k &= -D_k B_k, \\ G_k &= -D_k C_k, \end{aligned}\right\}. \tag{5}$$

3. NUMERICAL IMPLEMENTATION OF THE RECURRENCE SCHEME TO GENERATE THE TWO-DIMENSIONAL ORTHOGONAL POLYNOMIALS

For the numerical implementation of the procedure, the first polynomial is defined as

$$\phi_1^{(1)} = g(X, Y), \tag{6}$$

where $g(X, Y)$ satisfies the essential boundary conditions. As an example the function $g(X, Y)$ is defined for an elliptical boundary as

$$g(X, Y) = \left(1 - \frac{X^2}{a^2} - \frac{Y^2}{b^2}\right)^s,$$

where a and b are the semi major and minor axes of the ellipse, respectively, and s takes the value of 0, 1 or 2 in order to define free, simply-supported or clamped conditions, respectively, at the boundary.

In order to incorporate the recurrence scheme of Kowalski [13, 14], the polynomials are organised in to the following classes forming a pyramid given by

$$\begin{array}{cccccccccc}
 & & & & \phi_1^{(1)} & & & & & \\
 & & & \phi_2^{(2)} & & \phi_3^{(2)} & & & & \\
 & & \phi_4^{(3)} & & \phi_5^{(3)} & & \phi_6^{(3)} & & & \\
 & \phi_7^{(4)} & & \phi_8^{(4)} & & \phi_9^{(4)} & & \phi_{10}^{(4)} & & \\
 & - & - & & - & & - & & - & \\
 - & & - & & - & & - & & - & - \\
\phi^{(n)}_{\{n(n+1)/2)(-(n-1)\}} & \phi^{(n)}_{\{(n(n+1)/2)-(n-2)\}} & - & \cdot & \cdot & & - & & \phi^{(n)}_{\{(n(n+1)/2)-1\}} & \phi^{(n)}_{\{n(n+1)/2\}}
\end{array}$$

where the superscript in $\phi_i^{(j)}$ denotes the class number j to which it belongs.

The inner product of two functions $\phi_i^{(j)}(X, Y)$ and $\phi_k^{(r)}(X, Y)$ is defined as

$$\langle \phi_i^{(j)}, \phi_k^{(r)} \rangle = \iint_R \phi_i^{(j)}(X, Y)\, \phi_k^{(r)}(X, Y)\, \mathrm{d}x\, \mathrm{d}y. \tag{7}$$

The norm of $\phi_i^{(j)}$ is therefore given by

$$\| \phi_i^{(j)} \| = \langle \phi_i^{(j)}, \phi_i^{(j)} \rangle^{1/2}. \tag{8}$$

It is interesting to note here that in each class of polynomials, the number of orthogonal polynomials is equal to the class number. For example, class number 2 contains 2 orthogonal polynomials and those can be obtained as

$$\phi_2^{(2)} = F_2 = X\phi_1^{(1)} - \alpha_{21}\phi_1^{(1)}, \tag{9}$$

and

$$\phi_3^{(2)} = F_3 = Y\phi_1^{(1)} - \alpha_{31}\phi_1^{(1)} - \alpha_{32}\phi_2^{(2)}, \tag{10}$$

where

$$\alpha_{21} = \frac{\langle X\phi_1^{(1)}, \phi_1^{(1)}\rangle}{\langle \phi_1^{(1)}, \phi_1^{(1)}\rangle}, \qquad \alpha_{31} = \frac{\langle Y\phi_1^{(1)}, \phi_1^{(1)}\rangle}{\langle \phi_1^{(1)}, \phi_1^{(1)}\rangle}, \qquad \alpha_{32} = \frac{\langle Y\phi_1^{(1)}, \phi_2^{(2)}\rangle}{\langle \phi_2^{(2)}, \phi_2^{(2)}\rangle},$$

and so on. It is to be noted here that the above relations (9) and (10) are the same as Kowalski's relations given in equation (4), where

$$A_k = \begin{bmatrix} 1 & 0 \\ \alpha_{32} & 1 \end{bmatrix}, \qquad B_k = \begin{bmatrix} \alpha_{21} \\ \alpha_{31} \end{bmatrix}, \qquad \mathrm{D_k} = \begin{bmatrix} 1 & 0 \\ -\alpha_{32} & 1 \end{bmatrix}, \qquad E_k = \begin{bmatrix} -\alpha_{21} \\ -\alpha_{31} + \alpha_{32}\alpha_{21} \end{bmatrix},$$

and $C_k = G_k = 0$.

Moreover, the above matrices also satisfy equations (5). Now, in general, class j will have j orthogonal polynomials, and those can be generated by the recurrence scheme.

$$\phi_i^{(j)} = X\phi_{\{i-(j-1)\}}^{(j-1)} - \sum_{k=l}^{i-1} \alpha_{ik}\phi_k^{(j)} - \sum_{k=m}^{l-1} \alpha_{ik}\phi_k^{(j-1)} - \sum_{k=p}^{m-1} \alpha_{ik}\phi_k^{(j-2)};\ i = \{L-(j-1)\}, \ldots, \{L-1\},$$

$$Y\phi_{(i-j)}^{(j-1)} - \sum_{k=l}^{i-1} \alpha_{ik}\phi_k^{(j)} - \sum_{k=m}^{l-1} \alpha_{ik}\phi_k^{(j-1)} - \sum_{k=p}^{m-1} \alpha_{ik}\phi_k^{(j-2)};\ i = L, \qquad j = 2, 3, \ldots, N, \tag{11}$$

where

$$l = \left\{\frac{j(j+1)}{2} - (j-1)\right\}$$

describes the first orthogonal polynomial of the jth class,

$$L = \frac{j(j+1)}{2}$$

describes the last orthogonal polynomial of the jth class,

$$m = \left\{\frac{(j-1)j}{2} - (j-2)\right\}$$

describes the first orthogonal polynomial of the $(j-1)$th class, and

$$p = \left\{\frac{(j-2)(j-1)}{2} - (j-3)\right\} \geqslant 0$$

describes the first orthogonal polynomial of the $(j-2)$th class. Also,

$$\alpha_{ik} = \begin{cases} \dfrac{\langle X\phi^{(j-1)}_{\{i-(j-1)\}}, \phi^{(r)}_k \rangle}{\langle \phi^{(r)}_k, \phi^{(r)}_k \rangle}, \ i = \{L-(j-1)\}, \ldots, \{L-1\}, \\[2ex] \dfrac{\langle Y\phi^{(j-1)}_{(i-j)}, \phi^{(r)}_k \rangle}{\langle \phi^{(r)}_k, \phi^{(r)}_k \rangle}, \ i = L, \end{cases} \tag{12}$$

where $r = j$, $(j-1)$ and $(j-2)$, respectively, corresponding to α_{ik} appearing in the first, second and third summation terms in equation (11). These three summation terms in equation (11) clearly demonstrate that all the orthogonal polynomials in the jth class can be constructed using only those previously generated in the jth class so far, and those of the previous two classes i.e., $(j-1)$th and $(j-2)$th.

The orthogonal polynomials are then normalized by the condition

$$\hat{\phi}^{(j)}_i = \frac{\phi^{(j)}_i}{\|\phi^{(j)}_i\|}.$$

When the structure is undergoing simple harmonic motion, equating the maximum strain energy, V_{max}, and the maximum kinetic energy, T_{max}, of the deformed plate, the Rayleigh quotient is obtained as

$$\omega^2 = \frac{D \iint_R [W^2_{xx} + 2\nu W_{xx} W_{yy} + W^2_{yy} + 2(1-\nu) W^2_{xy}] \, \mathrm{d}y \, \mathrm{d}x}{\varrho h \iint_R W^2 \, \mathrm{d}y \, \mathrm{d}x}, \tag{13}$$

where $W(x, y)$ is the deflection of the plate, subscripts on W denote differentiation with respect to the subscripted variable, $D = Eh^3/(12(1-\nu^2))$ is the flexural rigidity, E is Young's modulus, ν is the Poisson ratio, h is the uniform plate thickness, ϱ is the density of the plate material and ω is the radian frequency of vibration.

The plate deflection is assumed in the form

$$W(x, y) = \sum_{i=1}^{N} c_i \phi_i(x, y), \tag{14}$$

where the superscript describing the class of the orthogonal polynomial is dropped since it is no longer needed.

Applying the condition for stationarity of ω^2 with respect to the coefficients c_j, in the form $\partial\omega^2/\partial c_j = 0$, results in the eigenvalue problem

$$\sum_{i=1}^{N} (a_{ji} - \lambda^2 b_{ji}) c_i = 0, \qquad j = 1, 2, \ldots, N, \tag{15}$$

where

$$a_{ji} = \iint_R \{(\phi_i)_{XX}(\phi_j)_{XX} + (\phi_i)_{YY}(\phi_j)_{YY} + \nu[(\phi_i)_{XX}(\phi_j)_{YY} + (\phi_i)_{YY}(\phi_j)_{XX}]$$

$$+ 2(1-\nu)(\phi_i)_{XY}(\phi_j)_{XY}\} \, \mathrm{d}Y \, \mathrm{d}X, \tag{16}$$

$$b_{ji} = \iint_R \phi_i \phi_j \, \mathrm{d}Y \, \mathrm{d}X = \delta_{ji}, \qquad \lambda^2 = \frac{a^4 \varrho h \omega^2}{D}, \tag{17, 18}$$

and $X = x/a$, $Y = y/a$.

It may be noted that $(\phi_i)_{XX}$, $(\phi_i)_{YY}$, etc. are second derivatives of ϕ_i with respect to X and Y, and δ_{ji} is given by $\delta_{ji} = 0$, if $j \neq i$; $\delta_{ji} = 1$, if $j = i$.

4. DISCUSSIONS AND CONCLUDING REMARKS

The recurrence scheme presented above is quite convenient for computer implementation. Two-dimensional boundary characteristic orthogonal polynomials have been generated using the present scheme for a variety of geometries. The orthonormality of the generated polynomials have also been verified. As already discussed, generation of any class of orthogonal polynomials requires only the orthogonal polynomials of the previous two classes and the orthogonal polynomials of the current class that have been already generated so far. Hence, the undue labour of orthogonalization with all the previous orthogonal polynomials is saved. Although the orthogonal polynomials generated by the present method are identical to those obtained in references [8–11], the present algorithm makes the generation quite efficient and straight forward in comparison to the previous method and the execution time is also greatly reduced.

The present scheme can be applied to vibration analysis of plates with various geometries and complicating effects. It is important to mention here that this scheme can easily be extended to study problems in three (or more) dimensions in vibration, fluid mechanics, diffusion etc.

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