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FREE VIBRATION ANALYSIS OF RECTANGULAR PLATES WITH VARIABLE THICKNESS

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An approximate method for analyzing the free vibration of thin and moderately thick rectangular plates with arbitrary variable thickness is proposed. The approximate method is based on the Green function of a rectangular plate. The Green function of a rectangular plate with arbitrary variable thickness is obtained as a discrete form solution for deflection of the plate with a concentrated load. The discrete form solution is obtained at each discrete point equally distributed on the plate. It is shown that the numerical solution for the Green function has good convergency and accuracy. By applying the Green function, the free vibration problem of the plate is translated into the eigenvalue problem of the matrix. The convergency and accuracy of the numerical solutions for the natural frequency parameter calculated by the proposed method are investigated, and the frequency parameters and their modes of free vibration are shown for some rectangular plates.

1. INTRODUCTION

The fundamental differential equations of free vibration of plates with variable thickness have variable coefficients concerning the flexural rigidity and thickness of the plate, and generally it seems almost impossible to get the analytical solution.

For some cases of variable thickness of a rectangular plate, investigations have been made and solutions have been obtained. Appl and Byers [1] investigated the case when the thickness varied only in one direction, and calculated the fundamental frequency of a simply supported rectangular plate having a linear thickness variation, and Plunkett [2] investigated the free vibration of linearly tapered rectangular cantilever plates. After that, a number of papers were published on the free vibration of rectangular plates with stepped thickness [3, 4], with linearly varying thickness [5–13], with bilinearly varying thickness [14, 15] and with thickness varying in two directions [16].

In this paper, an approximate method for generally analyzing the free vibration of thin and moderately thick rectangular plates with arbitrary varying thickness is proposed. At first the approximate solutions of a rectangular plate with variable thickness for a concentrated load are obtained at the discrete points equally

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distributed on the plate. The solution for deflection gives the discrete-type Green function of the plate. It is shown that the numerical solution for the Green function has good convergency and accuracy.

By applying the Green function, the free vibration problem of the plate is translated into the eigenvalue problem of the matrix. For some rectangular plates with various boundary conditions and a plate with variable thickness the convergency and accuracy of the numerical solutions for the natural frequency parameter calculated by the proposed method are investigated, and the lowest 16 frequency parameters and their modes of free vibration are shown.

2. DISCRETE GREEN FUNCTION OF A PLATE WITH VARIABLE THICKNESS

The Green function of the plate bending problem is given by the displacement function of the plate with a unit concentrated load, so the Green function $w(x, y, x_q, y_r)/\bar{P}$ of plates with variable thickness can be obtained from the fundamental differential equations of the plate with a concentrated load $\bar{P}$ at a point (x_q, y_r), which are given by following equations

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{P}\delta(x - x_q)\delta(y - y_r) = 0, \qquad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \quad \text{(1a, b)}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \qquad \frac{\partial \theta_x}{\partial x} + \nu\frac{\partial \theta_y}{\partial y} = \frac{M_x}{D}, \qquad \frac{\partial \theta_y}{\partial y} + \nu\frac{\partial \theta_x}{\partial x} = \frac{M_y}{D} \quad \text{(1c–e)}$$

$$\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2}{(1-\nu)}\frac{M_{xy}}{D}, \qquad \frac{\partial w}{\partial x} + \theta_x = \frac{Q_x}{Gt_s}, \qquad \frac{\partial w}{\partial y} + \theta_y = \frac{Q_y}{Gt_s} \quad \text{(1f–h)}$$

where Q_x, Q_y are the shearing forces, M_{xy} is the twisting moment, M_x, M_y are the bending moments, θ_x, θ_y are the slopes, w is the deflection, $D = Eh^3/12(1 - \nu^2)$ is the bending rigidity, E, G are the modulus and shear modulus of elasticity, ν is the Poisson ratio, $h = h(x, y)$ is the thickness of the plate, and $t_s = h/1{\cdot}2$, $\delta(x - x_q)$, $\delta(y - y_r)$ represent Dirac's delta functions.

By introducing the following non-dimensional expressions,

$$[X_1, X_2] = \frac{a^2}{D_0(1-\nu^2)}[Q_y, Q_x], \qquad [X_3, X_4, X_5] = \frac{a}{D_0(1-\nu^2)}[M_{xy}, M_y, M_x],$$

$$[X_6, X_7, X_8] = [\theta_y, \theta_x, w/a],$$

the differential equations (1a)–(1h) can be rewritten as follows:

$$\sum_{e=1}^{8}\left[F_{1te}\frac{\partial X_e}{\partial \zeta} + F_{2te}\frac{\partial X_e}{\partial \eta} + F_{3te}X_e\right] + P\delta(\eta - \eta_q)\delta(\zeta - \zeta_r)\delta_{1t} = 0, \qquad (2)$$

where $t = 1 \sim 8$, $\mu = b/a$, $\eta = x/a$, $\zeta = y/b$, $D_0 = Eh_0^3/12(1 - \nu^2)$ is the standard bending rigidity, h_0 is the standard thickness of the plate, a, b are the breadth and length of the rectangular plate, $P = \bar{P}a/D_0(1 - \nu^2)$, δ_{ft} is Kronecker's delta and F_{1te}, F_{2te}, F_{3te} are given in Appendix A.

3. DISCRETE SOLUTION OF FUNDAMENTAL DIFFERENTIAL EQUATION

With a rectangular plate divided vertically into m equal-length parts and horizontally n equal-length parts, as shown in Figure 1, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. In this paper, the rectangular area, $0 \leqslant \eta \leqslant \eta_i$, $0 \leqslant \zeta \leqslant \zeta_j$, corresponding to the arbitrary intersection (i, j), as shown in Figure 1, is denoted as the area $[i, j]$, the intersection (i, j) denoted by ⊚ is called the main piont of the area $[i, j]$, the intersections denoted by ○ are called the inner dependent points of the area, and the intersections denoted by ● are called the boundary dependent points of the area.

By integrating equation (2) over the area $[i, j]$, the following equation is obtained.

$$\sum_{e=1}^{8} \left\{ \begin{aligned} & F_{1te} \int_0^{\eta_i} [X_e(\eta, \zeta_j) - X_e(\eta, 0)] \, \mathrm{d}\eta + F_{2te} \int_0^{\zeta_j} [X_e(\eta_i, \zeta) - X_e(0, \zeta)] \, \mathrm{d}\zeta \\ & + F_{3te} \int_0^{\eta_i} \int_0^{\zeta_j} X_e(\eta, \zeta) \, \mathrm{d}\eta \, \mathrm{d}\zeta \end{aligned} \right\}$$

$$+ Pu(\eta - \eta_q) u(\zeta - \zeta_r) \delta_{1t} = 0, \tag{3}$$

where $u(\eta - \eta_q)$, $u(\zeta - \zeta_r)$ is the unit step function.

Next, by applying the numerical method, the simultaneous equation for the unknown quantities $X_{eij} = X_e(\eta_i, \zeta_j)$ at the main point (i, j) of the area $[i, j]$ is obtained as follows.

$$\sum_{e=1}^{8} \left\{ F_{1te} \sum_{k=0}^{i} \beta_{ik} (X_{ekj} - X_{ek0}) + F_{2te} \sum_{\ell=0}^{j} \beta_{j\ell} (X_{ei\ell} - X_{e0\ell}) + F_{3te} \sum_{k=0}^{i} \sum_{\ell=0}^{j} \beta_{ik} \beta_{j\ell} X_{ek\ell} \right\}$$

$$+ P u_{iq} u_{jr} \delta_{1t} = 0, \tag{4}$$

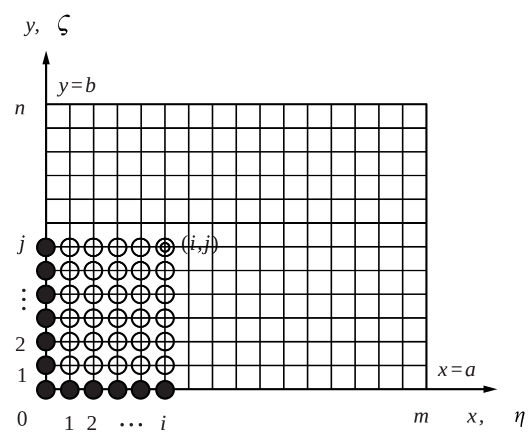

Figure 1. Discrete points on a plate.

where

$$u_{iq} = \begin{cases} 0 & (i < q) \\ 0{\cdot}5 & (i = q), \\ 1 & (i > q) \end{cases} \qquad u_{jr} = \begin{cases} 0 & (j < r) \\ 0{\cdot}5 & (j = r), \\ 1 & (j > r) \end{cases}$$

$$\alpha_{ik} = \begin{cases} 0{\cdot}5 & (k = 0, i) \\ 1 & (k \neq 0, i) \end{cases}, \qquad \alpha_{j\ell} = \begin{cases} 0{\cdot}5 & (\ell = 0, j) \\ 1 & (\ell \neq 0, j) \end{cases}$$

$$\beta_{ik} = \alpha_{ik}/m, \qquad \beta_{j\ell} = \alpha_{j\ell}/n.$$

The solution X_{pij} of the simultaneous equation (4) is obtained as follows.

$$X_{pij} = \sum_{e=1}^{8} \left\{ \begin{array}{l} \sum_{k=0}^{i} \beta_{ik} A_{pe}[X_{ek0} - X_{ekj}(1 - \delta_{ki})] + \sum_{\ell=0}^{j} \beta_{j\ell} B_{pe}[X_{e0\ell} - X_{ei\ell}(1 - \delta_{\ell j})] \\ + \sum_{k=0}^{i} \sum_{\ell=0}^{j} \beta_{ik}\beta_{j\ell} C_{pek\ell} X_{ek\ell}(1 - \delta_{ki}\delta_{\ell j}) \end{array} \right\} - \gamma_{p1} P u_{iq} u_{jr}, \tag{5}$$

where $p = 1, 2, \ldots, 8$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, A_{pe}, B_{pe}, $C_{pek\ell}$, γ_{p1} are given in Appendix B.

In equation (5), the quantity X_{pij} at the main point (i, j) of the area $[i, j]$ is related to the quantities X_{ek0} and $X_{e0\ell}$ at the boundary dependent paints of the area and the quantities X_{ekj}, $X_{ei\ell}$ and $X_{ek\ell}$ at the inner dependent points of the area. With the spreading of the area $[i, j]$ according to the regular order as [1, 1], [1, 2], . . . ,[1, n], [2, 1], [2, 2], . . . ,[2, n], . . . ,[m, 1], [m, 2], . . . ,[m, n], a main point of smaller area becomes one of the inner dependent points of the following larger areas. Whenever the quantity X_{pij} at the main point (i, j) is obtained by using equation (5) in the above mentioned order, the quantities X_{ekj}, $X_{ei\ell}$ and $X_{ek\ell}$ at the inner dependent points of the following larger areas can be eliminated by substituting the results obtained into the corresponding terms of the right side of equation (5). By repeating this process, the equation X_{pij} at the main point is related to only the quantities X_{rk0}, ($r = 1, 3, 4, 6, 7, 8$) and $X_{s0\ell}$, ($s = 2, 3, 5, 6, 7, 8$) which are six independent quantities at each boundary dependent point along the horizontal axis and the vertical axis in Figure 1 respectively. The result is as follows.

$$X_{pij} = \sum_{k=0}^{i} \left\{ \begin{array}{l} a_{1pijk1}(Q_y)_{k0} + a_{1pijk2}(M_{xy})_{k0} + a_{1pijk3}(M_y)_{k0} \\ + a_{1pijk4}(\theta_y)_{k0} + a_{1pijk5}(\theta_x)_{k0} + a_{1pijk6}(w)_{k0} \end{array} \right\} + \sum_{\ell=0}^{j} \left\{ \begin{array}{l} a_{2pij\ell 1}(Q_x)_{0\ell} + a_{2pij\ell 2}(M_{xy})_{0\ell} + a_{2pij\ell 3}(M_x)_{0\ell} \\ + a_{2pij\ell 4}(\theta_y)_{0\ell} + a_{2pij\ell 5}(\theta_x)_{0\ell} + a_{2pij\ell 6}(w)_{0\ell} \end{array} \right\} + \bar{q}_{pij} P \tag{6}$$

where $(Q_y) = X_1$, $(Q_x) = X_2$, $(M_{xy}) = X_3$, $(M_y) = X_4$, $(M_x) = X_5$, $(\theta_y) = X_6$, $(\theta_x) = X_7$, $(w) = X_8$, $\bar{q}_{pij}$ is given in Appendix C. Equation (6) gives the discrete solution [17] of the fundamental differential equation (2) of the plate bending problem, and the discrete Green function of the plate is obtained from $X_{8ij} = G(x_i, y_j, x_q, y_r) \cdot [\bar{P}a/D_0(1-\nu^2)]$ which is the displacement at a point (x_i, y_j) of a plate with a concentrated load $\bar{P}$ at a point (x_q, y_r).

4. INTEGRAL CONSTANT AND BOUNDARY CONDITION OF A RECTANGULAR PLATE

The integral constants $(Q_y)_{k0}$, $(M_{xy})_{k0}, \ldots, (w)_{k0}$, $(Q_x)_{0\ell}$, $(M_{xy})_{0\ell}, \ldots, (w)_{0\ell}$ being involved in the discrete solution (6) are to be evaluated by the boundary conditions of a rectangular plate. The combinations of the integral constants and the boundary conditions for some cases are shown in Figures 2–4, in which the integral constants and the boundary conditions at the four corners are shown in the boxes. The integral constants and the boundary conditions along the four edges are given at each equally-spaced discrete point. In this paper simply supported, fixed and free edges are denoted by a solid line, a thick solid line and a dotted line, respectively.

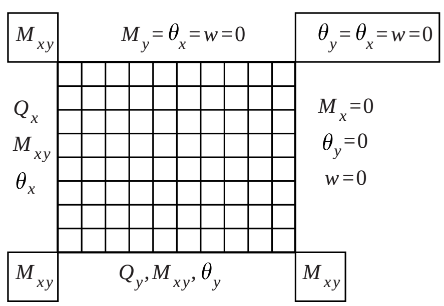

Figure 2. Simply supported plate.

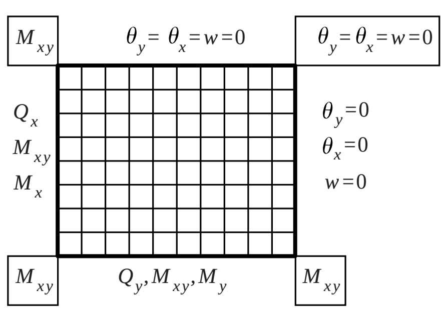

Figure 3. Fixed plate.

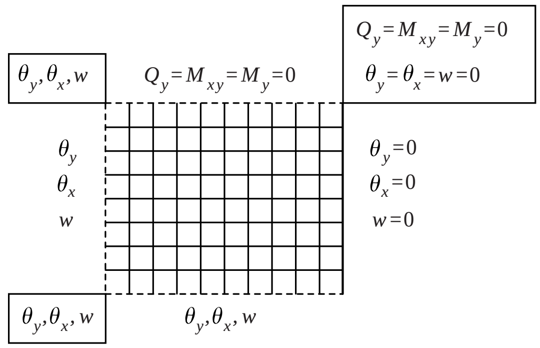

Figure 4. Cantilever plate.

5. CHARACTERISTIC EQUATION OF FREE VIBRATION OF A RECTANGULAR PLATE WITH VARIABLE THICKNESS

By applying the Green function $w(x_0, y_0, x, y)/\bar{P}$ which is the displacement at a point (x_0, y_0) of a plate with a concentrated load $\bar{P}$ at a point (x, y), the displacement amplitude $\hat{w}(x_0, y_0)$ at a point (x_0, y_0) of a rectangular plate during the free vibration is given as:

$$\hat{w}(x_0, y_0) = \int_0^b \int_0^a \rho h \omega^2 \hat{w}(x, y)[w(x_0, y_0, x, y)/\bar{P}]\, dx\, dy, \tag{7}$$

where ρ is the mass density of the plate material.

By using the non-dimensional expressions,

$$\lambda^4 = \frac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu^2)}, \qquad H(\eta, \zeta) = \frac{\rho(x, y)}{\rho_0}\frac{h(x, y)}{h_0}, \qquad W(\eta, \zeta) = \frac{\hat{w}(x, y)}{a},$$

$$G(\eta_0, \zeta_0, \eta, \zeta) = \frac{w(x_0, y_0, x, y)}{a}\frac{D_0(1-\nu^2)}{\bar{P}a},$$

ρ_0: standard mass density, the integral equation (7) can be rewritten as:

$$W(\eta_0, \zeta_0) = \int_0^1 \int_0^1 \mu \lambda^4 H(\eta, \zeta) G(\eta_0, \zeta_0, \eta, \zeta) W(\eta, \zeta)\, d\eta\, d\zeta. \tag{8}$$

By applying the numerical integration method mentioned in section 3, equation (8) is discretely expressed as

$$\kappa W_{k\ell} = \sum_{i=0}^{m} \sum_{j=0}^{n} \beta_{mi} \beta_{nj} H_{ij} G_{k\ell ij} W_{ij}, \qquad \kappa = 1/(\mu \lambda^4). \tag{9}$$

From equation (9), homogeneous linear equations in $(m+1)\times(n+1)$ unknowns $W_{00}, W_{01}, \ldots, W_{0n}, W_{10}, W_{11}, \ldots, W_{1n}, \sim, W_{m0}, W_{m1}, \ldots, W_{mn}$ are obtained as follows:

$$\sum_{i=0}^{m}\sum_{j=0}^{n}(\beta_{mi}\beta_{nj}H_{ij}G_{k\ell ij}-\kappa\delta_{ik}\delta_{j\ell})W_{ij}=0, \qquad (k=0,1,\ldots,m, \quad \ell=0,1,\ldots,n). \tag{10}$$

The characteristic equation of the free vibration of a rectangular plate with variable thickness is obtained from equation (10):

$$\begin{vmatrix} \mathbf{K}_{00} & \mathbf{K}_{01} & \mathbf{K}_{02} & \cdots & \mathbf{K}_{0m} \\ \mathbf{K}_{10} & \mathbf{K}_{11} & \mathbf{K}_{12} & \cdots & \mathbf{K}_{1m} \\ \mathbf{K}_{20} & \mathbf{K}_{21} & \mathbf{K}_{22} & \cdots & \mathbf{K}_{2m} \\ \cdots & & \cdots & & \\ \mathbf{K}_{m0} & \mathbf{K}_{m1} & \mathbf{K}_{m2} & \cdots & \mathbf{K}_{mm} \end{vmatrix} = 0, \tag{11}$$

where

$$\mathbf{K}_{ij}=\beta_{mj}\begin{bmatrix} \beta_{n0}H_{j0}G_{i0j0}-\kappa\delta_{ij} & \beta_{n1}H_{j1}G_{i0j1} & \beta_{n2}H_{j2}G_{i0j2} & \cdots & \beta_{nn}H_{jn}G_{i0jn} \\ \beta_{n0}H_{j0}G_{i1j0} & \beta_{n1}H_{j1}G_{i1j1}-\kappa\delta_{ij} & \beta_{n2}H_{j2}G_{i1j2} & \cdots & \beta_{nn}H_{jn}G_{i1jn} \\ \beta_{n0}H_{j0}G_{i2j0} & \beta_{n1}H_{j1}G_{i2j1} & \beta_{n2}H_{j2}G_{i2j2}-\kappa\delta_{ij} & \cdots & \beta_{nn}H_{jn}G_{i2jn} \\ & \cdots & & & \\ \beta_{n0}H_{j0}G_{inj0} & \beta_{n1}H_{j1}G_{inj1} & \beta_{n2}H_{j2}G_{inj2} & \cdots & \beta_{nn}H_{jn}G_{injn}-\kappa\delta_{ij} \end{bmatrix}.$$

6. NUMERICAL WORK

The convergency and accuracy of numerical solutions have been investigated for the free vibration problem of some rectangular plates with uniform thickness and a rectangular plate with variable thickness.

The convergent values of numerical solutions of frequency parameter for these plates have been obtained by using Richardson's extrapolation formula for two cases of combinations of divisional numbers m and n.

6.1. CONVERGENCY AND ACCURACY OF NUMERICAL RESULTS FOR PLATES WITH UNIFORM THICKNESS

6.1.1. *Simply supported square plate and rectangular plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a simply supported square plate and a rectangular plate of aspect $b/a=2$ are

TABLE 1

Natural frequency parameter λ for a simple rectangular plate

Mode	b/a = 1, m = 12	b/a = 1, m = 16	b/a = 1, Extrapolation	b/a = 1, Reference [18]	b/a = 2, m = 12	b/a = 2, m = 16	b/a = 2, Extrapolation	b/a = 2, Reference [18]
1	4·574	4·563	4·548	4·549	3·617	3·608	3·596	3·596
2	7·333	7·270	7·188	7·192	4·615	4·585	4·547	4·549
3	9·306	9·211	9·089	9·098	6·029	5·924	5·789	5·799
4	10·672	10·442	10·146	–	6·778	6·712	6·627	6·631
5	10·672	10·442	10·146	–	7·793	7·511	7·148	7·192
6	12·110	11·873	11·569	11·597	7·359	7·284	7·187	7·192
7	14·530	13·931	13·161	13·262	8·318	8·192	8·030	8·041
8	14·374	14·037	13·603	13·647	9·961	9·326	8·511	8·661
9	15·614	15·032	14·282	–	9·672	9·403	9·056	9·098
10	15·614	15·032	14·282	–	12·691	11·402	9·745	10·172
11	17·424	16·789	15·972	16·083	10·299	10·062	9·798	9·782
12	19·083	17·769	16·079	–	10·689	10·451	10·146	10·172
13	19·083	17·769	16·079	–	11·489	10·906	10·156	10·298
14	19·918	18·642	17·001	17·322	11·370	11·102	10·757	10·789
15	20·009	19·145	18·033	18·196	13·921	12·725	11·187	11·597
16	21·360	20·081	18·436	–	12·393	12·022	11·545	11·597

shown in Table 1. The thickness ratio h_0/a of these plates is 0·01. The convergent values of numerical solutions were obtained by using Richardson's extrapolation formula for the two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the rectangular plate. Table 1 includes the past theoretical values of Leissa [18], and it shows the good convergency and satisfactory accuracy of the numerical solutions using the present method. The nodal lines of 16 modes of free vibration of the two plates are shown in Figure 5.

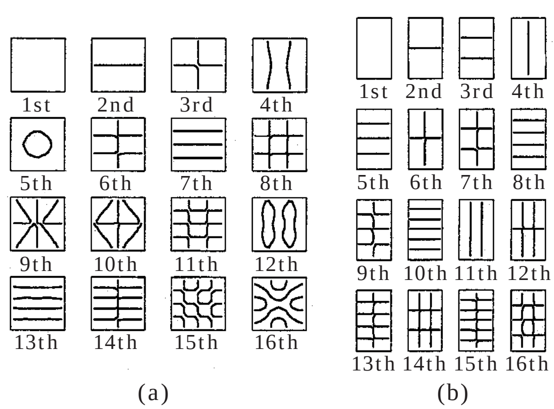

Figure 5. Nodal patterns for a simply supported plate. (a) Square plate; (b) rectangular plate ($b/a = 2$).

TABLE 2

Natural frequency parameter λ for a fixed rectangular plate

	$b/a = 1$					$b/a = 2$			
	m			Reference		m			
Mode	12	16	Extra-polation	[19]	[20]	12	16	Extra-polation	Reference [19]
1	6·205	6·175	6·138	6·142	6·066	5·133	5·107	5·073	5·076
2	9·030	8·911	8·756	8·771	8.742	5·883	5·834	5·771	5·776
3	10·985	10·829	10·629	10·651	10·614	7·175	7·024	6·829	6·851
4	12·533	12·162	11·686	11·745	11·747	8·970	8·573	8·064	8·148
5	12·563	12·191	11·714	11·772	–	8·475	8·344	8·176	8·190
6	13·910	13·552	13·091	13·152	13·127	8·923	8·789	8·617	8·632
7	16·690	15·803	14·663	14·856	14·849	11·305	10·430	9·305	9·564
8	16·194	15·717	15·105	–	15·163	9·743	9·558	9·320	9·343
9	17·655	16·817	15·741	15·933	15·934	11·023	10·667	10·209	10·279
10	17·697	16·856	15·774	–	–	14·372	12·611	10·347	–
11	19·421	18·551	17·432	–	17·606	18·600	15·193	10·813	11·044
12	21·676	19·825	17·446	–	–	12·892	12·134	11·160	–
13	21·684	19·833	17·454	–	–	12·196	11·808	11·309	–
14	22·409	20·633	18·351	–	–	12·505	12·123	11·632	–
15	22·063	20·929	19·471	–	19·712	15·565	13·992	11·970	–
16	23·707	21·982	19·765	–	–	13·074	12·672	12·155	–

6.1.2. *Fixed square plate and rectangular plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a fixed square plate and a rectangular plate of aspect ratio $b/a = 2$ are shown in Table 2. The thickness ratio h_0/a of these plates is 0·01. The convergent values of numerical solutions were obtained for the two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the plate. Table 2 includes the other theoretical values by Claassen and Thorne [19] and Bolotin [20]. The numerical solutions using the present method have good convergency and satisfactory

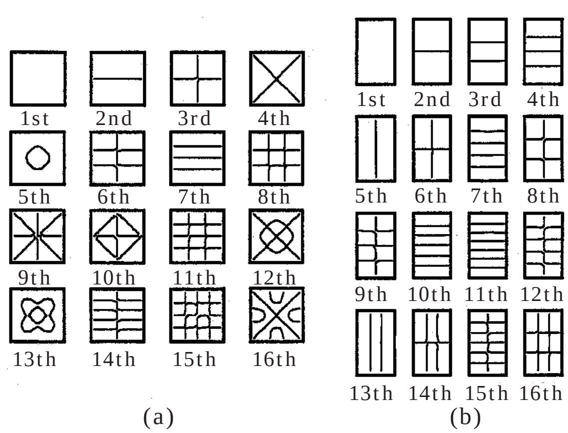

Figure 6. Nodal patterns for a fixed plate. (a) Square plate; (b) rectangular plate ($b/a = 2$).

TABLE 3

Natural frequency parameter λ for a cantilever rectangular plate

	b/a = 1					b/a = 2			
	m			Reference		m			
Mode	12	16	Extra-polation	[21]	[22]	12	16	Extra-polation	Reference [21]
1	1·909	1·908	1·906	1·908	1·914	1·915	1·914	1·913	1·914
2	2·990	2·987	2·982	2·994	2·993	2·372	2·370	2·366	2·375
3	4·783	4·756	4·721	4·724	4·741	3·294	3·281	3·264	3·273
4	5·399	5·372	5·337	5·340	5·365	4·605	4·543	4·465	4·463
5	5·756	5·724	5·684	5·710	5·716	4·849	4·820	4·783	4·791
6	7·640	7·588	7·521	7·545	–	5·155	5·122	5·080	5·099
7	8·305	8·171	7·999	8·016	–	5·846	5·800	5·741	5·746
8	8·478	8·354	8·195	8·204	–	6·354	6·173	5·939	5·979
9	8·913	8·776	8·601	8·633	–	6·907	6·818	6·705	6·729
10	10·179	10·027	9·830	–	–	8·310	7·912	7·400	–
11	10·389	10·244	10·059	–	–	8·192	8·011	7·779	–
12	11·999	11·614	11·119	–	–	8·333	8·190	8·007	–
13	12·259	11·863	11·355	–	–	8·531	8·345	8·107	–
14	12·404	12·043	11·578	–	–	8·943	8·803	8·623	–
15	12·606	12·348	12·016	–	–	10·764	9·938	8·876	–
16	13·338	12·952	12·457	–	–	9·704	9·445	9·111	–

accuracy. The nodal lines of 16 modes of free vibration of the two plates are shown in Figure 6.

6.1.3. *Cantilever square plate and rectangular plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a cantilever square plate and a rectangular plate of aspect ratio $b/a = 2$ with uniform thickness are shown in Table 3. The thickness ratio h_0/a of these plates is 0·01. The convergent values of the numerical solutions were obtained for the

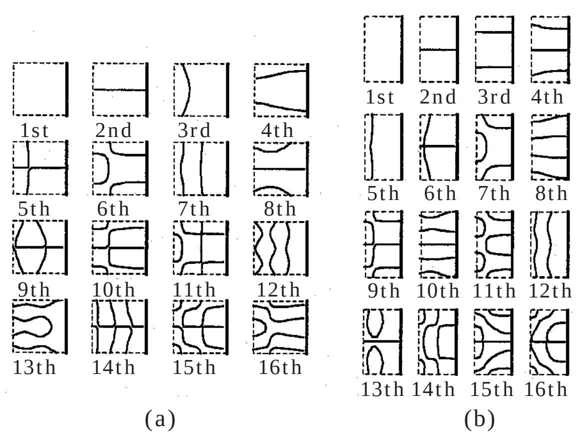

Figure 7. Nodal patterns for a cantilever plate. (a) Square plate; (b) rectangular plate ($b/a = 2$).

two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the plate. Table 3 involves the other theoretical values by Claassen and Thorne [21] and Young [22]. The numerical solutions by the present method have good convergency and satisfactory accuracy. The nodal lines of 16 modes of free vibration of the two plates are shown in Figure 7.

6.1.4. *Moderately thick square plate with uniform thickness*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a simply supported square plate and a fixed square plate of thickness ratio $h_0/a = 0{\cdot}2$ are shown in Table 4. The convergent values of numerical solutions were obtained by using Richardson's extrapolation formula for the two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the plate. Table 4 involves the other theoretical values by Liew *et al*. [23] and it shows good convergency and satisfactory accuracy of the numerical solutions by the present method. The nodal lines of free vibration of these two plates with moderate thickness are the same as those in Figure 5(a) or Figure 6(a).

6.2. NUMERICAL RESULTS FOR PLATES WITH VARIABLE THICKNESS

6.2.1. *Simply supported square plate and rectangular plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a simply supported square plate and a rectangular plate of aspect ratio $b/a = 2$ with a linear thickness variation in the η direction given by $h(\eta, \zeta) = h_0(1 + \alpha\eta)$,

TABLE 4

Natural frequency parameter λ for a moderately thick square plate

	Simple plate				Fixed plate			
	m				*m*			
Mode	12	16	Extra-polation	Reference [23]	12	16	Extra-polation	Reference [23]
1	4·346	4·336	4·323	4·274	5·349	5·332	5·310	5·266
2	6·536	6·490	6·431	6·318	7·177	7·121	7·048	6·954
3	7·885	7·832	7·744	7·593	8·341	8·273	8·185	8·057
4	8·727	8·589	8·413	8·251	9·077	8·932	8·747	8·600
5	8·727	8·589	8·413	8·251	9·119	8·975	8·789	8·659
6	9·545	9·415	9·247	9·068	9·836	9·700	9·526	9·367
7	10·789	10·495	10·117	–	10·995	10·698	10·316	–
8	10·713	10·547	10·334	–	10·921	10·751	10·532	–
9	11·302	11·030	10·679	–	11·463	11·188	10·835	–
10	11·302	11·030	10·679	–	11·494	11·218	10·863	–
11	12·109	11·832	11·477	–	12·248	11·969	11·610	–
12	12·800	12·256	11·556	–	12·910	12·366	11·667	–
13	12·800	12·256	11·556	–	12·918	12·375	11·676	–
14	13·133	12·620	11·960	–	13·237	12·724	12·064	–
15	13·169	12·824	12·382	–	13·273	12·927	12·481	–
16	13·688	13·197	12·565	–	13·769	13·278	12·647	–

TABLE 5

Natural frequency parameter λ for a simple square plate with variable thickness

	α = 0·1				α = 0·8			
	m				m			
Mode	12	16	Extra-polation	Reference [1]	12	16	Extra-polation	Reference [1]
1	4·687	4·675	4·660	4·661	5·386	5·372	5·354	5·355
2	7·512	7·446	7·363	–	8·576	8·501	8·404	–
3	7·513	7·447	7·363	–	8·612	8·535	8·437	–
4	9·534	9·436	9·311	–	10·944	10·829	10·680	–
5	10·927	10·692	10·389	–	13·342	12·080	11·742	–
6	10·932	10·696	10·393	–	12·512	12·238	11·886	–
7	12·405	12·162	11·851	–	14·210	13·926	13·560	–
8	12·407	12·164	11·852	–	14·265	13·977	13·607	–
9	14·870	14·269	13·496	–	16·581	15·918	15·066	–
10	14·724	14·258	13·659	–	16·889	16·308	15·562	–
11	14·883	14·378	13·729	–	17·018	16·481	15·791	–
12	15·993	15·396	14·628	–	18·294	17·602	16·712	–
13	15·997	15·400	14·631	–	18·402	17·701	16·799	–
14	17·846	17·195	16·358	–	20·428	19·670	18·695	–
15	17·848	17·197	16·360	–	20·513	19·742	18·752	–
16	19·514	18·174	16·451	–	21·467	20·047	18·222	–

TABLE 6

Natural frequency parameter λ for a simple rectangular plate with variable thickness ($b/a = 2$)

	α = 0·1				α = 0·8			
	m				m			
Mode	12	16	Extra-polation	Reference [1]	12	16	Extra-polation	Reference [1]
1	3·705	3·696	3·684	3·684	4·245	4·234	4·220	4·221
2	4·728	4·698	4·659	–	5·433	5·398	5·353	–
3	6·176	6·069	5·930	–	7·078	6·955	6·797	–
4	6·944	6·876	6·789	–	7·955	7·876	7·775	–
5	7·982	7·693	7·322	–	9·100	8·776	8·359	–
6	7·539	7·462	7·362	–	8·641	8·552	8·436	–
7	8·521	8·392	8·226	–	9·775	9·624	9·430	–
8	10·200	9·633	8·903	–	11·547	10·831	9·910	–
9	9·909	9·551	9·091	–	11·377	11·055	10·641	–
10	10·549	10·307	9·995	–	14·574	13·145	11·308	–
11	12·695	11·673	10·360	–	12·072	11·791	11·430	–
12	10·949	10·706	10·393	–	12·533	12·250	11·886	–
13	11·647	11·173	10·565	–	13·529	12·833	11·938	–
14	11·770	11·373	10·862	–	13·336	13·017	12·813	–
15	12·992	12·315	11·445	–	16·407	14·984	13·154	–
16	14·193	13·036	11·550	–	14·545	14·102	13·532	–

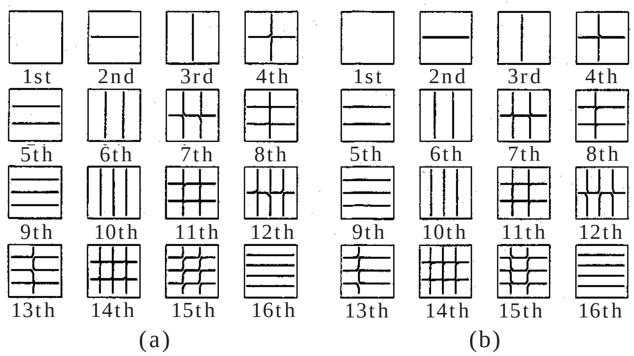

Figure 8. Nodal patterns for a simply supported square plate with variable thickness. (a) $\alpha = 0{\cdot}1$; (b) $\alpha = 0{\cdot}8$.

are shown in Tables 5 and 6 for two cases of $\alpha = 0{\cdot}1$ and $0{\cdot}8$. The thickness ratio h_0/a of these plates is 0·01. The convergent values of numerical solutions were obtained for the two cases of divisional numbers $m\ (=n)$ of 12 and 16 for the whole part of the plate. Tables 5 and 6 involve the other theoretical values of the fundamental frequency by Appl and Byers [1]. The numerical solutions by the present method have good convergency and satisfactory accuracy of fundamental frequency. The nodal lines of 16 modes of free vibration of the four plates of $b/a = 1, 2$ and $\alpha = 0{\cdot}1, 0{\cdot}8$ are shown in Figures 8 and 9.

6.2.2. *Fixed square plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a fixed square plate with a sinusoidal thickness variation in the η, ζ directions given by $h(\eta, \zeta) = h_0(1 - \alpha \sin \pi\eta)(1 - \alpha \sin \pi\zeta)$, are shown in Table 7 for two cases of $\alpha = 0{\cdot}3$ and $0{\cdot}5$. The thickness ratio h_0/a of these plates is 0·01. The convergent values of numerical solutions were obtained for the two cases of divisional numbers $m\ (=n)$ of 12 and 16 for the whole part of the plate. The nodal lines of 16 modes of free vibration of the two plates of $\alpha = 0{\cdot}3$, $0{\cdot}5$ are shown in

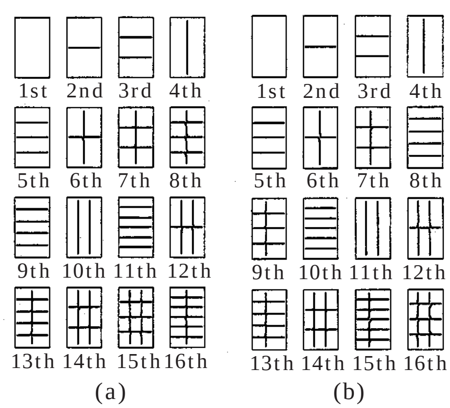

Figure 9. Nodal patterns for a simply supported rectangular plate with variable thickness ($b/a = 2$). (a) $\alpha = 0{\cdot}1$; (b) $\alpha = 0{\cdot}8$.

TABLE 7

Natural frequency parameter λ for a fixed square plate with variable thickness

	$\alpha = 0{\cdot}3$			$\alpha = 0{\cdot}5$		
	m			m		
Mode	12	16	Extra-polation	12	16	Extra-polation
1	1·990	1·989	1·988	2·166	2·164	2·162
2	2·799	2·794	2·789	2·675	2·671	2·665
3	4·324	4·300	4·270	3·745	3·720	3·687
4	4·603	4·576	4·540	4·033	4·006	3·971
5	5·014	4·985	4·948	4·337	4·307	4·269
6	6·523	6·476	6·415	5·148	5·046	4·914
7	6·860	6·747	6·601	5·378	5·323	5·254
8	7·274	7·153	6·997	6·178	6·062	5·914
9	7·694	7·575	7·421	6·671	6·422	6·103
10	8·675	8·538	8·361	6·446	6·331	6·183
11	8·868	8·734	8·562	6·992	6·842	6·650
12	9·591	9·281	8·884	7·328	7·188	7·009
13	10·464	10·115	9·665	8·433	7·904	7·224
14	10·662	10·378	10·013	8·701	8·378	7·963
15	10·790	10·520	10·172	8·941	8·596	8·154
16	11·373	11·032	10·593	8·765	8·521	8·207

TABLE 8

Natural frequency parameter λ for a cantilever square plate with variable thickness

	$\alpha = 1/2$			$\alpha = 1/8$		
	m			m		
Mode	12	16	Extra-polation	12	16	Extra-polation
1	1·743	1·742	1·741	1·541	1·540	1·540
2	2·459	2·456	2·451	1·886	1·883	1·880
3	3·866	3·846	3·819	2·579	2·561	2·537
4	4·096	4·070	4·037	3·057	3·035	3·006
5	4·532	4·506	4·472	3·266	3·229	3·182
6	5·990	5·891	5·764	3·472	3·411	3·333
7	5·918	5·873	5·816	4·079	4·005	3·909
8	6·655	6·542	6·396	4·391	4·243	4·052
9	7·032	6·919	6·775	4·867	4·755	4·610
10	7·897	7·768	7·602	5·230	5·003	5·712
11	8·064	7·902	7·695	5·067	4·944	4·784
12	8·352	8·121	7·825	5·544	5·271	4·921
13	9·615	9·286	8·862	6·563	6·038	5·363
14	9·782	9·525	9·195	5·801	5·655	5·467
15	9·889	9·617	9·266	6·850	6·401	5·825
16	10·339	10·022	9·614	7·018	6·680	6·245

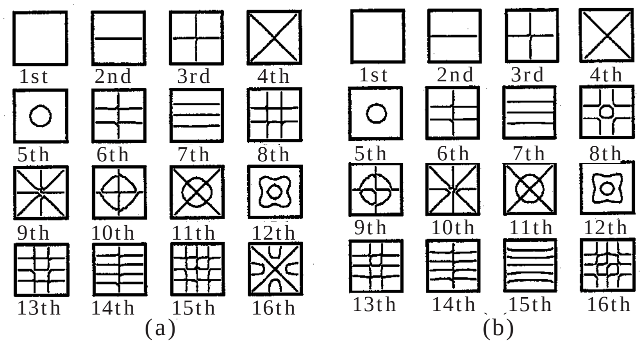

Figure 10. Nodal patterns for a fixed square plate with variable thickness. (a) $\alpha = 0{\cdot}3$; (b) $\alpha = 0{\cdot}5$.

Figure 10. There are some changes of mode order in the 9th, 10th, 15th and 16th modes.

6.2.3. *Cantilever square plate*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a cantilever square plate with a linear thickness variation in the η directions given by $h(\eta, \zeta) = h_0[\alpha + (1 - \alpha)\eta]$, are shown in Table 8 for two cases of $\alpha = 1/2$ and 1/8. The thickness ratio h_0/a of these plates is 0·01. The convergent values of numerical solutions were obtained for the two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the plate. The nodal lines of 16 modes of free vibration of the three plates of $\alpha = 1/2$ and 1/8 are shown in Figure 11. There are some differences of mode shape and mode order between these three cantilever plates.

6.2.4. *Moderately thick simple square plate with variable thickness*

Numerical solutions for the lowest 16 values of the natural frequency parameter λ of a moderately thick simply supported square plate with a linear thickness variation in the η direction given by $h(\eta, \zeta) = h_0(1 - \alpha\eta)$, $\alpha = 1$, are shown in

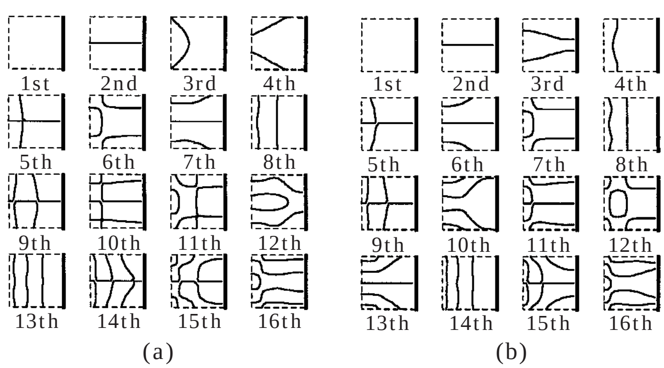

Figure 11. Nodal patterns for a cantilever square plate with variable thickness. (a) $\alpha = 1/2$; (b) $\alpha = 1/8$.

TABLE 9

Natural frequency parameter λ for a moderately thick simple square plate with variable thickness

	$h_0/a = 0{\cdot}1$				$h_0/a = 0{\cdot}2$			
	m				m			
Mode	12	16	Extra-polation	Reference [7]	12	16	Extra-polation	Reference [7]
1	5·327	5·314	5·298	5·238	4·828	4·818	4·806	4·758
2	8·054	7·995	7·920	7·975	6·791	6·751	6·700	6·873
3	8·056	7·998	7·922	7·994	6·802	6·762	6·711	6·882
4	9·761	9·683	9·582	9·649	7·916	7·866	7·801	8·048
5	10·835	10·658	10·431	10·654	8·615	8·501	8·355	8·671
6	10·852	10·673	10·444	–	8·618	8·507	8·364	–
7	11·879	11·712	11·497	–	9·285	9·177	9·039	–
8	11·882	11·715	11·502	–	9·292	9·183	9·044	–
9	13·487	13·105	12·244	–	10·247	10·016	9·719	–
10	13·540	13·156	12·291	–	10·315	10·069	9·753	–
11	13·381	13·166	12·511	–	10·254	10·115	9·936	–
12	14·148	13·793	13·337	–	10·742	10·512	10·217	–
13	14·150	13·796	13·340	–	10·766	10·534	10·236	–
14	15·191	14·832	14·371	–	11·806	11·381	10·834	–
15	15·191	14·832	14·371	–	11·420	11·185	10·882	–
16	16·097	15·386	14·472	–	11·435	11·197	10·891	–

Table 9 for two cases of thickness ratio $h_0/a = 0{\cdot}1$ and $0{\cdot}2$. The convergent values of numerical solutions were obtained by using Richardson's extrapolation formula for the two cases of divisional numbers m $(=n)$ of 12 and 16 for the whole part of the plate. Table 9 involves the other theoretical values by Aksu and Al-Kaabi [7] and it shows good convergency and satisfactory accuracy of the numerical solutions by the present method. The nodal lines of free vibration of these two moderately thick plates with variable thickness are shown in Figure 12.

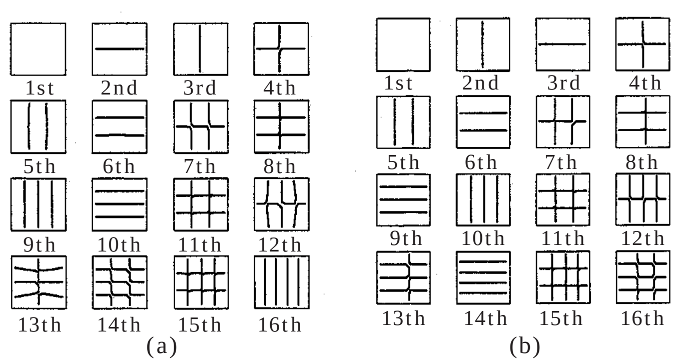

Figure 12. Nodal patterns for a moderately thick simple square plate with variable thickness. (a) $h_0/a = 0{\cdot}1$; (b) $h_0/a = 0{\cdot}2$.

7. CONCLUSIONS

By adopting the concept that the behaviour of a rectangular plate can be analyzed from the geometrical, material and mechanical properties at the discrete points uniformly distributed on the plate, an approximate method was proposed for analyzing the free vibration problem of various types of rectangular plates with uniform or non-uniform thickness. As a result of numerical work, it was shown that the numerical solutions rendered by the proposed method had good convergency and satisfactory accuracy for various types of rectangular plates with uniform or non-uniform thickness.

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APPENDIX A

$$F_{111} = F_{123} = F_{134} = F_{146} = F_{167} = F_{178} = F_{188} = 1,$$

$$F_{212} = F_{225} = F_{233} = F_{257} = F_{266} = \mu,$$

$$F_{156} = \nu, \qquad F_{247} = \nu\mu, \qquad F_{322} = F_{331} = -\mu,$$

$$F_{344} = F_{355} = -I, \qquad F_{363} = -J, \qquad F_{372} = -\kappa,$$

$$F_{377} = 1, \qquad F_{381} = -\mu\kappa, \qquad F_{386} = \mu, \qquad \text{other } F_{1te}, \qquad F_{2te}, \qquad F_{3te} = 0,$$

$$I = \mu(1-\nu^2)(h_0/h)^3, \qquad J = 2\mu(1+\nu)(h_0/h)^3, \qquad \kappa = (1/10)(E/G)(h_0/a)^2(h_0/h).$$

APPENDIX B

$$\mathrm{A}_{p1} = \gamma_{p1}, \qquad \mathrm{A}_{p2} = 0, \qquad \mathrm{A}_{p3} = \gamma_{p2}, \qquad \mathrm{A}_{p4} = \gamma_{p3},$$

$$\mathrm{A}_{p5} = 0, \qquad \mathrm{A}_{p6} = \gamma_{p4} + \nu\gamma_{p5}, \qquad \mathrm{A}_{p7} = \gamma_{p6},$$

$$A_{p8} = \gamma_{p7}, \qquad B_{p1} = 0, \qquad B_{p2} = \mu\gamma_{p1}, \qquad B_{p3} = \mu\gamma_{p3},$$

$$B_{p4} = 0, \qquad B_{p5} = \mu\gamma_{p2}, \qquad B_{p6} = \mu\gamma_{p6},$$

$$B_{p7} = \mu(\nu\gamma_{p4} + \gamma_{p5}), \qquad B_{p8} = \gamma_{p8},$$

$$C_{p1k\ell} = \mu(\gamma_{p3} + \kappa_{k\ell}\gamma_{p7}), \qquad C_{p2k\ell} = \mu\gamma_{p2} + \kappa_{k\ell}\gamma_{p8},$$

$$C_{p3k\ell} = J_{k\ell}\gamma_{p6}, \quad C_{p4k\ell} = I_{k\ell}\gamma_{p4}, \quad C_{p5k\ell} = I_{k\ell}\gamma_{p5}, \quad C_{p6k\ell} = -\mu\gamma_{p7}, \quad C_{p7k\ell} = -\gamma_{p8},$$

$$C_{p8k\ell} = 0, \qquad [\gamma_{pk}] = [\bar{\gamma}_{pk}]^{-1}, \qquad \bar{\gamma}_{11} = \beta_{ii}, \qquad \bar{\gamma}_{12} = \mu\beta_{jj},$$

$$\bar{\gamma}_{22} = -\mu\beta_{ij}, \qquad \bar{\gamma}_{23} = \beta_{ii}, \qquad \bar{\gamma}_{25} = \mu\beta_{jj},$$

$$\bar{\gamma}_{31} = -\mu\beta_{ij}, \qquad \bar{\gamma}_{33} = \mu\beta_{jj}, \qquad \bar{\gamma}_{34} = \beta_{ii}, \qquad \bar{\gamma}_{44} = -I_{ij}\beta_{ij},$$

$$\bar{\gamma}_{46} = \beta_{ii}, \qquad \bar{\gamma}_{47} = \mu\nu\beta_{jj}, \qquad \bar{\gamma}_{55} = -I_{ij}\beta_{ij},$$

$$\bar{\gamma}_{56} = \nu\beta_{ii}, \qquad \bar{\gamma}_{57} = \mu\beta_{jj}, \qquad \bar{\gamma}_{63} = -J_{ij}\beta_{ij}, \qquad \bar{\gamma}_{66} = \mu\beta_{jj},$$

$$\bar{\gamma}_{67} = \beta_{ii}, \qquad \bar{\gamma}_{71} = -\mu\kappa_{ij}\beta_{ij}, \qquad \bar{\gamma}_{76} = \mu\beta_{ij},$$

$$\bar{\gamma}_{78} = \beta_{ii}, \quad \bar{\gamma}_{82} = -\kappa_{ij}\beta_{ij}, \quad \bar{\gamma}_{87} = \beta_{ij}, \quad \bar{\gamma}_{88} = \beta_{jj}, \qquad \text{other } \bar{\gamma}_{pk} = 0, \quad \beta_{ij} = \beta_{ii}\beta_{jj}.$$

APPENDIX C

$$a_{11i0i1} = a_{13i0i2} = a_{14i0i3} = a_{16i0i4} = a_{17i0i5} = a_{i18i0i6} = 1, \qquad a_{15i0i3} = \nu,$$

$$a_{220jj1} = a_{230jj2} = a_{250jj3} = a_{260jj4} = a_{270jj5} = a_{280jj6} = 1, \qquad a_{240jj3} = \nu, \qquad a_{230002} = 0,$$

$$a_{hpijuv} = \sum_{e=1}^{8} \left\{ \begin{array}{l} \sum_{k=0}^{i} \beta_{ik} A_{pe}[a_{hek0uv} - a_{hekjuv}(1-\delta_{ki})] + \sum_{\ell=0}^{j} \beta_{j\ell} B_{pe}[a_{he0\ell uv} - a_{hei\ell uv}(1-\delta_{\ell j})] \\ + \sum_{k=0}^{i}\sum_{\ell=0}^{j} \beta_{ik}\beta_{j\ell} C_{pek\ell} a_{hek\ell uv}(1-\delta_{ki}\delta_{\ell j}), \end{array} \right\}$$

where $h = 1, 2,\ p = 1, 2, \ldots, 8,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ v = 1, 2, \ldots, 6$ $u = 0, 1, \ldots, i\ (h = 1),\ 0, 1, \ldots, j\ (h = 2)$.

$$\bar{q}_{pij} = \sum_{e=1}^{8} \left\{ \begin{array}{l} \sum_{k=0}^{i} \beta_{ik} A_{pe}[\bar{q}_{ek0} - \bar{q}_{ekj}(1-\delta_{ki})] + \sum_{\ell=0}^{j} \beta_{j\ell} B_{pe}[\bar{q}_{e0\ell} - \bar{q}_{ei\ell}(1-\delta_{\ell j})] \\ + \sum_{k=0}^{i}\sum_{\ell=0}^{j} \beta_{ik}\beta_{j\ell} C_{pek\ell} \bar{q}_{ek\ell}(1-\delta_{ki}\delta_{\ell j}) \end{array} \right\}$$

$$-\gamma_{p1} u_{iq} u_{jr}.$$