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ANALYSIS OF VIBRATION ISOLATION SYSTEMS USING A GRAPH MODEL

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A power-flow graph model is introduced for the analysis of the mechanical isolation system. Based on this method, the relationships of the velocity and force at both ends of each component of the isolation system, including machine, isolator and foundation, are expressed by the two-way power-flow model. The transmissions of both velocity and force between each component are displayed in the graph model of the isolation system, which can be generated by directly assembling each of the components. In addition, the transmissibility of force and velocity can be calculated according to the assembled graph model. In this investigation, one velocity isolation system and two force isolation systems are studied. Finally, an isolator modelled on a series connection of a MDOF mechanical system and an SDOF model of a flexible foundation are examined to illustrate the advantage of this method in practical use.

1. INTRODUCTION

A vibration isolator is usually installed between a dynamic system and an excitation source to protect the system from undesirable vibration. The two primary purposes in using the isolators are: (1) to reduce the acting force transmitted from the excitation force generated by an unbalanced revolution of the machine, called a force isolation system, and (2) to avoid the damage of a sensitive instrument on a vibrating foundation, called a velocity isolation system. The design objective is to obtain a lower force transmissibility for the former system and a velocity (or displacement acceleration) transmissibility for the latter system. Since a dynamic coupling between the machine, isolator and foundation exists, all of the components in the total system are considered simultaneously in the analysis, which may work well when the configuration of the isolator is very simple [1]. However, certain commonly used isolators, such as natural rubber or rubberlike material, are rationally modelled in practice as a series of connected multi-degree-of-freedom (MDOF) dynamic systems [2]. For these isolators, the computation is quite complex. In the classical method, the total system was modelled as a set of transfer matrices. The transmissibility can be computed by successive multiplication [3]. Munjal *et al*. developed a standard sub-program and

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an approximation method based on a circuit analogy to allow the computation work to be more suitable for computer calculation but the processes are still complicated [4, 5]. Paipetis and Vakakis derived the analytic solution of a uniform isolator. However, it is difficult to apply that scheme to the analysis of the total isolation system [6]. In fact, the isolator generally designed under the given condition of the machine and the foundation is fixed. If the interaction between the machine and isolator as well as the isolator and foundation can be known, the isolation system would be analyzed and designed more efficiently. Based on this approach, Busch-Vishniac proposed that at least one of the system components must be represented by its impedance. However, her method of analysis lacks an algorithm calculation [7].

In this article, a two-way power-flow model was introduced to deal with this problem. The relationship between the force and velocity at each component is established by a power-flow graph. From the graph model, we can see the interaction between each connected component. The total system can be configured by directly assembling each of the components. In addition, a closed form solution of the force and velocity transmissibility of the isolation systems can be calculated according to the graph model of the total isolation system. Finally, an isolator with a uniform multiple layered structure and a single-degree-of-freedom (SDOF) flexible foundation are examined to realize its feasibility.

2. POWER-FLOW MODEL

When an isolator is used to support a rotary machine with a periodical unbalanced force on the foundation, as shown in Figure 1, this system can be

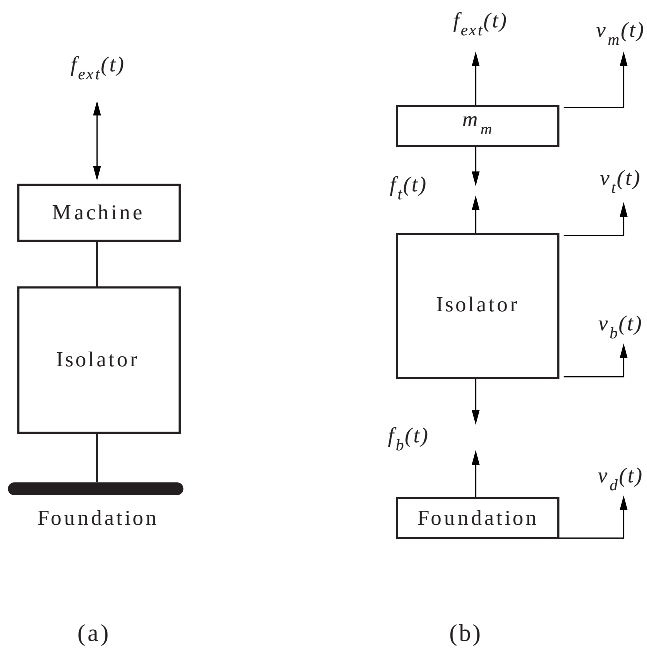
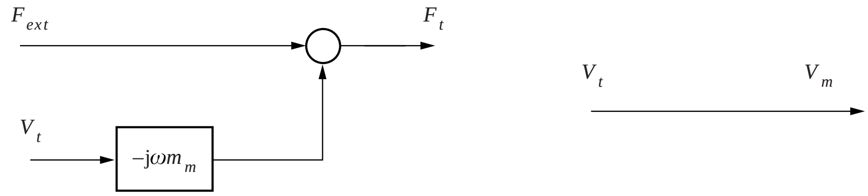


Figure 1. (a) Physical model and (b) variables definition of an isolation system.



(a) (b)

Figure 2. Power-flow graphical representation for (a) equation (5), and (b) equation (6).

modelled as a connection of three parts—a mass for the machine, a dynamic system for the isolator and a foundation. For the first case, Case I, a system with rigid foundation is considered. Based on Newton's law, the dynamic equation of the machine is given by

$$f_{ext}(t) - f_t(t) = m_m \frac{\mathrm{d}v_m(t)}{\mathrm{d}t}, \tag{1}$$

where $f_{ext}(t)$ and $f_t(t)$ are the excitation force and the acting force on the machine from the isolator, m_m is the mass of the machine, and $v_m(t)$ is the velocity of the machine. Assume that the machine is fixed on the top of the isolator. Then, the velocity of the machine is equal to that of the top of the isolator, denoted by $v_t(t)$. If $f_{ext}(t)$ and $v_t(t)$ are given and assigned to the input variables, the variables $f_m(t)$ and $v_m(t)$ should be a function of both input variables expressed as

$$f_t(t) = f_{ext}(t) - m_m \frac{\mathrm{d}v_t(t)}{\mathrm{d}t}, \qquad v_m(t) = v_t(t). \tag{2, 3}$$

Assuming a periodical excitation force with frequency ω, the steady state response of the force $f_t(t)$ and velocities $v_m(t)$ and $v_t(t)$ should also be periodical and have the same frequency, but different phase angles. Let the excitation be given as the real part of

$$f_{ext}(t) = F_{ext}\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{4}$$

where F_{ext} is the complex amplitude with units of force containing information about the phase angle, j is defined by $\mathrm{j} = (-1)^{1/2}$, $f_t(t)$, $v_m(t)$ and $v_t(t)$ may also be expressed in the same form as Equation (4). Equations (2) and (3) can be rewritten as

$$F_t = F_{ext} - \mathrm{j}\omega m_m V_t, \qquad V_m = V_t, \tag{5, 6}$$

where F_m, V_m and V_t are the complex amplitude of $f_t(t)$, $v_m(t)$ and $v_t(t)$. These F_{ext}, V_t, F_t and V_m are defined as the power-flow variables with respect to the dynamic system of the machine model. If F_{ext} and V_t, are selected as the input variables, F_t and V_m should be a function of the input variables defined as the output variables. Equations (5) and (6) can be represented as a power-flow graph, as in Figures 2(a) and (b). Then, a two-way power-flow graph model of the mass system can be formed by the combination of (a) and (b) in Figure 2 as shown in Figure 3.

Assume the isolator is modelled as a linear dynamic system. The steady state response of the velocity and force at the top and bottom of the system are also periodical with the same frequency as an excitation. If the complex amplitude of the acting force on the top F_t and the velocity of the bottom of the isolator V_b are selected as the input power-flow variables, the complex amplitude of the acting force on the bottom F_b and the velocity of the top of the isolator V_t can be expressed as

$$F_b = TR_{f,t,b}F_t + Z_b V_b, \qquad V_t = Y_t F_t + TR_{v,b,t} V_b, \tag{7, 8}$$

where $TR_{f,t,b}$ is the complex force gain of the isolator from the top to the bottom and $TR_{v,b,t}$ is the complex velocity gain of the isolator from the bottom to the top of the isolator defined as

$$TR_{f,t,b} = \left.\frac{F_b}{F_t}\right|_{V_b=0}, \qquad TR_{v,b,t} = \left.\frac{V_t}{V_b}\right|_{F_t=0}. \tag{9, 10}$$

Z_b and Y_t are the impedance and the mobility of the isolator at the bottom and the top defined as

$$Z_b = \left.\frac{F_b}{V_b}\right|_{F_t=0}, \qquad Y_t = \left.\frac{V_t}{F_t}\right|_{V_b=0}. \tag{11, 12}$$

The two-way dynamic flow model of the isolator can be configured by equations (7) and (8) illustrated in Figure 4. From Figures (3) and (4), one can see that the variables and directions of the power-flow model of the machine at the lower end are the same as that of the isolator at the top end. Thus, both power-flow models

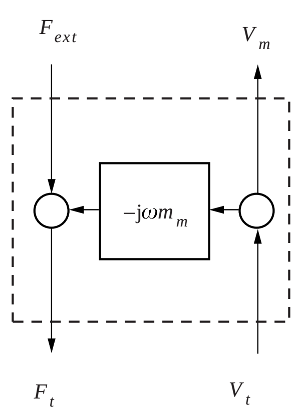


Figure 3. Two-way power-flow model of the machine component.

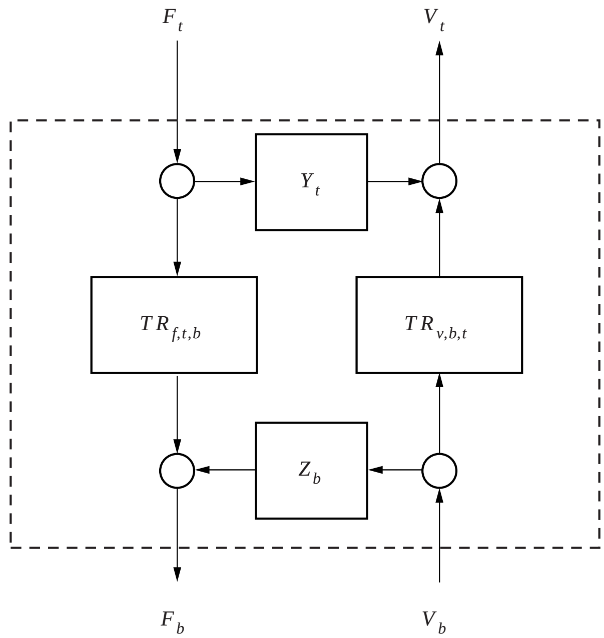


Figure 4. Power-flow model of the isolator component.

can be connected directly to form the combined model. In the same way, the two-way power-flow models of the foundation can be constructed according to their dynamic characteristics. Since the foundation is rigid, zero displacement and velocity response should be maintained. The boundary conditions are given by

$$F_d = F_b, \qquad V_b = V_d = 0, \tag{13, 14}$$

where F_d and V_d are the complex amplitude of the reaction force and velocity of the foundation. If the foundation is flexible with mobility Y_d, called Case II, the

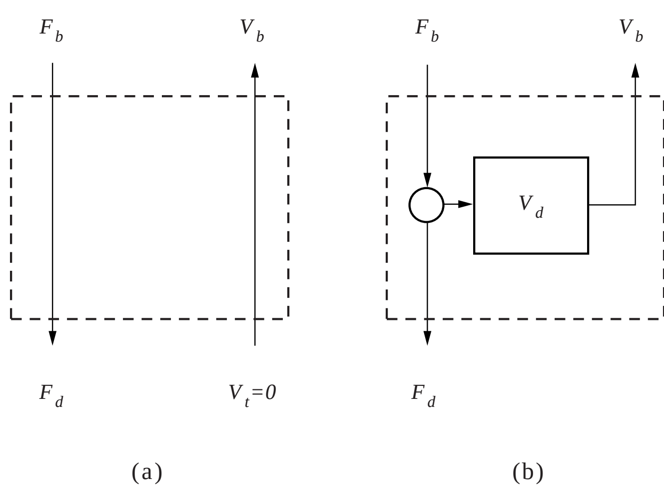


Figure 5. Power-flow model of the foundation: (a) rigid condition, and (b) flexible condition.

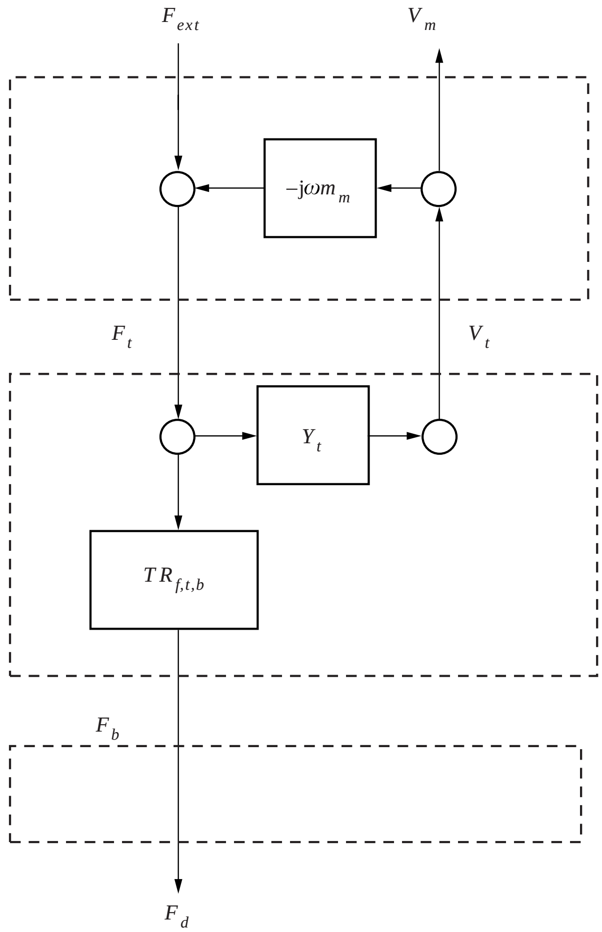


Figure 6. Power-flow model of the Case I system.

boundary condition in equation (14) is changed to

$$V_b = Y_d F_b. \tag{15}$$

According to equations (13)–(15), the two-way power-flow models of the foundation with respect to the flexible and rigid condition can be configured as Figures 5(a) and (b). The power-flow model of the total system including machine, isolator and foundation can be constructed by the series connection of the power-flow model of these three subsystems. For the Case I system, the power-flow model can be reduced since the power-flow is induced by a zero velocity input from the foundation, as shown in Figure 6. If the flexible foundation is considered, the power-flow model can be configured by the same method. Figure 7 illustrates the total power-flow model of the Case II system.

In addition to isolating an excitation force from the foundation, the isolator may also be typically used to isolate an instrument from a vibration foundation, such as Case III shown in Figure 8. In the same way, the two-way power-flow model may be configured as Figure 9.

3. TRANSMISSIBILITY FORMULATION

After the power-flow model is configured, the force and velocity transmissibility of these systems can be calculated by the model reduction method of gain formula [8] expressed as

$$G = \frac{\sum_i P_i D_i}{D}, \tag{16}$$

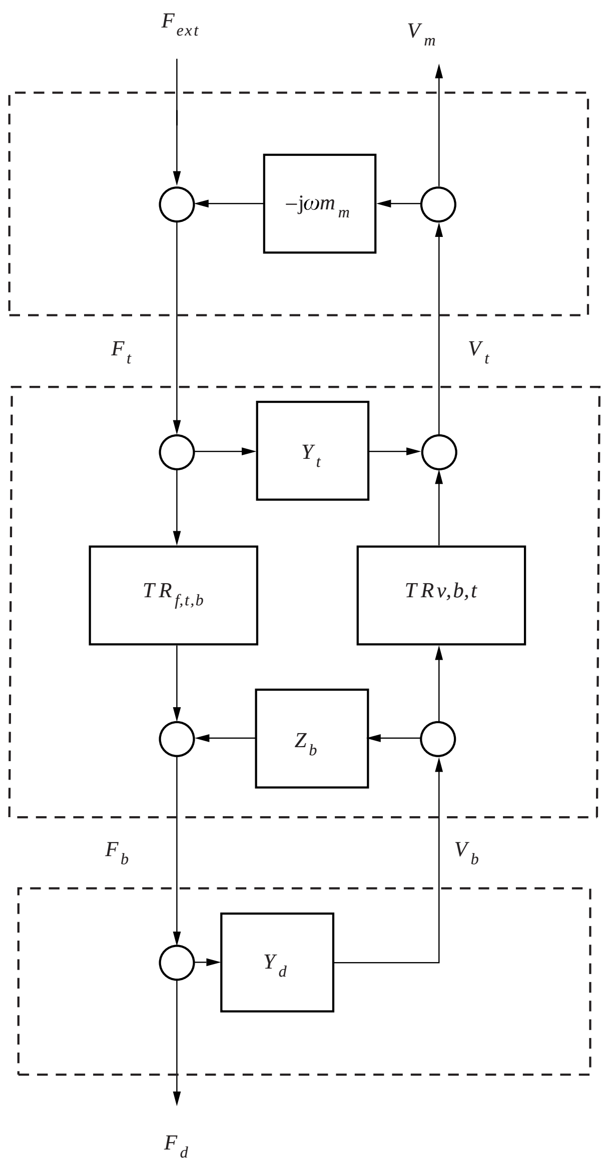


Figure 7. Power-flow model of the Case II system.

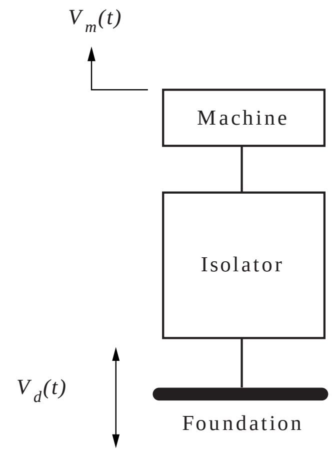


Figure 8. Mechanical model of the Case III system.

where G is the transfer function, P_i is the path gain of the ith forward path, D is the determinant of the graph, and D_i is the cofactor of the ith forward path determinant of the graph with the loops touching the ith forward path removed. The determinant of a graph is defined by

$$D = 1 - \text{(sum of all individual loop gains)}$$
$$+\text{(sum of gain products of all possible combinations of two non-touching loops)}$$
$$-\text{(sum of gain products of all possible combinations of three non-touching loops)}$$
$$+\ldots \tag{17}$$

When a rigid foundation is considered, there is only one closed loop in the graph model of the total system. The force transmissibility from F_{ext} to F_b can be calculated by the model reduction method or the gain formula. The result is

$$TR_{1,m,b} = \left|\frac{TR_{f,t,b}}{1 + \mathrm{j}\omega m_m Y_t}\right|. \tag{18}$$

For the Case II system, the force transmissibility $TR_{1,m,b,f}$ is defined by the ratio of the amplitude of F_{ext} to F_b. From Figure 7, one can see that there are three counter-clockwise loops in the power-flow model. The number of the forward path from F_{ext} to F_b is only one. The forward path gain is $TR_{f,t,b}$. Based on the gain formula, the transmissibility leads to

$$TR_{2,m,b} = \left|\frac{TR_{f,t,b}}{1 + \mathrm{j}\omega m_m (Y_t + Y_d TR_{f,t,b} TR_{v,b,t} - Y_t Z_b Y_d) - Z_b Y_d}\right|. \tag{19}$$

From Figure 7, it is known that the force transmissibility from F_{ext} to F_d should be equal to equation (18). The transmissibility from F_{ext} to V_b is $|Y_d| TR_{2,m,b}$.

The velocity transmissibility of the Case III system, $TR_{3,d,m}$, is defined by the amplitude ratio of V_m by V_d. There is only one closed loop and one forward path

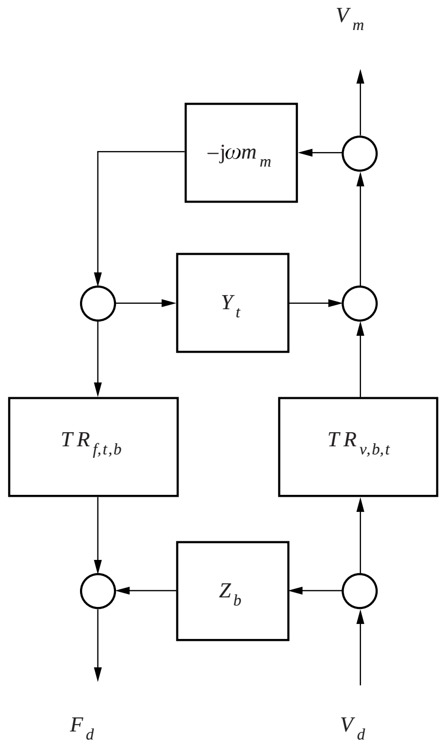


Figure 9. Power-flow model of the Case III system.

in the model shown in Figure 9. Based on the same process, the transmissibility can be calculated as

$$TR_{3,d,m} = \left| \frac{TR_{v,b,t}}{1 + j\omega m_m Y_t} \right|. \tag{20}$$

4. ANALYSIS OF THE ISOLATOR AND FOUNDATION

Assume that the isolator is modelled as an n series connection of identical mechanical subsystems, as illustrated in Figure 10. For each layer, the subsystem includes one mass series to a parallel connection of a spring and a damper. If the force and velocity of each component in the ith subsystem from the base are defined as Figure 11(a), the dynamic equations expressed by the complex amplitude of these force and velocity variables are given by

$$F_{i-1} = F_i - j\omega m V_i, \qquad V_i = V_{i-1} + \frac{F_{i-1}}{\left(c - \dfrac{k}{\omega} j\right)}. \tag{21, 22}$$

According to equations (20) and (21), the two-way power-flow model of the ith subsystem can be formed, as shown in Figure 11(b). Thus, the power-flow model

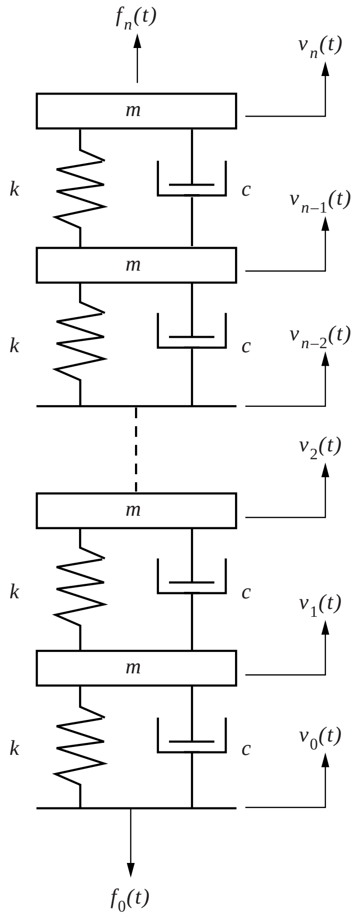


Figure 10. Mechanical model of the MDOF isolator.

of the isolator can be formed by a series connecting the power-flow model of each subsystem from the bottom to the top. The constructed power-flow model of the isolator should form a ladder shape having $2n$ transverse flows with alternating direction. By the gain formula, only one forward path follows the direction of the power-flow from F_t to F_b. The path gain of this forward path is 1. Moreover, there are $C(n, 2)$ closed loops in the model, where C is the selection function. All loops are touching the forward path. As shown in Appendix A, the force complex gain of the isolator leads to

$$TR_{f,t,b} = \left(\sum_{i=0}^{n} C(n+i, 2i) L^i \right)^{-1}, \qquad L = -\frac{\omega^2 m}{j\omega c + k}. \tag{23, 24}$$

The complex transmissibility from V_b to V_t can be computed by the same formula. The number of the forward path is one and the gain of the forward path is also 1. So the velocity complex gain $TR_{v,b,t}$ is equal to the force complex gain $TR_{f,t,b}$ expressed as

$$TR_{v,b,t} = TR_{f,t,b}. \tag{25}$$

The driving mobility at the top of the isolator, defined in equation (12), can also be calculated by the gain formula. However, there are n forward paths from the force F_t to the velocity response V_t. Each gain of the forward paths is identical. For the forward path passing through the ith left to right transverse path counted from bottom, the $C(i-1, 2)$ closed loops are not touching the forward path. According to Appendix B, the mobility at the top is given by

$$Y_t = \frac{\sum_{i=0}^{n-1} C(n+1, 2i+1)L^i}{\left(c - \dfrac{k}{\omega}\mathrm{j}\right)\left(\sum_{i=0}^{n} C(n+i, 2i)L^i\right)}. \tag{26}$$

By a similar process, the driving impedance at the bottom of the isolator leads to

$$Z_b = \frac{-\mathrm{j}\omega m \sum_{i=0}^{n-1} C(n+1, 2i+1)L^i}{\left(\sum_{i=0}^{n} C(n+i, 2i)L^i\right)}. \tag{27}$$

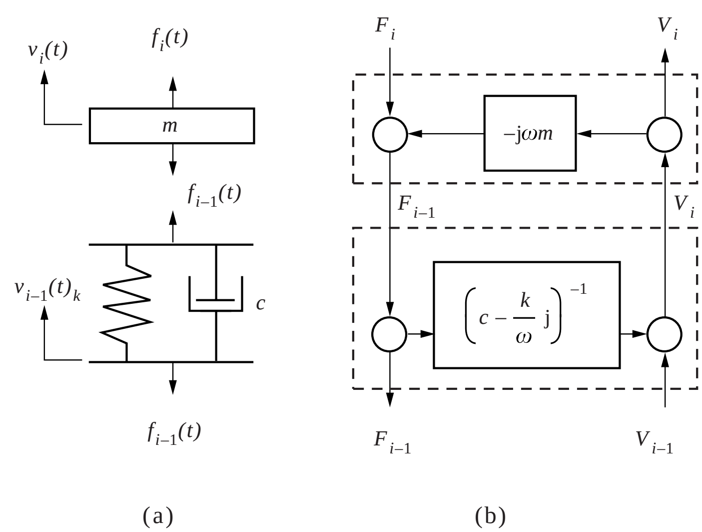


Figure 11. (a) Variables definition, and (b) power-flow model of the ith subsystem of the MDOF isolator.

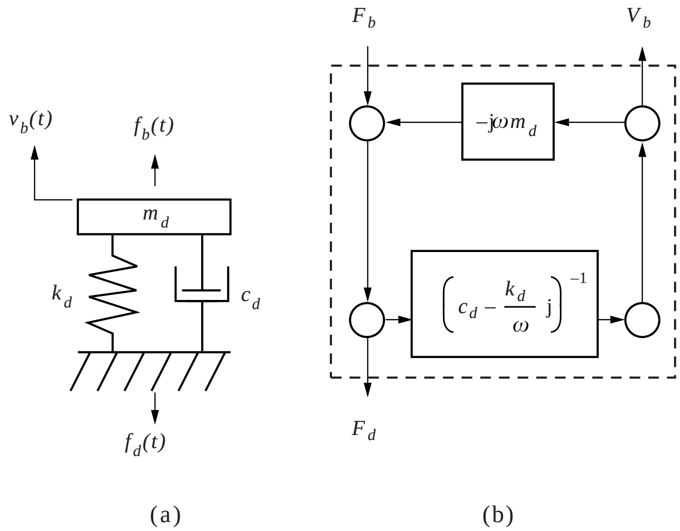


Figure 12. (a) Mechanical model, and (b) power-flow model of the SDOF foundation.

If the flexible foundation of Case II is modelled as an SDOF mass–spring–dashpot component, as illustrated in Figure 12(a), the two-way power-flow of the foundation can be configured by the same scheme shown in Figure 12(b). The mobility of the foundation leads to

$$Y_d = \frac{\omega}{c_d\omega + \mathrm{j}(\omega^2 m_d - k_d)}. \tag{28}$$

5. CONCLUSIONS

In this study, the two-way power-flow method has been proposed to express the transmission of the force and velocity of each component in the isolator systems. The interaction between the machine and isolator, as well as the isolator and foundation, was also illustrated in the graph model. Moreover, the analytical solutions of the force and velocity transmissibility were calculated based on the gain formula. The solution of the three cases of the isolation systems, including the force isolation systems with rigid and flexible foundations subject to a periodical excitation from the machine and the velocity isolation system subject to foundation vibrations, have been investigated. Finally, the closed form solutions of these isolation systems with a MDOF series mechanical model of the isolator and an SDOF mechanical model of the foundation were derived. The results show that this method has been successfully provided for the analysis of an isolator in unidirectional motion. Based on this concept, advance research for more complex systems will be extended in the near future.

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APPENDIX A

Based on the gain formula shown in equation (16), the force complex gain can be expressed as

$$TR_{f,t,b} = \frac{1}{D_0}, \tag{A1}$$

where

$$D_0 = 1 + \sum_{i_2=1}^{n} \sum_{i_1=1}^{i_2} L + \sum_{i_4=2}^{n} \sum_{i_3=2}^{i_4} \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} L^2 + \sum_{i_6=3}^{n} \sum_{i_5=3}^{i_6} \sum_{i_4=2}^{i_5-1} \sum_{i_3=2}^{i_4} \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} L^3$$
$$+ \sum_{i_{2n-2}=n-1}^{n} \sum_{i_{2n-3}=n-1}^{i_{2n-2}} \sum_{i_{2n-4}=n-2}^{i_{2n-3}-1} \sum_{i_{2n-5}=n-2}^{i_{2n-4}} \cdots \sum_{i_2=1}^{i_3-1} \sum_{i_1=1}^{i_2} L^{n-1} + L^n, \tag{A2}$$

where

$$L = -\frac{\omega^2 m}{\mathrm{j}\omega c + k}. \tag{A3}$$

Each complex summing term in equation (A2) can be calculated layer by layer. The results become

$$D_0 = 1 + \frac{(n+1)n}{2} L + \frac{(n+2)(n+1)n(n-1)}{4} L^2$$
$$+ \frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{6} L^3$$
$$+ \cdots + (2n-1)L^{n-1} + L^n. \tag{A4}$$

Equation (A4) can be rewritten as

$$D_0 = \sum_{i=0}^{n} C(n+i, 2i)L^i. \tag{A5}$$

APPENDIX B

The mobility at the top is given by

$$Y_t = \frac{\sum_{i=1}^{n} \left(c - \frac{k}{\omega}\mathrm{j}\right)^{-1} N_{ni}}{D_0}, \tag{B1}$$

where

$$N_{ni} = 1 + \sum_{i_2=1}^{i-1}\sum_{i_1=1}^{i_2} L + \sum_{i_4=2}^{i-1}\sum_{i_3=2}^{i_4}\sum_{i_2=1}^{i_3-1}\sum_{i_1=1}^{i_2} L^2 - \sum_{i_6=3}^{i-1}\sum_{i_5=3}^{i_6}\sum_{i_4=2}^{i_5-1}\sum_{i_3=2}^{i_4}\sum_{i_2=1}^{i_3-1}\sum_{i_1=1}^{i_2} L^3$$

$$+ \sum_{i_{2i-4}=i-2}^{i-1}\sum_{i_{2i-5}=i-2}^{i_{2n-4}}\sum_{i_{2i-6}=i-3}^{i_{2n-5}-1}\sum_{i_{2i-7}=i-3}^{i_{2n-6}} \cdots \sum_{i_2=1}^{i_3-1}\sum_{i_1=1}^{i_2} L^{i-2} + L^{i-1},$$

$$= \sum_{r=0}^{i-1} C(i+r-1, 2r)L^r. \tag{B2}$$

Substituting equation (B2) into equation (B1) gives

$$Y_t = \frac{\sum_{i=0}^{n-1} C(n+i, 2i+1)L^i}{D_0\left(c - \frac{k}{\omega}\mathrm{j}\right)}. \tag{B3}$$