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FLUCTUATIONS OF VORTICITY AND ENTROPY AS SOURCES OF ACOUSTICAL EXERGY

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In the case of a perfect fluid in an adiabatic process with irrotational mean flow, a second order wave equation of convected type is given for the acoustical exergy. The source terms are linked to fluctuating entropy and to fluctuating Coriolis acceleration. Propagation equations for these fluctuations are given and solved in natural co-ordinates associated with the mean flow.

1. INTRODUCTION

Recently, Doak [1, 2] suggested the idea that the fluctuating total enthalpy is the unique generalized acoustical field. Previously, Howe [3] also used this variable in the context of vortex sound. In this paper the use of the acoustical exergy as a natural acoustical variable is suggested, in the restricted case where the fluid is ideal, the main flow is irrotational and homentropic.

The availability of a fluid particle to produce work depends not only on its intrinsic state but also on the surrounding temperature and pressure. That is why the use of an extrinsic state variable, like exergy, is proposed. The exergy (also called, in thermodynamical literature, available enthalpy or external mechanical energy) is defined by $\Pi = B - T_0 s$, where B is the total enthalpy, s is the entropy and T_0 is the surrounding temperature (extrinsic variable). In this paper, T_0 is the local mean temperature so that Π is related to the potential, of a given fluid particle, to produce acoustical work from its stationary state.

We derive a theory of propagation for the three variables: fluctuations of entropy, Coriolis deceleration and acoustical exergy in the particular case of an adiabatic process of a perfect fluid assuming that the main flow is irrotational and isentropic. In this case, the entropy fluctuations are first determined (section 2). Then vorticity fluctuations (by the mean of Coriolis deceleration) are computed with entropy fluctuations as a source (section 3). These fluctuations can be easily found in analytical form when co-ordinates linked to the main flow are used (section 4). The propagation equation for acoustical exergy is given (section 5).

Some applicative examples of this theory can be found in reference [4] and a

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short illustrative example of the use of the acoustical exergy is given in reference [5].

2. ENTROPY FLUCTUATIONS

Consider the propagation and generation of sound in an inviscid fluid without any energy exchange with the exterior of the domain (adiabatic process). For all the quantities, stationary q_0, and small fluctuating q' parts ($q = q_0 + q'$) are separated. Moreover, it is assumed that the stationary entropy remains constant, $\nabla s_0 = 0$, and that the stationary flow is irrotational, $\mathbf{\Omega}_0 = 0$.

In the motion of this perfect fluid, the entropy s of a fluid particle remains constant:

$$\partial s/\partial t + (\mathbf{v} \cdot \nabla)s = 0. \tag{1}$$

The linearization of equation (1),

$$\frac{\partial s'}{\partial t} + \mathbf{v}_0 \cdot \nabla s' = \frac{D_0 s'}{D_0 t} = 0, \tag{2}$$

leads to the solution

$$s' = s'(\mathbf{x} - \mathbf{v}_0 t). \tag{3}$$

Thus, in this case, entropy fluctuations do not depend on acoustic and vorticity waves and are only convected with the fluid particles.

3. VORTICITY FLUCTUATIONS

Euler's equation in Crocco's form is

$$\partial \mathbf{v}/\partial t + \nabla B - \mathbf{v} \wedge \mathbf{\Omega} = T\nabla s, \tag{4}$$

where B is the stagnation or total enthalpy.

The equation for transport of vorticity fluctuations $\mathbf{\Omega}' = \mathrm{curl}\,(\mathbf{v}')$ can be deduced from linearization of the curl of equation (4):

$$\partial \mathbf{\Omega}'/\partial t - \mathrm{curl}\,(\mathbf{v}_0 \wedge \mathbf{\Omega}') = -\nabla s' \wedge \nabla T_0. \tag{5}$$

Variations of $\mathbf{\Omega}'$ are defined by transported fluctuations of vorticity and by entropy fluctuations s'. If $s' = 0$, the vortex lines are attached to fluid particles and are convected by the main flow [6].

The fluctuations s' and $\mathbf{\Omega}'$, which propagate with mean flow velocity, are related to "pseudo-sound". So they are not directly involved in the "true-sound" propagation (fluctuations which propagate with sound velocity respect to the main flow). They are, however, governing the "true-sound" generation. For this purpose, it is useful to consider only components of these fluctuations influencing this generation. The linearization of the equation (4) and of the continuity equation gives the two main equations governing the sound propagation:

$$\partial \mathbf{v}'/\partial t + \nabla \Pi' = -\mathscr{F}'(s', \mathbf{\Omega}'), \tag{6}$$

$$\partial\rho'/\partial t + \nabla \cdot \mathbf{m}' = 0, \tag{7}$$

where

$$\Pi' = p'/\rho_0 + \mathbf{v}_0 \cdot \mathbf{v}' = B' - T_0 s' \tag{8}$$

is the acoustical exergy,

$$\mathbf{m}' = \rho_0 \mathbf{v}' + \rho' \mathbf{v}_0 \tag{9}$$

is the fluctuating mass velocity and

$$\mathscr{F}'(s', \mathbf{\Omega}') = \mathbf{\Omega}' \wedge \mathbf{v}_0 + s' \nabla T_0. \tag{10}$$

is a force function.

The second member in the definition of $\mathscr{F}'$ comes from entropy fluctuations which are easy to find explicitly by equation (3). The first member results from vorticity fluctuations but only two of its components are relevant: those which are perpendicular to the stationary velocity. The emphasis should therefore be less on vorticity fluctuations than on the product $\mathbf{\Upsilon}' = \mathbf{\Omega}' \wedge \mathbf{v}_0$ itself, which is the fluctuating Coriolis deceleration.

The equation describing the transport of $\mathbf{\Upsilon}'$ can be found by making the vectorial product of equation (5) with $\mathbf{v}_0$:

$$\partial \mathbf{\Upsilon}'/\partial t - \mathbf{v}_0 \wedge \operatorname{curl}(\mathbf{\Upsilon}') = \nabla s'(\mathbf{v}_0 \cdot \nabla T_0) - \nabla T_0(\mathbf{v}_0 \cdot \nabla s') = \mathscr{K}'. \tag{11}$$

Studying directly $\mathbf{\Upsilon}'$ is more efficient and simpler than methods based on direct use of vorticity fluctuations $\mathbf{\Omega}'$.

4. SOLUTIONS FOR CORIOLIS DECELERATION FLUCTUATIONS

For solving equation (11), curvilinear "natural" co-ordinates $(\alpha_1, \alpha_2, \xi)$ are introduced. The co-ordinate surfaces $\xi = \text{const}$ are the equipotential surfaces for the mean flow and domains at constant α_1 and α_2 are current lines (normal to equipotential surfaces). In those co-ordinates the projection of $\mathbf{\Upsilon}'$ on the axis ξ is zero and the vectorial equation (11) induces only two scalar equations of first order (for Υ_1' and Υ_2'). In the case of orthogonality of surfaces $\alpha_1 = \text{const}$ and $\alpha_2 = \text{const}$, the two equations are independent and could be written ($\mathbf{v}_0 = v_0 \mathbf{k}_\xi$)

$$\frac{\partial h_i \Upsilon_i'}{\partial t} + v_0 \frac{\partial h_i \Upsilon_i'}{h_\xi \partial \xi} = h_i \mathscr{K}_i', \tag{12}$$

where h_1, h_2 and h_ξ are the scale factors of curvilinear co-ordinates α_1, α_2 and ξ. It is very difficult to verify, in general, that it is possible to build an orthogonal co-ordinate system on the equipotential surfaces. However, when the main flow depends on only one or two "natural" co-ordinates, the problem can be solved easily. This kind of problem includes, for instance, quasi-unidimensional flow (like that of a nozzle), cylindrical vortex and helicoidal flow in cylindrical channels. In the first case, main flow parameters depend only on ξ while in the two other cases they do not depend on ξ. Two dimensional flows are another example of this kind of problems.

The scale factor of "natural" co-ordinates is defined by the structure of the stationary field. Relations binding these co-ordinates come directly from kinematic equation. In that way, the equation, for an irrotational main flow, $\mathbf{v}_0 = \nabla\phi_0$ leads to

$$v_0 = \frac{1}{h_\xi}\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi} \tag{13}$$

and $h_\xi v_0$ is a function of only ξ.

From the equation of mass conservation $\nabla \cdot (\rho_0 \mathbf{v}_0) = 0$, the spreading of a current tube is given by

$$h_1 h_2 \rho_0 v_0 = h_1^0 h_2^0 \rho_0^0 v_0^0 = M_0^0, \tag{14}$$

where $M_0^0(\alpha_1, \alpha_2)$ is the mass flow through the unit current tube $(\mathrm{d}\alpha_1, \mathrm{d}\alpha_2)$ on the base surface $(\xi = 0)$.

The source term on the right side of equation (11) is

$$h_i \mathscr{K}'_i = \frac{\partial s'}{\partial \alpha_i} v_0 \frac{\partial T_0}{h_\xi \partial \xi} + \frac{\partial T_0}{\partial \alpha_i}\frac{\partial s'}{\partial t}. \tag{15}$$

If $\mathscr{K}'_i = 0$ (i.e., for $s' = 0$), the solution of equation (12) is

$$h_i \Upsilon'_i(\mathbf{x}, t) = h_i^0 \Upsilon'^0_i(\alpha_1, \alpha_2, \tau), \tag{16}$$

where τ, the phase of the signal, is

$$\tau = t - \int_0^\xi \frac{h_\xi \,\mathrm{d}\xi}{v_0}. \tag{17}$$

$\Upsilon'^0_i(\alpha_1, \alpha_2, t)$ is the distribution given on the base surface.

The couple $h_i \Upsilon'_i$ is transported along the current line without any change. The projection Υ'_i increases (respectively decreases) according to the increase (respectively the decrease) of the scale factor along the related axis in the current tube deformation.

If $\mathscr{K}'_i \neq 0$, the solutions of equation (11) can also be found by direct integration of this equation. The function $s'(\mathbf{x}, t)$ is given by equation (2) for transport of entropy in "natural" co-ordinates,

$$\frac{\partial s'}{\partial t} + v_0 \frac{\partial s'}{h_\xi \partial \xi} = 0, \tag{18}$$

which has the analytical solution

$$s'(\mathbf{x}, t) = s'^0(\alpha_1, \alpha_2, \tau). \tag{19}$$

The function $h_i \Upsilon'_i$ is calculated as

$$h_i \Upsilon'_i(\mathbf{x}, t) = h_i^0 \Upsilon'^0_i(\alpha_1, \alpha_2, \tau) - \frac{1}{C_p}\int_0^\xi \varphi_i(\xi, \alpha_1, \alpha_2, \tau)\,\mathrm{d}\xi, \tag{20}$$

where C_p is the specific heat coefficient at constant pressure and

$$\varphi_i(\xi, \alpha_1, \alpha_2, \tau) = v_0 \frac{\partial v_0}{\partial \xi} \frac{\partial s'^0(\alpha_1, \alpha_2, \tau)}{\partial \alpha_i} + h_\xi \frac{\partial v_0}{\partial \alpha_i} \frac{\partial s'^0(\alpha_1, \alpha_2, \tau)}{\partial \tau}. \tag{21}$$

The solution of equation (20) is very simple and, for $\tau > 0$, is given by only the distribution of $s'^0(\alpha_1, \alpha_2, \tau)$ and $\Upsilon_i'^0(\alpha_1, \alpha_2, \tau)$, for $i = 1, 2$ on the base surface.

If it is required, the distribution of vorticity can be calculated with $\boldsymbol{\Upsilon}' = \boldsymbol{\Omega}' \wedge \mathbf{v}_0$ and equation (5):

$$M_0^0 \frac{h_1 \Omega_1'}{\rho_0} = h_2 \Upsilon_2', \qquad M_0^0 \frac{h_2 \Omega_2'}{\rho_0} = -h_1 \Upsilon_1',$$

$$-\frac{\partial}{\partial t}\left(\frac{h_\xi \Omega_\xi'}{\rho_0}\right) = h_\xi v_0 \left(\frac{\partial h_2 \Upsilon_2'}{\partial \alpha_1} - \frac{\partial h_1 \Upsilon_1'}{\partial \alpha_2} + \frac{\partial s'}{\partial \alpha_1}\frac{\partial T_0}{\partial \alpha_2} - \frac{\partial s'}{\partial \alpha_2}\frac{\partial T_0}{\partial \alpha_1}\right). \tag{22}$$

5. PROPAGATION EQUATION FOR ACOUSTICAL EXERGY

Upon making use of equation (6), the material derivative of Π' is given by

$$\frac{D_0 \Pi'}{D_0 t} = \frac{1}{\rho_0}\frac{\partial p'}{\partial t} - s' \mathbf{v}_0 \cdot \nabla T_0, \tag{23}$$

which can be transformed, by using equation (2), into

$$\frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 \Pi'}{D_0 t}\right) = \frac{\partial}{\partial t}\left(\frac{D_0}{D_0 t}\left(\frac{\rho'}{\rho_0}\right)\right) - s' \mathbf{v}_0 \cdot \nabla \left(\frac{\mathbf{v}_0 \cdot \nabla T_0}{c_0^2}\right). \tag{24}$$

Transformation of the continuity equation (7) leads to

$$\frac{D_0}{D_0 t}\left(\frac{\rho'}{\rho_0}\right) = -\frac{1}{\rho_0} \nabla \cdot (\rho_0 \mathbf{v}'). \tag{25}$$

The elimination of the term in $\mathbf{v}'$ between equations (24, 25) and the divergence of ρ_0 times equation (6) leads to a second order equation for the acoustical exergy Π':

$$\frac{1}{\rho_0} \nabla(\rho_0 \nabla \Pi') - \frac{D_0}{D_0 t}\left(\frac{1}{c_0^2}\frac{D_0 \Pi'}{D_0 t}\right) = -\frac{1}{\rho_0} \nabla \cdot (\rho_0 \boldsymbol{\Upsilon}') - \frac{1}{\rho_0} \nabla \cdot (\rho_0 s' \nabla T_0)$$

$$+ s' \mathbf{v}_0 \cdot \nabla \left(\frac{\mathbf{v}_0 \cdot \nabla T_0}{c_0^2}\right). \tag{26}$$

The generation of pressure waves is given by the solution of equation (26) where the right side is defined by the solutions (19) and (20).

Equation (26) is very close to Howe's equation for stagnation enthalpy B', small differences appears in the right side caused by the difference in variables

($B' = \Pi' + T_0 s'$). In the case of isentropic flows B' equals Π'. On the other hand, in non-isentropic flows the physical and thermodynamical meanings of B' and Π' are different.

Π' characterizes the work that can be produced by a gas when it is transformed from equilibrium condition. Π' is then related to the availability of gas to generate an energy for acoustical waves.

B' characterizes the work that can be produced by a gas when it is transformed from zero absolute temperature (this is an hypothetical process with no link to the real process of sound waves).

The non-equivalence of the variables B' and Π' is especially clear for adiabatic non-isentropic processes (for instance in viscous gas). In this case, the stagnation enthalpy in a volume without any energy exhange with the exterior is constant. But, according to the second law of thermodynamics, as the entropy of the system is increasing (some energy is irreversibly transformed in this volume), there is a decrease in the real work that the gas can produce. This implies that the potential acoustical radiation decreases also.

Even for isentropic processes, the use of B' for estimating the sound intensity can also lead to important disadvantages mainly caused by the invariability of entropy along velocity lines. For instance, in the case of stationary flow (without any gradient) generated in a point source $r = 0$, the value Π' decreases $1/r$ and s' is kept constant on velocity lines. Far enough from the source, B' depends practically only on entropy fluctuations. This is the reason why the use of B' is not convenient for estimating the sound radiation with approximative methods, e.g., numerical methods.

Classical variables (p', $\mathbf{v}'$ and s') are connected to variables (Π', $\boldsymbol{\Upsilon}'$ and s') by

$$\partial \mathbf{v}'/\partial t = -\nabla \Pi' - \boldsymbol{\Upsilon}' - s'\nabla T_0 + \mathbf{F}', \tag{27}$$

$$\partial p'/\partial t = \rho_0(D_0\Pi'/D_0 t) + \rho_0 s'\mathbf{v}_0 \cdot \nabla T_0 - \rho_0 \mathbf{v}_0 \cdot \mathbf{F}', \tag{28}$$

where $\mathbf{F}'$ is the fluctuating external force (which is not always equal to zero). The distribution of p', $\mathbf{v}'$ and s' on a surface G_0, which is not a surface of current, can be defined by distribution of Π', $\boldsymbol{\Upsilon}'$, s' and by the normal derivative $\partial\Pi'/\partial n$ on the same surface.

It is convenient to give the boundary conditions for Π', $\boldsymbol{\Upsilon}'$ and s' on an equipotential surface $\xi = \text{const}$ and a current surface $f(\alpha_1, \alpha_2) = \text{const}$. The distribution of $\boldsymbol{\Upsilon}'$ and s' has to be given on a surface $\xi = \text{const}$ at the upstream side of the domain.

When the current pipe is closed by a non-moving impermeable solid surface, the boundary condition is (with $\mathbf{F}' = 0$):

$$\partial \Pi'/\partial n + \Upsilon'_n + s'\,\partial T_0/\partial n = 0. \tag{29}$$

On an oscillating wall where the normal velocity is $v'_n(\mathbf{x}, t)$, this boundary condition becomes

$$\partial \Pi'/\partial n = -D_0 v'_n/D_0 t - \Upsilon'_n - s'\,\partial T_0/\partial n. \tag{30}$$

If the flow is limited by an impedance surface, the boundary condition is

$$\frac{\partial p'_H}{\partial t} = \frac{D_0 \Pi'}{D_0 t} + \rho_0 s'(\mathbf{v}_0 \cdot \nabla T_0), \qquad \frac{\partial \Pi'}{\partial n} = -\frac{\partial v'_n}{\partial t} - \Upsilon'_n - s' \frac{\partial T_0}{\partial n}, \tag{31}$$

where p'_H and $v'_H(p'_H) = -v'_n$ are the pressure and normal velocity in wall direction near the impedance wall outside of the flow.

6. CONCLUSION

The results presented show that, in the case of perfect fluid in adiabatic process with irrotational mean flow, the acoustical exergy satisfies a second order wave equation of convected type. In this restrictive case, the acoustical exergy is a possible alternative to the total enthalpy to describe the acoustical field. In such a case, acoustical exergy has the advantage of being a real availability of energy for acoustical waves.

The source terms, in the propagation equation for acoustical exergy, correspond to excitation by entropy waves and by fluctuating Coriolis acceleration waves. Both of these waves propagate with the mean flow and can be computed if they are given on an upstream section.

It can be noticed that the fluctuating vorticity is not really needed: only the fluctuating Coriolis acceleration need be computed. This calculation is easy to make in natural co-ordinates linked to the mean flow.

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