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THE STABILITY AND CONVERGENCE CHARACTERISTICS OF THE DELAYED-X LMS ALGORITHM IN ANC SYSTEMS

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The Delayed-x LMS algorithm is a simplified version of the Filtered-x LMS algorithm, in which the model $\hat{C}$ of the secondary path C from the adaptive filter output to the error sensor is represented by a pure delay of k samples (the delayed model D) in order to reduce system complexity. However, the simplification produces a modelling error, which deteriorates the ANC performance. In this paper, the stability, especially that of the convergence characteristics of the feedforward active noise control system with the Delayed-x LMS algorithm, is investigated. It is shown that the stability condition is one in which the phase error (the phase difference between the secondary path C and its delayed model D) is in the range between $-\pi/2$ and $\pi/2$. Furthermore, it is shown that if the phase error is large, a small step-size parameter μ should be adopted to achieve stable noise cancellation, though the convergence speed becomes slow in the frequency domain. The theoretical results are verified by computer simulations.

1. INTRODUCTION

The Filtered-x LMS algorithm [1] is widely used in feedforward active noise control (ANC) systems [2–4]. This adaptive algorithm is an alternate version of the LMS algorithm which is used when the secondary path C from the adaptive filter output to the error sensor is represented by a non-unitary transfer function. The Filtered-x LMS algorithm requires a filtered reference signal, which is the convolution of an input signal and the impulse response of model $\hat{C}$ of a secondary

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path C, making computation of convolution a heavy burden for real-time controllers.

The Delayed-x LMS algorithm is a simplified form of the Filtered-x LMS algorithm, in which model $\hat{C}$ of secondary path C from the adaptive filter output to the error sensor is represented by a pure delay of k samples (the delayed model D) to reduce computation and reduce system complexity. This simplified version of the algorithm has been used in telecommunications applications [5, 6]. In the ANC system, the simplification produces a modelling error [7, 8], which causes deterioration in the ANC performance. The ANC system with the Delayed-x LMS algorithm was empirically studied [9–11] in the time domain. On the contrary, stability has been evaluated by using a frequency domain model of the "filtered" LMS algorithm [12]. But theoretical study of the convergence characteristics has not been carried out.

In this paper, the effects of the modelling error on the stability, especially that of the convergence speed of the Delayed-x LMS algorithm, are theoretically examined by use of a frequency domain analysis method [13, 14]. It is shown that the stability requires that the phase error (the phase difference between the secondary path C and its delayed model D) should be in the range between $-\pi/2$ and $\pi/2$ at all frequencies in the frequency range concerned. It is found that under the stable state, the convergence speed is affected by the phase error. That is, if the phase error is large, a small step-size parameter μ should be adopted to achieve stable noise cancellation, though the convergence speed becomes slow.

2. ADAPTIVE ALGORITHM

2.1. FILTERED-X LMS ALGORITHM

A block diagram of the active noise control system with the Filtered-x LMS algorithm is shown in Figure 1, assuming that the primary and the secondary paths are represented by finite impulse response (FIR) systems. The adaptive filter for noise control has an FIR-type filter.

The filtered reference signal $u(n)$ is given by

$$u(n) = \mathbf{x}^{\mathrm{T}}(n)\hat{\mathbf{c}}, \tag{1}$$

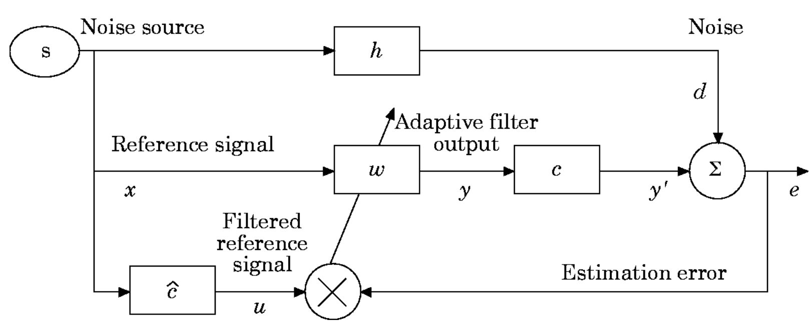

Figure 1. Block diagram of an ANC system with the Filtered-x LMS algorithm.

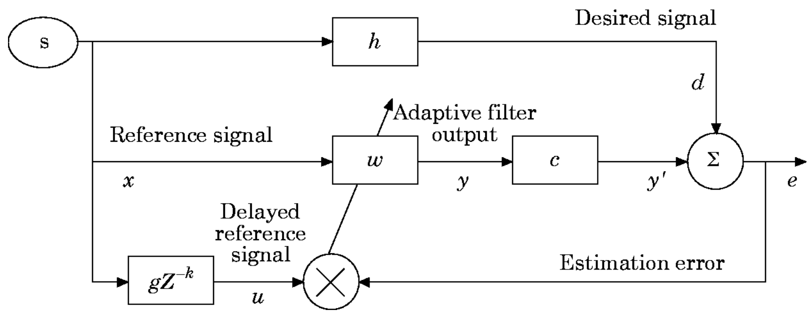

Figure 2. Block diagram of an ANC system with the Delayed-x LMS algorithm.

where

$$\hat{\mathbf{c}} = [\hat{c}_1 \hat{c}_2 \ldots \hat{c}_{N_c}]^{\mathrm{T}}. \tag{2}$$

$\hat{c}_j (j = 1, \ldots, N_c)$ is the coefficient of the FIR filter, which is model $\hat{C}$ of the secondary path C; N_c is its length. In this paper, the secondary path C is assumed to be time invariant.

The estimation error $e(n)$ and recursive relation for updating the tap-weight vector $\boldsymbol{w}(n)$ are

$$e(n) = d(n) + \boldsymbol{u}^{\mathrm{T}}(n)\boldsymbol{w}(n), \qquad \boldsymbol{w}(n+1) = \boldsymbol{w}(n) - 2\mu\boldsymbol{u}(n)e(n), \tag{3, 4}$$

where

$$\boldsymbol{w}(n) = [w_1(n) \quad w_2(n) \ldots w_{N_w}(n)]^{\mathrm{T}}. \tag{5}$$

N_w is the length of the adaptive filter $\boldsymbol{w}(n)$, $\boldsymbol{u}(n)$ is a vector of the filtered reference signal $u(n)$ shown by equation (1), and μ is the step-size parameter.

2.2. DELAYED-X LMS ALGORITHM

In the Delayed-x LMS algorithm, the model of the secondary path C is replaced by a delayed model D, which requires no convolution to obtain the filtered reference signal. A block diagram of the ANC system with the Delayed-x LMS algorithm is shown in Figure 2. The filtered reference signal is given by

$$u'(n) = gx(n-k), \tag{6}$$

where the gain g is usually 1 and k is the number of points from 0 to the peak of the impulse response of secondary path C.

Using equation (6), the estimation error and recursive relation for updating of the Delayed-x LMS algorithm are expressed as follows:

$$e(n) = d(n) + \boldsymbol{u}^{\mathrm{T}}(n)\boldsymbol{w}(n), \qquad \boldsymbol{w}(n+1) = \boldsymbol{w}(n) - 2\mu\boldsymbol{u}'(n)e(n), \tag{7, 8}$$

where $\boldsymbol{u}'(n)$ is the vector of the filtered reference signal $u'(n)$ defined by equation (6).

3. ADAPTIVE CONDITION

The Delayed-x LMS algorithm can reduce computation load significantly by eliminating the convolution. However, the modelling error caused by simplifying the filtered reference signal deteriorates the performance of the adaptive control system. The stability and convergence characteristics of the Delayed-x LMS algorithm will be discussed below.

3.1. EVALUATION IN THE TIME DOMAIN

The method used here for evaluating the performance of the Delayed-x LMS algorithm is similar to that described elsewhere [13, 14]. Substituting equation (7) into (8), and taking the mathematical expectation on both sides, one obtains

$$E[\boldsymbol{w}(n+1)] = (\boldsymbol{I} - 2\mu E[\boldsymbol{u}'(n)\boldsymbol{u}^{\mathrm{T}}(n)])E[\boldsymbol{w}(n)] - 2\mu E[\boldsymbol{u}'(n)\, d(n)], \tag{9}$$

where $\boldsymbol{u}(n)$ is a vector of the filtered reference signal $u(n)$ shown in equation (1) in the case of $\hat{C} = C$, and $\boldsymbol{u}'(n)$ is a vector of the filtered reference signal $u'(n)$ shown in equation (6) in the case of $\hat{C} = D$.

From equation (9), the convergence time in the adaptive algorithm is determined by the eigenvalues of the matrix, $E[\boldsymbol{u}'(n)\boldsymbol{u}^{\mathrm{T}}(n)]$ [15, 16].

The stability condition of the Delayed-x algorithm is assured if

$$0 < |1 - 2\mu\lambda(i)| < 1, \tag{10}$$

where $\lambda(i)$ is the ith eigenvalue of the matrix, $\mathbf{E}[\boldsymbol{u}'(n)\boldsymbol{u}^{\mathrm{T}}(n)]$.

In the case of $\hat{C} = C$, the eigenvalues of $\boldsymbol{E}[\boldsymbol{u}(n)\boldsymbol{u}^{\mathrm{T}}(n)]$ of the Filtered-x LMS algorithm are always postive reals. Then, the stability of the ANC system with the Filtered-x LMS algorithm can be achieved by adopting a positive value for the step-size parameter μ. However, in the Delayed-x LMS algorithm, the eigenvalues of $\mathbf{E}[\boldsymbol{u}'(n)\boldsymbol{u}^{\mathrm{T}}(n)]$ are complex in general, and thus written as

$$\lambda(i) = |\lambda(i)| \exp(\mathrm{j}\theta(i)), \tag{11}$$

where $|\lambda(i)|$ represents the magnitude, and $\theta(i)$ is the phase error.

The stability condition shown in equation (10) is rewritten as

$$0 < |1 - 2\mu|\lambda(i)| \exp(\mathrm{j}\theta(i))| < 1. \tag{12}$$

Thus, if the step-size parameter μ is a positive value, the stability of the ANC system with the Delayed-x LMS algorithm is assured for phase $\theta(i)$ shown in the following equations:

$$-\pi/2 \pm 2\pi < \theta(i) < \pi/2 \pm 2\pi, \tag{13}$$

or

$$-\pi/2 < \theta(i)(\mathrm{mod}\ 2\pi) < \pi/2. \tag{14}$$

In other words, the stability condition is not satisfied when the absolute value of the phase error $|\theta(i)|$ is larger than $\pi/2$, *i.e.* $|\theta(i)| > \pi/2$. The theoretical result is the same as that obtained by analyzing the Filtered-x LMS algorithm when the modelling error caused ($\hat{C} \neq C$) [8, 17, 18]. However, evaluation in the time domain does not lead to easy understanding of the physical meaning. Namely, the relation between phase $\theta(i)$ and the delayed model D is still unclear, and it is not easy to describe the influence of the phase error on the convergence characteristics of the Delayed-x LMS algorithm.

3.2. FREQUENCY DOMAIN FORMULATION

In this section, the stability condition and convergence characteristics are analyzed in the frequency domain. The method employed for evaluating the behavior of the adaptive algorithm in the frequency domain facilitates the understanding of its physical meaning and requires fewer computations [13].

When the convergence speed of the Delayed-x LMS algorithm is slow, the adaptive filters are considered as time invariant linear filters for a given period [14, 19]. Thus, the equation described in equation (9) in the time domain can approximately be expressed in the frequency domain as:

$$\mathbf{E}[W(\omega, n+1)] = (1 - 2\mu E[\overline{U'(\omega, n)}U(\omega, n)])E[W(\omega, n)] - 2\mu E[U'(\omega, n)D(\omega, n)], \tag{15}$$

where $U(\omega, n)$ is the DFT of the filtered reference signal $u(n)$, and $U(\omega, n) \approx X(\omega, n)C(\omega)$. $U'(\omega, n)$ is the DFT of the filtered reference signal $u'(n)$, $U'(\omega, n) \approx X(\omega, n)G(\omega)$ and $\overline{U'(\omega, n)}$ is the complex conjugate of the $U'(\omega, n)$. The transfer function of the delayed model D is defined as $G(\omega)$.

From equation (15), the convergence characteristics of the Delayed-x LMS algorithm are defined by matrix $\mathbf{E}[U'(\omega, n)U(\omega, n)]$ at each frequency bin ω. Assuming that the primary noise $x(n)$ is a white noise with zero mean and unit variance, the matrix can be written as

$$\mathbf{E}[\overline{U'(\omega, n)}U(\omega, n)] = E[\overline{G(\omega)}C(\omega)]. \tag{16}$$

It is clear that the stability of the Delayed-x LMS algorithm is assured by

$$0 < |1 - 2\mu\overline{G(\omega)}C(\omega)| < 1, \tag{17}$$

where

$$\overline{G(\omega)} = |G(\omega)| \exp(-j\theta_{\omega g}), \qquad C(\omega) = |C(\omega)| \exp(j\theta_{\omega c}). \tag{18, 19}$$

For convenience, the frequency bin ω will be omitted hereafter. From equations (18) and (19), equation (17) can be rewritten as

$$0 < |1 - 2\mu|G||C| \exp(j(\theta_c - \theta_g))| < 1. \tag{20}$$

It is clear that the change of the gain $|G|$ can be translated into the adjustment of the step-size parameter μ.

The stability condition of the Delayed-x LMS algorithm is defined by the phase error $\theta_s = \theta_c - \theta_g$ as in the following:

$$-\pi/2 < \theta_s (\text{mod } 2\pi) < \pi/2. \tag{21}$$

In other words, if the phase error θ_s is out of the above range, the Delayed-x LMS algorithm will not be stable. The theoretical result, is the same as that obtained by Feinutuch [12].

For easy understanding, the stability and convergence characteristics of the Delayed-x LMS algorithm are discussed in the complex plane [20]. From equation (20), the stability and convergence speed of the Delayed-x LMS algorithm are shown in the polar form as follows:

$$\lim_{n\to\infty} (1 - 2\mu|G||C| \exp(\mathrm{j}\theta_s))^n \to 0. \tag{22}$$

If

$$|B| = 2\mu|G||C|, \tag{23}$$

the bracketed term in equation (22) can be expressed as

$$1 - 2\mu|G||C| \exp(\mathrm{j}\theta_s) = 1 - |B| \exp(\mathrm{j}\theta_s) = 1 + |B| \exp(\mathrm{j}(\pi + \theta_s)) = |A| \exp(\mathrm{j}\theta_A), \tag{24}$$

where $|A|$ and $|B|$ are the absolute values of the complex numbers A and B respectively. Substituting equation (24) into (22), equation (22) is rewritten as follows:

$$\lim_{n\to\infty} |A|^n \exp(\mathrm{j}n\theta_A) \to 0. \tag{25}$$

It is clear that the adaptive filter with the Delayed-x LMS algorithm will converge when $|A| < 1$, and diverge when $|A| > 1$ as shown in Figure 3.

In case ①, the adaptive filter diverges because $|A| > 1$, since the absolute value of the phase error $|\theta_s|$ is larger than $\pi/2$. When the absolute value of the phase error $|\theta_s|$ is smaller than $\pi/2$, the stability is defined by μ, $|G|$ and $|C|$. If $|G|$ and $|C|$ are fixed, the step-size parameter μ can be given a small value to generally promote the stability of the Delayed-x LMS algorithm. In case ②, the Delayed-x LMS algorithm works stably and $|A|$ is in a unit circle when step-size parameter μ is set at a value smaller than that in case ③ where the adaptive filter is not stable as shown in Figure 3. It is clear from Figure 3 that the larger the phase error is, the smaller the μ which should be used. Then, the convergence speed of the Delayed-x LMS algorithm will be slower.

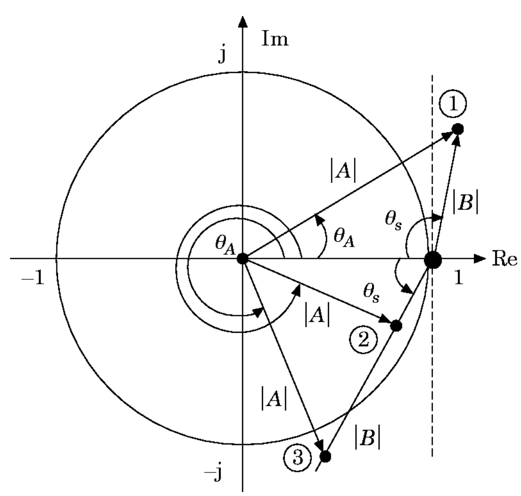

Figure 3. Evaluation of the stability on a complex plane.

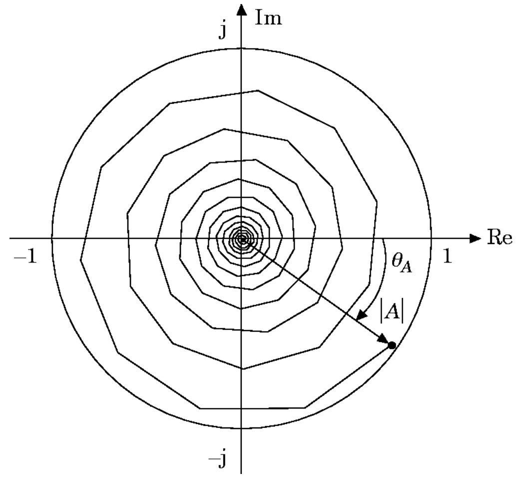

Figure 4. Example of convergence characteristics when $|A| < 1$, $\theta_A < 0$.

The convergence characteristics of the Delayed-x LMS algorithm is evaluated in terms of $|A|^n \exp(\mathrm{j}n\theta_A)$ shown in equation (25). The smaller the $|A|$ is, the faster the $|A|^n$ converges on 0 as n increases. $|A|$ does not converge on 0 on the real axis when θ_A is not 0, and the term $|A|^n \exp(\mathrm{j}n\theta_A)$ rotates clockwise when $\theta_A < 0$ on the complex plane as shown in Figure 4, and rotates counterclockwise when $\theta_A > 0$.

Next, the influence of the phase error on convergence of the Delayed-x LMS algorithm is discussed. In Figure 5, the convergence is faster for $|A_1|$ than for $|A_2|$, while $|B_1| = |B_2|$, because $\theta_{s2} > \theta_{s1}$. In other words, the nearer to 0 the phase error θ_s is, the faster the convergence speed.

It is clear from the above discussion on the complex plane that the stability of the Delayed-x LMS algorithm is influenced by the phase error. The stability condition is achieved when the phase error is in the range between $-\pi/2$ and $\pi/2$, and the convergence speed is slower when the phase error is large.

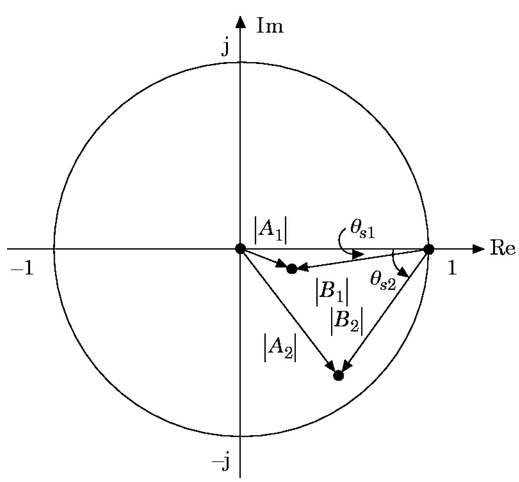

Figure 5. Convergence speed affected by the difference in phase, θ_s.

4. COMPUTER SIMULATIONS

Numerical simulations were carried out to confirm the theoretical results described in the previous section using the simple ANC system shown in Figure 2. All the simulations were performed in the time domain, while their evaluations were carried out both in time and frequency domains.

The experimental conditions were (a) The noise source was white noise with zero mean and unit variance. (b) The initial value $\mathbf{w}(0) = 0$. (c) The length N_w of the adaptive filter $\mathbf{w}(n)$ was 512. (d) The impulse response of the secondary path C was assumed to have an appropriate form for simulation, and its model had $g = 1$ and $k = 7$.

4.1. PHASE DIFFERENCE SATISFIES THE STABILITY CONDITION

To investigate the influence of the phase error on the filter convergence, two simulations were carried out when the absolute value of the phase error is less than $\pi/2$ over the whole frequency range concerned.

There are two kinds (case A and B) of secondary path C, simulation results for them being shown in Figures 6 and 7, respectively. Figures 6(a) and 7(a) show the two kinds of impulse responses of secondary path C, and Figures 6(b) and 7(b) show the two kinds of gain $|C|$ of that path respectively. The two kinds of phase error are shown in Figures 6(c) and 7(c) respectively. It is clear from Figures 6 and 7 that the variation in gain is small and that the phase error is clearly different. Figure 6(d) and 7(d) show the residual noise at the error sensor after 20 000, 100 000 and 500 000 iterations respectively, when step-size parameter μ was set at 0·0001.

In the simulation of case A (Figure 6), the step-size parameter μ was assumed to be 0·0001 to keep the system stable and to achieve a better convergence speed. The adaptive filter of case A diverged when μ was increased. However, in the simulation of case B (Figure 7), when μ was increased to 0·001 (ten times), the

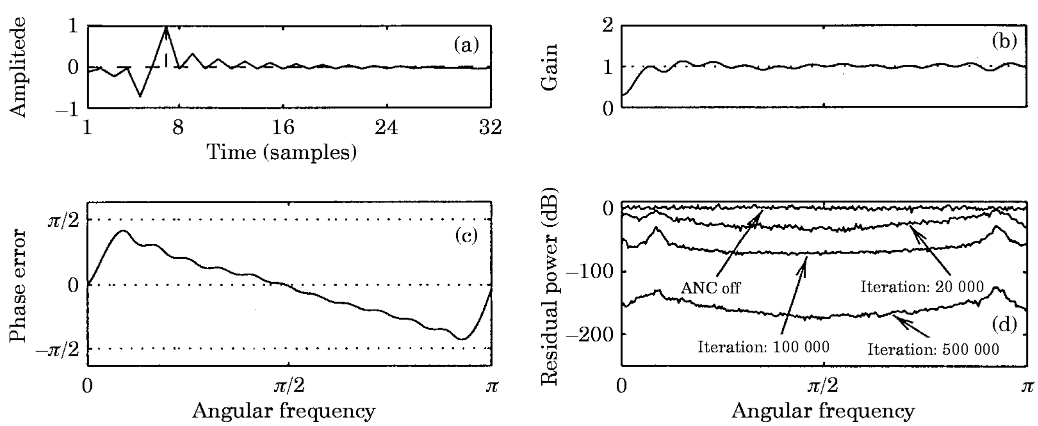

Figure 6. Simulated results of the phase error (case A) when the stability condition is satisfied. The impulse responses of secondary path C (solid line) and delayed model D (dashed line); (b) The gains of secondary path C (solid line) and delayed model D (dashed line); (c) The phase error; (d) The power spectra of residual noise (mean of 8 independent trials; 0, 20 000, 100 000 and 500 000 iterations).

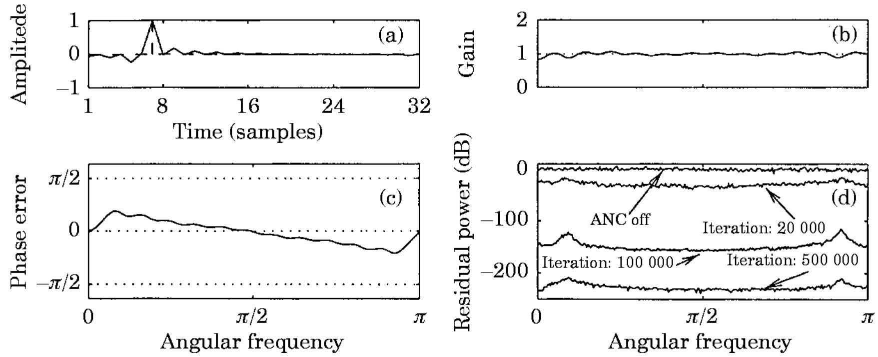

Figure 7. Simulated results of the phase error (case B) when the stability condition is satisfied. The impulse responses of secondary path *C* (solid line) and delayed model *D* (dashed line); (b) The gains of secondary path *C* (solid line) and delayed model *D* (dashed line); (c) The phase error; (d) The power spectra of residual noise (mean of 8 independent trials; 0, 20 000, 100 000 and 500 000 iterations).

convergence of the adaptive filter was still stable. It is clear that the phase error is larger at some frequencies, and so μ was set at a smaller value for all frequencies. Figure 8 shows the mean square error (MSE) defined by equation (26) at the error sensors.

$$MSE\ (\mathrm{dB}) = -10\log_{10}\left(E[e^2(n)]/E[d^2(n)]\right), \tag{26}$$

where $E[d^2(n)]$ is the average power of the desired signal due to primary noise, and $E[e^2(n)]$ is the average power of the residual at the error sensor. The result obtained by plotting MSE against the number of iterations is called the "learning curve" in the time domain.

It is difficult to evaluate the convergence speed by comparing the two impulse responses of secondary path C shown in Figures 6(a) and 7(a) in the time domain. It is found from Figures 6(b), 7(b) and Figures 6(d), 7(d) that if the phase error

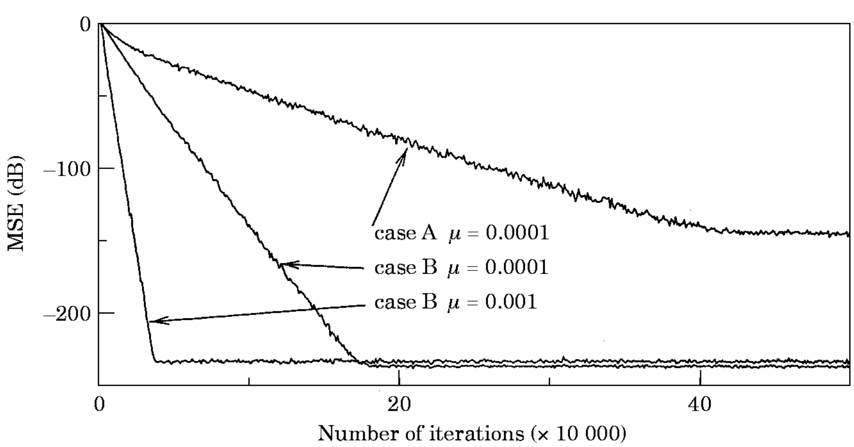

Figure 8. Learning curves of the simulation for two different cases (case A and case B) of the impulse response of secondary path *C*, as shown in Figures 6(a) and 7(a), respectively.

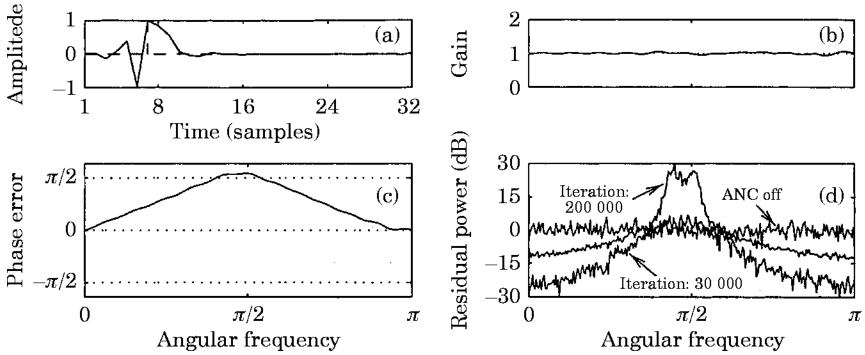

Figure 9. Simulated results of the phase difference when the stability condition does not satisfy. The impulse responses of secondary path C (solid line) and delayed model D (dashed line); (b) The gains of secondary path C (solid line) and delayed model D (dashed line); (c) The phase error; (d) The power spectra of residual noise (mean of 8 independent trials; 0, 30 000 and 200 000 iterations).

is large at a certain frequency, it results in a slow convergence speed, a large computation error, and less noise cancellation at that frequency.

In other words, if the phase error is relatively large, μ must be set to a smaller value for all frequencies in order to satisfy the stability condition. Thus, the convergence speed is slower and the cancellation is smaller at all frequencies. Furthermore, the total convergence speed is slower and the total cancellation is also smaller when evaluated in the time, as shown in Figure 8.

4.2. PHASE DIFFERENCE DOES NOT SATISFY THE STABILITY CONDITION

This simulation was carried out to investigate the influence of the phase error when the stability condition is not satisfied, that is, when the absolute of the phase error is larger than $\pi/2$ at some frequencies.

Figure 9(a) shows the impulse response of secondary path C, and Figure 9(b) shows its gain $|C|$, where the variation in gain $|C|$ is small. Figure 9(c) indicates the phase error. The step-size parameter μ was assumed to be 0·00005. Figure 9(d) represents the residual noise at the error sensor after 30 000 and 200 000 iterations. It is clear from Figure 9(d) that the adaptive filter diverges with an increase in iterations, which is also shown by the learning curve in the time domain, as shown in Figure 10.

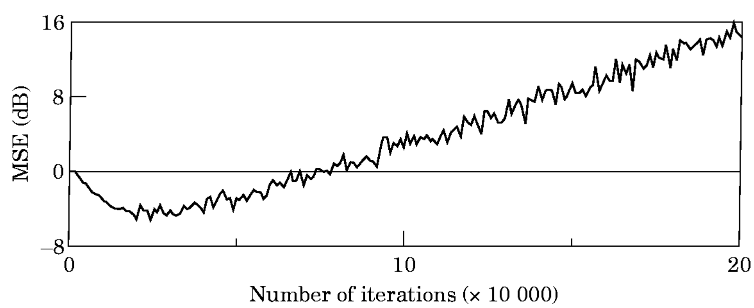

Figure 10. Learning curves of the simulation of the impulse response of secondary path C, as shown in Figure 9(a).

From Figure 9, it is found that the adaptive filter diverges with increasing iterations when the phase error does not satisfy the stability condition, that is, when the absolute of the phase error is larger than $\pi/2$ at some frequencies.

5. CONCLUSIONS

The stability condition and convergence characteristics of an adaptive filter with the Delayed-x LMS algorithm, which is a simplified version of the Filtered-x LMS algorithm, has been herein described. In that version, model $\hat{C}$ of secondary path C from the adaptive filter output to the error sensor is represented by a pure delay of k samples.

It is found that the Delayed-x LMS algorithm is stable when the phase error is in the range between $-\pi/2$ and $\pi/2$. It was also found that under the stability condition, the convergence speed of the adaptive filter is slower and cancellation is smaller when the phase error θ_s is large in the frequency domain. The results of computer simulations were given to verify the theoretical predictions.

Since secondary path C can be measured generally prior to active noise cancellation, stability and convergence characteristics are easily evaluated by calculating the phase error before performing the cancellation. A possible way to achieve good performance is to adjust the position of the loudspeaker and error microphone and to adopt the number of the delayed points.

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