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TWO PROBLEMS OF WAVEGUIDES CARRYING MEAN FLUID FLOW

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This work is concerned with the radiation of sound from a semi-infinite rigid duct. This duct is located symmetrically inside an infinite duct whose surfaces are (1) soft and (2) rigid. The whole fluid region inside these ducts is in motion with a constant fluid velocity. Because of the fluid flow vortex sheets are assumed to be attached to the two half planes which constitute the semi-infinite duct. The present mathematical problems can be considered as models for a duct with splitter plates.

1. INTRODUCTION

Noise is a major pollutant in an industrialized society and many people regard it as the most annoying one. Frequent sources of noise are ventilation fans, motor cars and heavy vehicles. For these producers of noise, as well as many others, a significant part of the noise propagates in a duct carrying a mean fluid flow. Normally, the interaction between the acoustic waves and the fluid flow is insignificant at low flow speeds: i.e., low Mach numbers. However, a technically important exception is at locations in the duct where the flow separates and, at these locations, the interaction can be significant even at low Mach numbers.

Nilsson and Brander [1, 2] studied a related problem concerning the propagation of sound in systems of cylindrical ducts with flow. Another recent application of a building block method for analyzing exhaust ducts (but without fluid flow) has been made by Rawlins [3].

Related work on flow acoustics has been carried for the diffraction of a sound wave by a strip in a moving fluid; see for example references [4–6].

The aim here is to analyze mathematically a model that could help in the design of a practical splitter plate exhaust system. At the trailing edge of these splitter plates the attached wake requires the imposition of a Kutta–Joukowski edge condition [5]. This requires that the velocity must be bounded at the trailing edge. Although a soft surface is a somewhat ideal concept of an absorbent surface, it can be used to give realistic design data for absorbent linings [7, 8]. Also honeycomb surfaces have been designed, constructed and used successfully for noise barriers surfaces [9]. The mathematical model simplifications can be regarded

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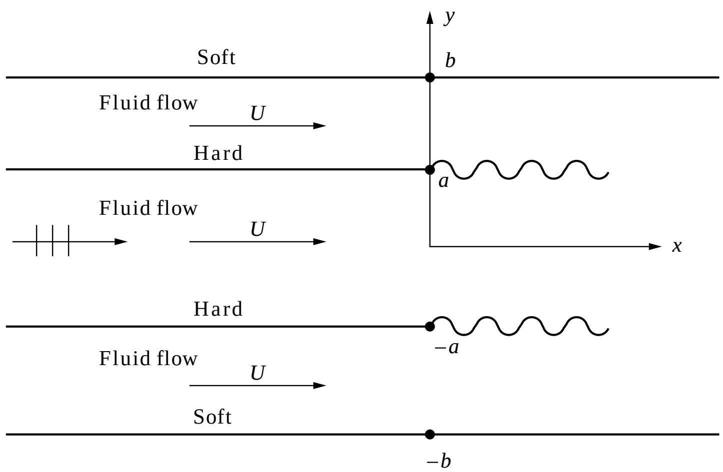

Figure 1. Sketch of the geometry of the problem with flow.

as first approximations to a more sophisticated mathematical model that involves finite length splitter structures and absorbent type boundary conditions.

Here an exact closed form solution will be presented for two new convected waveguide trifurcation problems with the aid of the Wiener–Hopf technique.

The trifurcated waveguide problems under consideration here are shown in Figures 1 and 2. The plates which make up the infinite duct are (1) soft and (2) hard. Located inside this infinite waveguide, in both cases, is a semi-infinite waveguide whose plates are rigid. The whole system is symmetric about a centre line of the system. A wake is attached to hard–hard semi-infinite plates. In order to simplify the formulae that will arise in the solution, it is assumed that only the practically important fundamental plane wave mode is assumed to propagate out of the mouth of the semi-infinite waveguide. Higher order modes offer no significant mathematical difficulties.

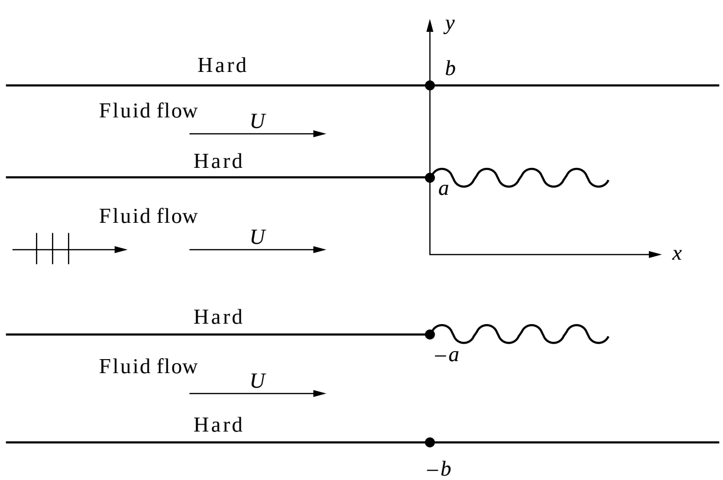

Figure 2. Geometry of the second diffraction problem with wake.

For each waveguide problem, the mathematical problem is formulated. Only the first problem is solved in detail. The solution of the second problem will be merely outlined since the solution procedure is similar to that of the first problem.

The solution will be expressed in the form of complex contour integrals. These integrals are analytically manipulated to reduce them to infinite series of modes which propagate in the waveguide system. In order not to disrupt the flow of the solution in the main text, various appendices have been included at the end of the paper. These include analytic details required in the main text. An expression for reflexion coefficients will be obtained, and the numerical and graphical results for the reflexion coefficients are discussed. Various graphical plots are given for the reflexion coefficients for some values of the Mach number M, which is the ratio of the fluid velocity over the speed of sound.

2. FORMULATION OF THE FIRST TRIFURCATED BOUNDARY VALUE PROBLEM

The effects on the distant sound field are to be considered when sound is propagated out of the open end of a semi-infinite hard duct into a moving stream. This semi-infinite duct is symmetrically placed inside an infinite soft duct. A wask is attached to the trailing edges of the hard duct (see Figure 1). The sound wave, of small amplitude, is superimposed on to a main stream which is moving with subsonic velocity U in the direction of the positive x-axis. The stream is assumed to have the same uniform velocity at all points of the regions inside the duct.

In the irrotational sound field the perturbation velocity u can be written in terms of a velocity potential ϕ, by $\mathbf{u} = \mathbf{grad}\phi$. Then the pressure in the sound field is given by

$$p = -\rho_0 \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \phi(x, y, t), \tag{1}$$

where ρ_0 is the density of the undisturbed stream.

The source field is located at $x = x_0 (x_0 < 0)$ and the acoustic field propagates along the semi-infinite duct. A time dependence $\exp(-\mathrm{i}\omega t)$ is assumed in the following work and this factor is suppressed throughout the remainder of this work. Thus, the problem becomes one of solving the convective wave equation

$$(1 - M^2)\phi_{xx} + 2\mathrm{i}M\phi_x + \phi_{yy} + k^2\phi = 0, \tag{2}$$

in which $M = U/c$ is the Mach number and $k = \omega/c$, where c is the constant speed of sound in the fluid, subject to the boundary conditions,

$$\phi = 0, \quad y = b, \quad -\infty < x < \infty, \qquad \phi_y = 0, \quad y = a, \quad -\infty < x < 0, \tag{3, 4}$$

$$\phi_y = 0, \quad y = -a, \quad -\infty < x < 0, \qquad \phi = 0, \quad y = -b, \quad -\infty < x < \infty. \tag{5, 6}$$

It is assumed here that $b > a$. The boundary conditions along the wakes are

$$\phi(x, -a^-) - \phi(x, -a^+) = F\, \mathrm{e}^{\mathrm{i}kx/M}, \quad x > 0, \tag{7}$$

$$\phi(x, a^-) - \phi(x, a^+) = F\, \mathrm{e}^{\mathrm{i}kx/M}, \quad x > 0. \tag{8}$$

The boundary conditions (7) and (8) are in the appropriate form to express a wake, which does not spread, trailing off the edges $y = \pm a$ of the duct, for subsonic flow, $0 < M < 1$ [10]. F is an unknown constant which is determined by the field behaviour at the trailing edges. The wake conditions (7) and (8) are obtained by direct integration of the boundary condition

$$\left(-\mathrm{i}k + M\frac{\partial}{\partial x}\right)\phi(x, \pm a^+) = \left(-\mathrm{i}k + M\frac{\partial}{\partial x}\right)\phi(x, \pm a^-), \quad x > 0, \tag{9}$$

which ensures the continuity of the pressure across the wake.

For the subsonic case $M < 1$, one can make the real transformations

$$x = (1 - M^2)^{1/2}X, \qquad k = (1 - M^2)^{1/2}K, \tag{10}$$

which, together with the substitution

$$\phi(x, y) = \psi(X, y)\,\mathrm{e}^{-\mathrm{i}KMX}, \tag{11}$$

reduces the boundary value problem (2)–(8) to

$$\tilde{\psi}_{XX} + \tilde{\psi}_{yy} + K^2\tilde{\psi} = 0, \qquad \tilde{\psi}_y = 0, \quad \text{on} \quad y = \pm a, \quad X < 0, \tag{12, 13}$$

$$\tilde{\psi} = 0, \quad \text{on} \quad y = \pm b, \quad -\infty < X < \infty, \tag{14}$$

$$\tilde{\psi}(X, a^+) + \tilde{\psi}(X, a^-) = F\,\mathrm{e}^{\mathrm{i}KX/M}, \quad X > 0, \tag{15}$$

$$\tilde{\psi}(X, -a^+) - \tilde{\psi}(X, -a^-) = F\,\mathrm{e}^{\mathrm{i}KX/M}, \quad X > 0. \tag{16}$$

To these boundary conditions those conditions at infinity relevant to the nature of the lowest propagating modes which the various duct regions can sustain are to be added.

One therefore has

For $X \to +\infty$, $-a \leqslant y \leqslant a$,

$$\tilde{\psi}(X, y) \Rightarrow \mathrm{e}^{\mathrm{i}\chi_0 X} + R\,\mathrm{e}^{-\mathrm{i}\chi_0 X}, \tag{17}$$

where $\chi_n = \sqrt{(K^2 - n^2\pi^2/4a^2)}$, $(n = 0, 1, 2, \ldots)$ are the roots of the equation

$$\sqrt{(K^2 - \chi_n^2)}\,\sin\left(2\sqrt{(K^2 - \chi_n^2)}a\right) = 0. \tag{18}$$

If $0 < Ka < \pi/2$ then $\chi_0 > 0$ and $\chi_1 = i\sqrt{(\pi^2/4a^2) - K^2}$ so that $\operatorname{Im}\chi_1 > 0$, $\operatorname{Re}\chi_1 = 0$. Thus, the semi-infinite duct region $-\infty < X < 0$, $-a \leqslant y \leqslant a$ can sustain only the lowest incident and reflected modes.

For $X \to +\infty$, $-b \leqslant y \leqslant b$

$$\tilde{\psi}(X, y) \Rightarrow T_1\,\mathrm{e}^{\mathrm{i}\sigma_1 X}\sin\left[(\pi/2b)(-y + b)\right], \tag{19}$$

where $\sigma_n = \sqrt{(K^2 - n^2\pi^2/4b^2)}$ $(n = 1, 2, \ldots)$ are the roots of the equation

$$\sin\left(2\sqrt{(K^2 - \sigma_n^2)}b\right) = 0. \tag{20}$$

For $X \to -\infty$, $a \leqslant y \leqslant b$

$$\tilde{\psi}(X, y) \Rightarrow \tilde{T}_1 \, e^{-i\tilde{\alpha}_1 X} \sin\left(\frac{\pi}{2(b-a)}(-y+b)\right), \tag{21}$$

where $\tilde{\alpha}_m = \sqrt{(K^2 - m^2\pi^2/4(b-a)^2)}(m = 2n-1, n = 1, 2, \dots)$.

For $X \to -\infty$, $-b \leqslant y \leqslant -a$

$$\tilde{\psi}(X, y) \Rightarrow \tilde{T}_1 \, e^{-i\tilde{\alpha}_1 X} \sin\left(\frac{\pi}{2(b-a)}(y+b)\right). \tag{22}$$

The edge conditions [11, 12] require that

$$\tilde{\psi}(X, \pm a) = O(1) \quad \text{and} \quad \tilde{\psi}_y(X, \pm a) = O(X^{1/2}) \quad \text{as} \quad X \to 0. \tag{23}$$

The edge condition (23) is a Kutta–Joukowski edge condition applied to the trailing edge, to which is attached a wake. This condition requires that the velocity be finite at the edge. When a trailing wake is present, it is necessary, for a unique solution to the boundary value problem, that one imposes a Kutta–Joukowski condition. However, it shall initially be assumed here that the edge condition at $y = \pm a$ is the usual one associated with diffraction theory: that is

$$\tilde{\psi}(X, \pm a) = O(1) \quad \text{and} \quad \tilde{\psi}_y(X, \pm a) = O(X^{-1/2}) \quad \text{as} \quad X \to 0, \tag{24}$$

and only in the end result will the condition (23) be imposed, in order to determine F, which occurs in the wake boundary conditions (15) and (16). If $F = 0$, the wake conditions would vanish and the Kutta–Joukowski condition would be replaced by the usual edge condition (24) used in diffraction theory.

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

For analytic convenience it is assumed that

$$K = \mathrm{Re}\, K + \mathrm{i}\,\mathrm{Im}\, K(\mathrm{Re}\, K > \mathrm{Im}\, K \geqslant 0).$$

A suitable representation for the total field $\tilde{\psi}(X, y)$ in all space $-\infty < X < \infty$, $|y| \leqslant b$, which satisfies equations (12)–(14) is given by an application of the Fourier transform approach as

$$\tilde{\psi}(X, y) = \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{e^{i\alpha X} \sin \kappa(y+b)\phi_1^-(\alpha)}{\kappa \cos \kappa(b-a)} \, d\alpha,$$

$$-b \leqslant y \leqslant -a, \quad -\infty < X < \infty, \tag{25}$$

$$\tilde{\psi}(X, y) = e^{iKX} + \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{e^{i\alpha X}}{\kappa \sin 2\kappa a}[\cos \kappa(y-a)\phi_1^-(\alpha) - \phi_2^-(\alpha) \cos \kappa(y+a)] \, d\alpha,$$

$$-a \leqslant y \leqslant a, \quad -\infty < X < \infty, \tag{26}$$

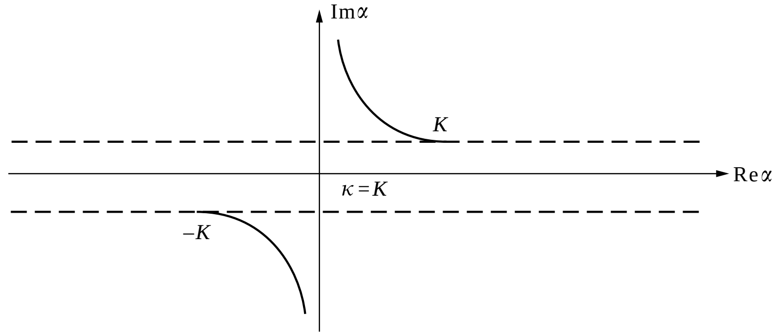

Figure 3. Branch cuts in the complex α-plane.

$$\tilde{\psi}(X, y) = \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{e^{i\alpha X}\sin\kappa(y-b)\phi_2^-(\alpha)}{\kappa\cos\kappa(b-a)}\,d\alpha,$$

$$a \leqslant y \leqslant b, \quad -\infty < X < \infty. \quad (27)$$

In expressions (25)–(27), $\kappa = \sqrt{(K^2 - \alpha^2)}$, and branch cuts are taken to be from K to $i\infty$ and $-K$ to $-i\infty$. The cut sheet is defined by $0 \leqslant \arg\kappa \leqslant \pi$; see Figure 3. The real parameter τ is restricted by requiring the asymptotic behaviour (17), (19) and (21) to be achieved. This necessitates that the contour of integration in expressions (25)–(27) lies in the strip

$$\text{Max}\{-\text{Im}\,K,\ -\text{Im}\,\tilde{\alpha}_1\} < \tau < \text{Min}\{\text{Im}\,K,\ \text{Im}\,\sigma_1\}.$$

It can be shown that this is achieved if $-\text{Im}\,K < \tau < \text{Im}\,K$. Then no singularities of the integrands of expressions (25)–(27) will lie within the strip $-\text{Im}\,K < \tau < \text{Im}\,K$. The unknown functions $\phi_{1,2}^-(\alpha)$ are functions which are analytic and regular in the region $\text{Im}\,\alpha < \text{Im}\,K$. By substituting expressions (25)–(27) into equations (15) and (16), one obtains

$$\frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} e^{i\alpha X}\left[-\frac{\cos\kappa(b+a)\phi_1^-(\alpha)}{\kappa\sin 2\kappa a\cos\kappa(b-a)} + \frac{\phi_2^-(\alpha)}{\kappa\sin 2\kappa a} - \frac{1}{(\alpha - K)} - \frac{F}{(\alpha - K/M)}\right]d\alpha = 0, \quad X > 0, \quad (28)$$

$$\frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} e^{i\alpha X}\left[-\frac{\cos\kappa(b+a)\phi_2^-(\alpha)}{\kappa\sin 2\kappa a\cos\kappa(b-a)} + \frac{\phi_1^-(\alpha)}{\kappa\sin 2\kappa a} + \frac{1}{(\alpha - K)} + \frac{F}{(\alpha - K/M)}\right]d\alpha = 0, \quad X > 0. \quad (29)$$

A solution of equations (28) and (29) is given by

$$-\frac{\cos\kappa(b+a)\phi_1^-(\alpha)}{\kappa\sin 2\kappa a\cos\kappa(b-a)}+\frac{\phi_2^-(\alpha)}{\kappa\sin 2\kappa a}-\frac{1}{(\alpha-K)}-\frac{F}{(\alpha-K/M)}=\phi_1^+(\alpha), \quad (30)$$

$$-\frac{\cos\kappa(b+a)\phi_2^-(\alpha)}{\kappa\sin 2\kappa a\cos\kappa(b-a)}+\frac{\phi_1^-(\alpha)}{\kappa\sin 2\kappa a}+\frac{1}{(\alpha-K)}+\frac{F}{(\alpha-K/M)}=\phi_2^+(\alpha), \quad (31)$$

where $\phi_{1,2}^+(\alpha)$ are functions that are analytical and regular in the region $\operatorname{Im}\alpha > -\operatorname{Im} K$.

By subtracting and adding equations (30) and (31), one obtains an uncoupled system of Wiener–Hopf equations,

$$-G(\alpha)\phi^-(\alpha)+\frac{2}{(\alpha-K)}+\frac{2F}{(\alpha-K/M)}=\phi^+(\alpha), \quad (32)$$

$$W(\alpha)\psi^-(\alpha)=\psi^+(\alpha), \quad (33)$$

where

$$\phi_2^\pm(\alpha)-\phi_1^\pm(\alpha)=\phi^\pm(\alpha), \qquad \phi_2^\pm(\alpha)+\phi_1^\pm(\alpha)=\psi^\pm(\alpha), \quad (34, 35)$$

$$G(\alpha)=\frac{\cos\kappa b}{\kappa\sin\kappa a\cos\kappa(b-a)}, \quad (36)$$

$$W(\alpha)=\frac{\sin\kappa b}{\kappa\cos\kappa a\cos\kappa(b-a)}. \quad (37)$$

The usual Wiener–Hopf technique can be applied in a straightforward manner to equations (32) and (33).

In Appendix A, it is shown that $G(\alpha)=G_-(\alpha)G_+(\alpha)$, and also that $W(\alpha)=W_-(\alpha)W_+(\alpha)$. The subscripts $\pm$ denote functions analytic and regular in $\operatorname{Im}\alpha > -K$ and $\operatorname{Im}\alpha < K$, respectively.

The Wiener–Hopf equations (32) and (33) may now be solved by using these product splits for $G(\alpha)$ and $W(\alpha)$. One rewrites equations (32) and (33) as

$$\begin{aligned}
&-G_-(\alpha)\phi^-(\alpha)+\frac{2}{(\alpha-K)G_+(K)}+\frac{2F}{(\alpha-K/M)G_+(K/M)}\\
&=\frac{\phi^+(\alpha)}{G_+(\alpha)}-\frac{2}{(\alpha-K)}\left(\frac{1}{G_+(\alpha)}-\frac{1}{G_+(K)}\right)\\
&-\frac{2F}{(\alpha-K/M)}\left(\frac{1}{G_+(\alpha)}-\frac{1}{G_+(K/M)}\right),
\end{aligned} \quad (38)$$

$$W_-(\alpha)\psi^-(\alpha)=\psi^+(\alpha)/W_+(\alpha). \quad (39)$$

The left sides of these equations are analytic in $\operatorname{Im}\alpha < \operatorname{Im} K$, and the right sides are analytic in $\operatorname{Im}\alpha > -\operatorname{Im} K$. Consequently the analytic continuation of both sides to the whole α-plane is an entire function. The determination of these entire functions depends on the asymptotic behaviour of various functions appearing in the equations. In Appendix A, it is shown that, as $|\alpha|\to\infty$,

$$G_{\pm}(\alpha) = O(|\alpha|^{-1/2}), \qquad W_{\pm}(\alpha) = O(|\alpha|^{-1/2}). \tag{40, 41}$$

The asymptotic behaviours of $\phi^{\pm}(\alpha)$ and $\psi^{\pm}(\alpha)$ may be found from the edge condition (24); it is not difficult to show that, as $|\alpha|\to\infty$,

$$\phi^{+}(\alpha) = O(|\alpha|^{-1}), \quad \psi^{+}(\alpha) = O(|\alpha|^{-1}), \quad \text{in} \quad \tau > -\operatorname{Im} K, \tag{42}$$

$$\phi^{-}(\alpha) = O(|\alpha|^{-1/2}), \quad \psi^{-}(\alpha) = O(|\alpha|^{-1/2}), \quad \text{in} \quad \tau < \operatorname{Im} K. \tag{43}$$

Similarly one can show that by using equations (40–43) in equations (38) and (39) and applying Liouville's theorem that the entire functions are constants which are identically equal to zero. Thus, one obtains

$$\phi_1^{-}(\alpha) = -\frac{1}{(\alpha - K)G_{+}(K)G_{-}(\alpha)} - \frac{F}{(\alpha - K/M)G_{+}(K/M)G_{-}(\alpha)}, \tag{44}$$

$$\phi_2^{-}(\alpha) = \frac{1}{(\alpha - K)G_{+}(K)G_{-}(\alpha)} + \frac{F}{(\alpha - K/M)G_{+}(K/M)G_{-}(\alpha)}. \tag{45}$$

One can now substitute equations (44) and (45) into the integral representations (25)–(27) and obtain the field representations for the different regions as follows: region $P1$ $(-b \leqslant y \leqslant -a,\ X < 0)$,

$$\tilde{\psi}(X, y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{-\mathrm{e}^{\mathrm{i}\alpha X}}{\cos \kappa(b - a)} \left\{ \frac{\sin \kappa(y + b)}{\kappa(\alpha - K)G_{+}(K)G_{-}(\alpha)} + \frac{F \sin \kappa(y + b)}{\kappa(\alpha - K/M)G_{+}(K/M)G_{-}(\alpha)} \right\} \mathrm{d}\alpha; \tag{46}$$

region $P2$ $(-a \leqslant y \leqslant -a,\ X < 0)$,

$$\tilde{\psi}(X, y) = \mathrm{e}^{\mathrm{i}KX} + \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{-\mathrm{e}^{\mathrm{i}\alpha X}}{\kappa \sin \kappa a} \left\{ \frac{\cos \kappa y}{(\alpha - K)G_{+}(k)G_{-}(\alpha)} + \frac{F \cos \kappa y}{(\alpha - K/M)G_{+}(K/M)G_{-}(\alpha)} \right\} \mathrm{d}\alpha; \tag{47}$$

region $P3$ $(a \leqslant y \leqslant b,\ X < 0)$,

$$\tilde{\psi}(X, y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\alpha X}}{\cos \kappa(b - a)} \left\{ \frac{\sin \kappa(y - b)}{\kappa(\alpha - K)G_{-}(K)G_{-}(\alpha)} + \frac{F \sin \kappa(y - b)}{\kappa(\alpha - K/M)G_{+}(K/M)G_{-}(\alpha)} \right\} \mathrm{d}\alpha; \tag{48}$$

region $Q3$ ($a \leqslant y \leqslant b$, $X > 0$),

$$\tilde{\psi}(X, y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\alpha X}}{\cos \kappa b} \left\{ \frac{\sin \kappa(y-b) \sin \kappa\alpha G_+(\alpha)}{(\alpha - K)G_+(K)} + \frac{F \sin \kappa(y-b) \sin \kappa\alpha G_+(\alpha)}{(\alpha - K/M)G_+(K/M)} \right\} \mathrm{d}\alpha; \tag{49}$$

region $Q2$ ($-a \leqslant y \leqslant a$, $X > 0$),

$$\tilde{\psi}(X, y) = \mathrm{e}^{\mathrm{i}KX} + \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{-\mathrm{e}^{\mathrm{i}\alpha X}}{\cos \kappa b} \left\{ \frac{\cos \kappa y \cos \kappa(b-a) G_+(\alpha)}{(\alpha - K)G_+(K)} + \frac{F \cos \kappa y \cos \kappa(b-a) G_+(\alpha)}{(\alpha - K/M)G_+(K/M)} \right\} \mathrm{d}\alpha; \tag{50}$$

region $Q1$ ($-b \leqslant y \leqslant -a$, $X > 0$),

$$\tilde{\psi}(X, y) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}\tau}^{\infty + \mathrm{i}\tau} \frac{-\mathrm{e}^{\mathrm{i}\alpha X}}{\cos \kappa b} \left\{ \frac{\sin \kappa(y+b) \sin \kappa a G_+(\alpha)}{(\alpha - K)G_+(K)} + \frac{F \sin \kappa(y+b) \sin \kappa a G_+(\alpha)}{(\alpha - K/M)G_+(K/M)} \right\} \mathrm{d}\alpha. \tag{51}$$

By using the relation between the behaviour of a function at the origin and its Fourier transform at infinity one can show from equations (46) and (48) that

$$\tilde{\psi}(X, \pm a) \sim \frac{\mathrm{i}^{1/2}}{\pi} X^{1/2} \left(\frac{1}{G_+(K)} + \frac{F}{G_+(K/M)} \right) + O(X^{2/3}) \quad \text{as} \quad X \to 0. \tag{52}$$

One must now impose the Kutta–Joukowski edge condition (23) to account for the wake when $0 < M < 1$. Thus, the requirement that the velocity will remain bounded at the edge requires that the coefficient of $X^{1/2}$ vanishes. The Kutta–Joukowski condition therefore requires that

$$F = -G_+(K/M)/G_+(k). \tag{53}$$

In expressions (46)–(51) the poles $\alpha = K, K/M$ lie above the contour of integration: that is, $\tau > -\mathrm{Im}\, K$. One can also note that the terms in the curly brackets $\{ \}$ of equations (46) and (48) have no branch points in $\tau < \mathrm{Im}\, K$, and those in equations (49)–(51) have no branch points in $\tau > -\mathrm{Im}\, K$. Thus, the only singularities in $\tau < \mathrm{Im}\, K$ of the integrands of equations (46) and (48) occur at the zeros of $\cos \kappa(b-a) = 0$, that is at

$$\alpha = -\tilde{\alpha}_m = -\sqrt{(K^2 - m^2\pi^2/4(b-a)^2)} \quad (m = 2n-1, n = 1, 2, \ldots).$$

The only singularities in $\tau < \text{Im}\, K$ of the integrand in equation (47) occur at the zeros of $\kappa \sin \kappa a = 0$, that is at $\alpha = -\chi_{2n} = -\sqrt{(K^2 - n^2\pi^2/a^2)}$, $(n = 0, 1, 2, \ldots)$. The only singularities in $\tau > -\text{Im}\, K$ of the integrands in equations (49)–(51) occur at the zeros of $\cos \kappa b = 0$, that is at $\alpha = \sigma_m = \sqrt{(K^2 - m^2\pi^2/4b^2)}$, $(n = 1, 2, \ldots)$; and the poles at $\alpha = K,\ K/M$.

4. MODAL FIELD REPRESENTATION

An application of Cauchy's residue theorem to the complex integrals in equations (46)–(51) then gives the field in various regions as a sum of waveguide modes.

Thus, in region $P1$ $(-b \leqslant y \leqslant -a,\ x < 0)$,

$$\phi(x, y) = -\sum_{n=1}^{\infty} (-)^n \frac{e^{-i\tilde{\alpha}_m x/(1-M^2)^{1/2}}\, e^{-ikMx/(1-M^2)}}{\tilde{\alpha}_m (b-a)(\tilde{\alpha}_m + k/(1-M^2)^{1/2})} \times \frac{\sin \delta_m (y+b)}{G_+(k/(1-M^2)^{1/2}) G_-(-\tilde{\alpha}_m)}$$
$$- F \sum_{n=1}^{\infty} (-)^n \frac{e^{-i\tilde{\alpha}_m x/(1-M^2)^{1/2}}\, e^{-ikMx/(1-M^2)}}{\tilde{\alpha}_m (b-a)(\tilde{\alpha}_m + k/M(1-M^2)^{1/2})} \times \frac{\sin \delta_m (y+b)}{G_+(k/M(1-M^2)^{1/2}) G_-(-\tilde{\alpha}_m)}, \tag{54}$$

where

$$\delta_m = \sqrt{\left(\frac{k^2}{(1-M^2)} - \tilde{\alpha}_m^2\right)} = \frac{m\pi}{2(b-a)}, \qquad F = -\frac{G_+(k/M(1-M^2)^{1/2})}{G_+(k/(1-M^2)^{1/2})}. \tag{55, 56}$$

In region $P2$ $(-a \leqslant y \leqslant a,\ x < 0)$,

$$\phi(x, y) = e^{ikx/(1+M)} - \frac{(1-M^2)\, e^{-ikx/(1-M)}}{4ak^2 G_+(k/(1-M^2)^{1/2}) G_-(-k/(1-M^2)^{1/2})}$$
$$- F \frac{M(1-M^2)\, e^{-ikx/(1-M)}}{2ak^2(1+M) G_+(k/M(1-M^2)^{1/2}) G_-(-k/(1-M^2)^{1/2})}$$
$$- \sum_{n=1}^{\infty} (-)^n \frac{e^{-i\chi_{2n} x/(1-M^2)^{1/2}}\, e^{-ikMx/(1-M^2)}}{a\chi_{2n}(\chi_{2n} + k/(1-M^2)^{1/2})} \times \frac{\cos \alpha_{2n} y}{G_+(k/(1-M^2)^{1/2}) G_-(-\chi_{2n})}$$
$$- F \sum_{n=1}^{\infty} (-)^n \frac{e^{-i\chi_{2n} x/(1-M^2)^{1/2}}\, e^{-ikMx/(1-M^2)}}{a\chi_{2n}(\chi_{2n} + k/M(1-M^2)^{1/2})} \times \frac{\cos \alpha_{2n} y}{G_+(k/M(1-M^2)^{1/2}) G_-(-\chi_{2n})}, \tag{57}$$

where

$$\alpha_{2n} = \sqrt{\left(\frac{k^2}{(1-M^2)} - \chi_{2n}^2\right)} = \frac{n\pi}{a}, \tag{58}$$

$(n = 0, 1, 2, \ldots)$.

In region $P3$ ($a \leqslant y \leqslant b$, $x < 0$),

$$\begin{aligned}\phi(x, y) = &-\sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_m x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{\tilde{\alpha}_m(b-a)(\tilde{\alpha}_m + k/(1-M^2)^{1/2})} \\ &\times \frac{\sin\delta_m(y-b)}{G_+(k/(1-M^2)^{1/2})G_-(-\tilde{\alpha}_m)} \\ &- F\sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_m x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{\tilde{\alpha}_m(b-a)(\tilde{\alpha}_m + k/M(1-M^2)^{1/2})} \\ &\times \frac{\sin\delta_m(y-b)}{G_+(k/M(1-M^2)^{1/2})G_-(-\tilde{\alpha}_m)}.\end{aligned} \tag{59}$$

In region $Q3$ ($a \leqslant y \leqslant b$, $x > 0$),

$$\begin{aligned}\phi(x, y) = &\,F\frac{\mathrm{e}^{\mathrm{i}kx/M}\sin\bar{\kappa}(y-b)\sin\bar{\kappa}a}{\cos\bar{\kappa}b} \\ &+ \sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{\mathrm{i}\sigma_m x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{b\sigma_m(\sigma_m - k/(1-m^2)^{1/2})} \\ &\times \frac{\beta_m \sin\beta_m(y-b)\sin\beta_m a G_+(\sigma_m)}{G_+(k/(1-M^2)^{1/2})} \\ &+ F\sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{\mathrm{i}\sigma_m x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kx/(1-M^2)}}{b\sigma_m(\sigma_m - k/M(1-M^2)^{1/2})} \\ &\times \frac{\beta_m \sin\beta_m(y-b)\sin\beta_m a G_+(\sigma_m)}{G_+(k/M(1-M^2)^{1/2})},\end{aligned} \tag{60}$$

where

$$\beta_m = \sqrt{\left(\frac{k^2}{(1-M^2)} - \sigma_m^2\right)} = \frac{m\pi}{2}, \tag{61}$$

$$\bar{\kappa} = \sqrt{\left((M^2-1)\frac{k^2}{M^2(1-M^2)}\right)} = \frac{\mathrm{i}k}{M}. \tag{62}$$

In region $Q2$ $(-a \leqslant y \leqslant a, x > 0)$,

$$\phi(x, y) = -F\frac{e^{ikx/M}\cos\bar{\kappa}(b-a)\cos\bar{\kappa}y}{\cos\bar{\kappa}b}$$

$$+\sum_{n=1}^{\infty}(-)^n\frac{e^{i\sigma_m x/(1-M^2)^{1/2}}\,e^{-ikMx/(1-M^2)}}{b\sigma_m(\sigma_m - k/(1-M^2)^{1/2})}$$

$$\times\frac{\beta_m\cos\beta_m(b-a)\cos\beta_m y G_+(\sigma_m)}{G_+(k/(1-M^2)^{1/2})}$$

$$+F\sum_{n=1}^{\infty}(-)^n\frac{e^{i\sigma_m x/(1-M^2)^{1/2}}\,e^{-ikx/(1-M^2)}}{b\sigma_m(\sigma_m - k/M(1-M^2)^{1/2})}$$

$$\times\frac{\beta_m\cos\beta_m(b-a)\cos\beta_m y G_+(\sigma_m)}{G_+(k/M(1-M^2)^{1/2})}. \qquad (63)$$

In region $Q1$ $(-b \leqslant y \leqslant -a, x > 0)$,

$$\phi(x, y) = -F\frac{e^{ikx/M}\sin\bar{\kappa}(y+b)\sin\bar{\kappa}a}{\cos\bar{\kappa}b}$$

$$+\sum_{n=1}^{\infty}(-)^n\frac{e^{i\sigma_m x/(1-M^2)^{1/2}}\,e^{-ikMx/(1-M^2)}}{b\sigma_m(\sigma_m - k/(1-M^2)^{1/2})}$$

$$\times\frac{\beta_m\sin\beta_m(y+b)\sin\beta_m a G_+(\sigma_m)}{G_+(k/(1-M^2)^{1/2})}$$

$$+F\sum_{n=1}^{\infty}(-)^n\frac{e^{i\sigma_m x/(1-M^2)^{1/2}}\,e^{-ikMx/(1-M^2)}}{b\sigma_m(\sigma_m - k/M(1-M^2)^{1/2})}$$

$$\times\frac{\beta_m\sin\beta_m(y+b)\sin\beta_m a G_+(\sigma_m)}{G_+(k/M(1-M^2)^{1/2})}. \qquad (64)$$

5. NUMERICAL AND GRAPHICAL RESULTS

It is of interest to know the amount of acoustic power in the incident mode $e^{ikx/(1+M)}$ which is distributed among the various waveguide regions. Of particular importance is the fundamental mode $n = 0$, and the reflexion coefficient in this case is given by the coefficient of the $e^{-ikx/(1-M)}$ in expression (57): that is,

$$R = -\frac{(1-M^2)(\frac{1}{2} - M/(1+M))}{2ak^2G_+^2(k/(1-M^2)^{1/2})}. \qquad (65)$$

$(1-|R|^2)$ represents the power that is radiated from the semi-infinite guide, and which is distributed among the other regions. Thus, $|R|$ determines the amount of energy of the incident wave which is distributed amongst the various other modes in the rest of the waveguide. One can obtain, after some algebra, simple closed form expressions for the modulus of the reflection coefficient $|R|$ which can be

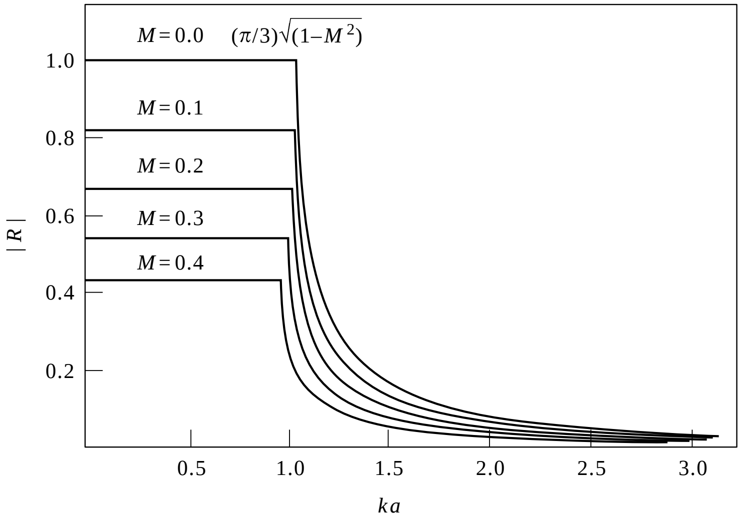

Figure 4. The absolute value of the modulus $|R|$ as a function of the wave number ka for $b = 3a/2$.

calculated numerically. The $\pm$sign in the term $(1 \pm M)$ that appears in the exponential terms in equation (57), signify the Doppler effect. That is the exponent $k/(1+M)$ shows there is an increase in the wavelength of the incident wave $e^{ikx/(1+M)}$ propagating along with the flow, whereas $k/(1-M)$ represents a decrease in wavelength of reflected wave. That is, flow compresses the wavelength of the reflection wave $e^{ikx/(1-M)}$ propagating against the flow. Clearly as more modes are excited in the various regions, the value of $|R|$ will suffer an abrupt change. Two situations are considered.

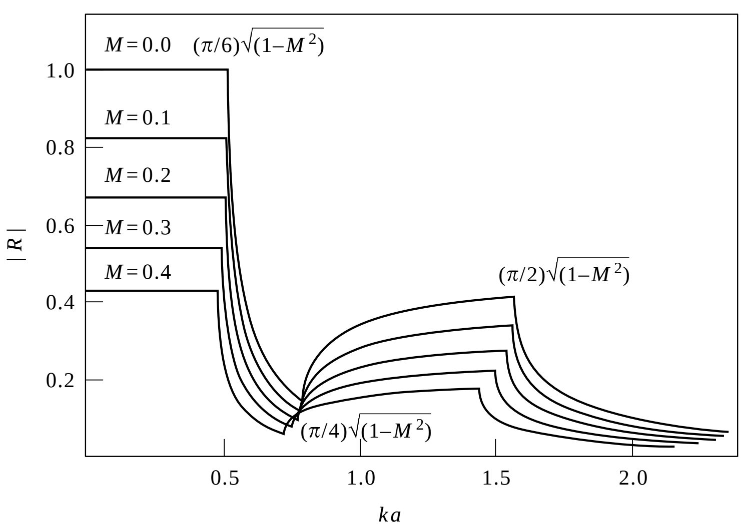

Figure 5. The absolute value of the modulus $|R|$ as a function of the wave number ka for $b = 3a$.

(1) Here the guide dimensions are such that one mode (the lowest) propagates in $|y|<a$, $x<0$, one mode propagates in $|y|<b$, $x<0$, no mode propagates in $-b<y<-a$, $x<0$, and no mode propagates in $a<y<b$, $x<0$. In this case it can be shown that

$$|R| = 2\pi^2(\tfrac{1}{2} - M/(1+M)) \Bigg/ \left| \frac{3ka}{(1-M^2)^{1/2}} + \sqrt{\left(\frac{9k^2a^2}{(1-M^2)} - \pi^2\right)} \right|^2. \tag{66}$$

The graph of $|R|$ is shown in Figure 4 for this situation, by choosing $b = 3a/2$, and the frequency range $0 < ka < \pi\sqrt{(1-M^2)}$.

(2) This is when the guide dimensions are such that only one mode propagates in $-b<y<-a$, $x<0$, one mode in $a<y<b$, $x<0$, and consequently two modes in $|y|<b$, $x>0$. In this situation it can be shown that

$$|R| = \frac{\pi^2(1-M^2)|2ka + \sqrt{(4k^2a^2 - \pi^2(1-M^2)/4)}|^2}{2|3ka + \sqrt{(9k^2a^2 - \pi^2(1-M^2)/4)}|^2} \times \frac{(\tfrac{1}{2} - M/(1+M))}{|ka + \sqrt{(k^2a^2 - \pi^2(1-M^2)/4)}|^2}. \tag{67}$$

The graph of $|R|$ is shown in Figure 5 for this situation by choosing $b = 3a$ and the frequency range $0 < ka < (3\pi/4)\sqrt{(1-M^2)}$.

From the graphs it can be seen that for $M = 0$, that is when the fluid is at rest, there is less energy radiating out of the duct than when $M = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$. It is also noted that $F = 0$ if $M = 0$. For $b = 3a/2$ (see Figure 4) only one mode can propagate in the forward direction $x>0$, $|y|<b$ and no modes in the backward direction $x<0$, $a<|y|<b$. The onset of this forward mode is at $ka = (\pi/3)\sqrt{(1-M^2)}$.

For $b = 3a$ (see Figure 5), three possible modes can propagate in the forward region $x>0$, $|y|<b$ and the backward direction $x<0$, $a<|y|<b$. Specifically for $0 < ka < (\pi/6)\sqrt{(1-M^2)}$ the only mode that propagates is the plane wave mode in $x<0$, $|y|<a$.

For $(\pi/6)\sqrt{(1-M^2)} < ka < (\pi/4)\sqrt{(1-M^2)}$ a mode propagates in the forward direction $x>0$, $|y|<b$. For $(\pi/4)\sqrt{(1-M^2)} < ka < (\pi/2)\sqrt{(1-M^2)}$ a further mode propagates in the backward direction $x<0$, $a<|y|<b$, so that only one mode is propagating in this direction. For $(\pi/2)\sqrt{(1-M^2)} < ka < (3\pi/4)\sqrt{(1-M^2)}$ a further mode propagates in the forward direction $x>0$, $|y|<b$, so that two modes are now propagating simultaneously in this direction.

It will be noticed from Figure 5 that power is transferred from the semi-infinite guide for only $ka > (\pi/3)\sqrt{(1-M^2)}$; similarly from Figure 5 no power is transmitted for $0 < ka < (\pi/6)\sqrt{(1-M^2)}$.

6. FORMULATION OF THE SECOND TRIFURCATED BOUNDARY VALUE PROBLEM

Consider how the acoustic diffraction of a wave mode propagating in a convective fluid flow out of the open end of the same semi-infinite duct which is situated symmetrically between the two infinite plates which are now soft. The geometry of the problem is shown in Figure 2.

To this end one requires a representation for the solution $\phi(x, y)$ of the two-dimensional convective Helmholtz equation

$$(1 - M^2)\phi_{xx} + 2\mathrm{i}kM\phi_x + \phi_{yy} + k^2\phi = 0, \tag{68}$$

in the convected trifurcated duct system which satisfies the following boundary conditions (see Figure 2)

$$\phi_y = 0, \quad y = b, \quad -\infty < x < \infty, \qquad \phi_y = 0, \quad y = a, \quad -\infty < x < 0, \tag{69, 70}$$

$$\phi_y = 0, \quad y = -a, \quad -\infty < x < 0, \qquad \phi_y = 0, \quad y = -b, \quad -\infty < x < \infty, \tag{71, 72}$$

where it is assumed that $b > a$. Also

$$\phi(x, -a^-) - \phi(x, -a^+) = F\,\mathrm{e}^{\mathrm{i}kx/M}, \quad x > 0,$$
$$\phi(x, a^+) - \phi(x, a^-) = F\,\mathrm{e}^{\mathrm{i}kx/M}, \quad x > 0. \tag{73, 74}$$

The wake conditions (73) and (74) are obtained by the continuity of the pressure

$$p = -\rho_0\left(-i\omega + U\frac{\partial}{\partial x}\right)\phi(x, y), \tag{75}$$

where ρ_0 is the density of the undisturbed stream.

For subsonic flow one can make the real substitutions

$$x = (1 - M^2)^{1/2}X. \qquad k = (1 - M^2)^{1/2}K, \tag{76}$$

which, together with the substitution

$$\phi(x, y) = \tilde{\psi}(X, y)\,\mathrm{e}^{-\mathrm{i}KMX}, \tag{77}$$

reduce the boundary value problem of equations (68)–(74) to

$$\tilde{\psi}_{XX} + \tilde{\psi}_{yy} + K^2\tilde{\psi} = 0, \qquad \tilde{\psi}_y = 0, \quad \text{on} \quad y = \pm a, \quad X < 0, \tag{78, 79}$$

$$\tilde{\psi}_y = 0, \quad \text{on} \quad y = \pm b, \quad -\infty < X < \infty, \tag{80}$$

$$\tilde{\psi}(X, a^+) - \tilde{\psi}(X, a^-) = F\,\mathrm{e}^{\mathrm{i}KX/M}, \quad X > 0, \tag{81}$$

$$\tilde{\psi}(X, -a^+) - \tilde{\psi}(X, -a^-) = F\,\mathrm{e}^{\mathrm{i}KX/M}, \quad X > 0. \tag{82}$$

To these boundary conditions one adds those conditions at infinity relevant to the nature of the lowest propagating modes which the various duct regions can sustain.

One therefore has, for $X \to +\infty$, $-a \leqslant y \leqslant a$,

$$\tilde{\psi}(X, y) \Rightarrow e^{\mathrm{i}\chi_0 X} + R\,\mathrm{e}^{-\mathrm{i}\chi_0 X}, \tag{83}$$

where $\chi_n = \sqrt{(K^2 - n^2\pi^2/4a^2)}$ $(n = 0, 1, 2, \ldots)$ are the roots of the equation

$$\sqrt{(K^2 - \chi_n^2)}\,\sin(2\sqrt{(K^2 - \chi_n^2)}a) = 0. \tag{84}$$

If $0 < Ka < \pi/2$ then $\chi_0 > 0$ and $\chi_1 = \mathrm{i}\sqrt{(\pi^2/4a^2) - K^2)}$ so that $\operatorname{Im}\chi_1 > 0$, $\operatorname{Re}\chi_1 = 0$. Thus, the semi-infinite duct region $-\infty < X < 0$, $-a \leqslant y \leqslant a$ can sustain only the lowest incident and reflected modes.

For $X \to +\infty$, $-b \leqslant y \leqslant b$,

$$\tilde{\psi}(X, y) \Rightarrow T\,\mathrm{e}^{\mathrm{i}\sigma_0 X}, \tag{85}$$

where $\sigma_n = \sqrt{(K^2 - n^2\pi^2/4b^2)}$ $(n = 0, 1, \dots)$ are the roots of the equation

$$\sqrt{(K^2 - \sigma_n^2)}\sin(2\sqrt{(K^2 - \sigma_n^2)}b) = 0. \tag{86}$$

For $X \to +\infty$, $a \leqslant y \leqslant b$,

$$\tilde{\psi}(X, y) \Rightarrow \tilde{T}\,\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_0 X}, \tag{87}$$

where $\tilde{\alpha}_n = \sqrt{(K^2 - n^2\pi^2/(b-a)^2)}$ $(n = 0, 1, \dots)$.

For $X \to -\infty$, $-b \leqslant y \leqslant -a$,

$$\tilde{\psi}(X, y) \Rightarrow \tilde{T}\,\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_0 X}. \tag{88}$$

The edge conditions require that

$$\tilde{\psi}(X, \pm a) = O(1) \quad \text{and} \quad \tilde{\psi}_y(X, \pm a) = O(X^{1/2}) \tag{89}$$

as $X \to 0$, which represent the Kutta–Joukowski edge condition.

7. SOLUTION OF THE BOUNDARY VALUE PROBLEM II

A suitable representation for the total field $\tilde{\psi}(X, y)$ in all space $-\infty < X < \infty$, $|y| \leqslant b$, which satisfies equations (78)–(80) is given by an application of the Fourier transform approach as

$$\tilde{\psi}(X, y) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\alpha X}\cos\kappa(y+b)\phi_1^-(\alpha)}{\kappa\sin\kappa(b-a)}\,\mathrm{d}\alpha,$$

$$-b \leqslant y \leqslant -a, \quad -\infty < X < \infty, \tag{90}$$

$$\tilde{\psi}(X, y) = \mathrm{e}^{\mathrm{i}KX} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty+\mathrm{i}\tau} + \mathrm{i}\tau\,\frac{\mathrm{e}^{\mathrm{i}\alpha X}}{\kappa\sin 2\kappa a}[\cos\kappa(y-a)\phi_1^-(\alpha) - \phi_2^-(\alpha)\cos\kappa(y+a)]\,\mathrm{d}\alpha,$$

$$-a \leqslant y \leqslant a, \quad -\infty < X < \infty, \tag{91}$$

$$\tilde{\psi}(X, y) = \frac{1}{2\pi i}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau} \frac{\mathrm{e}^{\mathrm{i}\alpha X}\cos\kappa(y-b)\phi_2^-(\alpha)}{\kappa\sin\kappa(b-a)}\,\mathrm{d}\alpha,$$

$$a \leqslant y \leqslant b, \quad -\infty < X < \infty. \tag{92}$$

In the expressions (90)–(92), $\kappa = \sqrt{(K^2 - \alpha^2)}$, and $K = \operatorname{Re}K + \mathrm{i}\operatorname{Im}K(\operatorname{Re}K > \operatorname{Im}K \geqslant 0)$, $\operatorname{Im}K < \tau < \operatorname{Im}K$. By following the same procedure as that in the previous problem, one eventually ends up with the following Wiener–Hopf equations:

$$-G(\alpha)\phi^{-}(\alpha)+\frac{2}{(\alpha-K)}+\frac{2F}{(\alpha-K/M)}=\phi^{+}(\alpha), \tag{93}$$

$$-W(\alpha)\psi^{-}(\alpha)=\psi^{+}(\alpha), \tag{94}$$

where

$$\phi_2^{\pm}(\alpha)-\phi_1^{\pm}(\alpha)=\phi^{\pm}(\alpha), \qquad \phi_2^{\pm}(\alpha)+\phi_1^{\pm}(\alpha)=\psi^{\pm}(\alpha), \qquad (95, 96)$$

$$G(\alpha)=\frac{\sin\kappa b}{\kappa\sin\kappa a\sin\kappa(b-a)}, \qquad W(\alpha)=\frac{\cos\kappa b}{\kappa\sin\kappa(b-a)\cos\kappa a}. \qquad (97, 98)$$

In Appendix B, it is shown that $G(\alpha)=G_{-}(\alpha)G_{+}(\alpha)$, and $W(\alpha)=W_{-}(\alpha)W_{+}(\alpha)$. The Wiener–Hopf equation (93) and (94) may now be solved following the same procedure as used for the previous problem. Thus, one obtains

$$\phi_1^{-}(\alpha)=-\frac{1}{(\alpha-K)G_{+}(K)G_{-}(\alpha)}-\frac{F}{(\alpha-K/M)G_{+}(K/M)G_{-}(\alpha)}, \tag{99}$$

$$\phi_2^{-}(\alpha)=\frac{1}{(\alpha-K)G_{+}(K)G_{-}(\alpha)}+\frac{F}{(\alpha-K/M)G_{+}(K/M)G_{-}(\alpha)}. \tag{100}$$

One now substitutes equations (99) and (100) into the integral representations (90)–(92) and obtains the field representations for the different regions as follows: region $P1$ $(-b\leqslant y\leqslant -a,\ X<0)$,

$$\tilde{\psi}(X,y)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}\alpha X}}{\kappa\sin\kappa(b-a)}\left\{\frac{\cos\kappa(y+b)}{(\alpha-K)G_{+}(K)G_{-}(\alpha)}+\frac{F\cos\kappa(y+b)}{(a-K/M)G_{+}(K/M)G_{-}(\alpha)}\right\}\mathrm{d}\alpha; \tag{101}$$

region $P2$ $(-a\leqslant y\leqslant -a,\ X<0)$,

$$\tilde{\psi}(X,y)=\mathrm{e}^{\mathrm{i}KX}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{-\mathrm{e}^{\mathrm{i}\alpha X}}{\kappa\sin\kappa a}\left\{\frac{\cos\kappa y}{(\alpha-K)G_{+}(k)G_{-}(\alpha)}+\frac{F\cos\kappa y}{(\alpha-K/M)G_{+}(K/M)G_{-}(\alpha)}\right\}\mathrm{d}\alpha; \tag{102}$$

region $P3$ $(a\leqslant y\leqslant b,\ X<0)$,

$$\tilde{\psi}(X,y)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\tau}^{\infty+\mathrm{i}\tau}\frac{\mathrm{e}^{\mathrm{i}\alpha X}}{\kappa\sin\kappa(b-a)}\left\{\frac{\cos\kappa(y-b)}{(\alpha-K)G_{+}(K)G_{-}(\alpha)}+\frac{F\cos\kappa(y-b)}{(\alpha-K/M)G_{+}(K/M)G_{-}(\alpha)}\right\}\mathrm{d}\alpha; \tag{103}$$

region $Q3$ ($a \leqslant y \leqslant b$, $X > 0$),

$$\tilde{\psi}(X, y) = \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{e^{i\alpha X}}{\kappa \sin \kappa b}\left\{\frac{\cos \kappa(y-b)\kappa \sin \kappa a G_+(\alpha)}{(\alpha - K)G_+(K)}\right.$$

$$\left. + \frac{F \cos \kappa(y-b)\kappa \sin \kappa a G_+(\alpha)}{(\alpha - K/M)G_+(K/M)}\right\} d\alpha; \tag{104}$$

region $Q2$ ($-a \leqslant y \leqslant a$, $X > 0$),

$$\tilde{\psi}(X, y) = e^{iKX} + \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{-e^{i\alpha X}}{\kappa \sin \kappa b}\left\{\frac{\kappa \cos \kappa y \sin \kappa(b-a) G_+(\alpha)}{(\alpha - K)G_+(K)}\right.$$

$$\left. + \frac{F\kappa \cos \kappa y \sin \kappa(b-a) G_+(\alpha)}{(\alpha - K/M)G_+(K/M)}\right\} d\alpha; \tag{105}$$

region $Q1$ ($-b \leqslant y \leqslant -a$, $X > 0$),

$$\tilde{\psi}(X, y) = \frac{1}{2\pi i}\int_{-\infty + i\tau}^{\infty + i\tau} \frac{e^{i\alpha X}}{\kappa \sin \kappa b}\left\{\frac{\kappa \cos \kappa(y+b) \sin \kappa a G_+(\alpha)}{(\alpha - K)G_+(K)}\right.$$

$$\left. + \frac{F\kappa \cos \kappa(y+b) \sin \kappa a G_+(\alpha)}{(\alpha - K/M)G_+(K/M)}\right\} d\alpha; \tag{106}$$

Without going into details one obtains, in order to satisfy the Kutta–Joukowski edge condition (89), that

$$F = -G_+(K/M)/G_+(K). \tag{107}$$

In expressions (101)–(106) the poles $\alpha = K,\ K/M$ lie above the contour of integration; that is $\tau > -\operatorname{Im} K$. One also notes that the only singularities in $\tau < \operatorname{Im} K$ of the integrands of equations (101) and (103) occur at the zeros of $\kappa \sin \kappa(b-a) = 0$, that is at $\alpha = -\tilde{\alpha}_n = -\sqrt{(K^2 - n^2\pi^2/(b-a)^2)}$ ($n = 0, 1, \ldots$). The only singularities in $\tau < \operatorname{Im} K$ of the integrand (102) occur at the zeros of $\kappa \sin \kappa a = 0$, that is at $\alpha = -\chi_{2n} = -\sqrt{(K^2 - n^2\pi^2/a^2)}$ ($n = 0, 1, 2, \ldots$). The only singularities in $\tau > \operatorname{Im} K$ of the integrands (104)–(106) occur at the zeros of $\kappa \sin kb = 0$, that is at $\alpha = +\sigma_{2n} = \sqrt{(K^2 - n^2\pi^2/b^2)}$ ($n = 0, 1, \ldots$), and the poles at $\alpha = K,\ K/M$.

8. MODAL FIELD REPRESENTATION

An application of Cauchy's residue theorem to the complex integrals (101)–(106) then gives the field in the various regions as a sum of waveguide modes:

region $P1(-b \leqslant y \leqslant -a, x < 0)$,

$$\phi(x, y) = -\sum_{n=0}^{\infty} (-)^{n+1} \frac{e^{-i\tilde{\alpha}_n x/(1-M^2)^{1/2}} e^{-ikMx/(1-M^2)}}{\tilde{\alpha}_n (b-a)(\tilde{\alpha}_n + k/(1-M^2)^{1/2})} \times \frac{\cos \delta_n (y+b)}{G_+(k/(1-M^2)^{1/2}) G_-(-\tilde{\alpha}_n)}$$
$$- F \sum_{n=0}^{\infty} (-)^{n+1} \frac{e^{-i\tilde{\alpha}_n x/(1-M^2)^{1/2}} e^{-ikMx/(1-M^2)}}{\tilde{\alpha}_n (b-a)(\tilde{\alpha}_n + k/M(1-M^2)^{1/2})} \times \frac{\cos \delta_m (y+b)}{G_+(k/M(1-M^2)^{1/2}) G_-(-\tilde{\alpha}_n)}, \tag{108}$$

where

$$\delta_n = \sqrt{\left(\frac{k^2}{(1-M^2)} - \tilde{\alpha}_n^2\right)} = \frac{n\pi}{(b-a)}, \qquad F = -\frac{G_+(k/M(1-M^2)^{1/2})}{G_+(k/(1-M^2)^{1/2})}; \tag{109, 110}$$

region $P2(-a \leqslant y \leqslant a, x < 0)$,

$$\phi(x, y) = e^{ikx/(1+M)} - \frac{(1-M^2)\, e^{-ikx/(1-M)}}{4ak^2 G_+(k/(1-M^2)^{1/2}) G_-(-k/(1-M^2)^{1/2})}$$
$$- F \frac{M(1-M^2)\, e^{-ikx/(1-M)}}{2ak^2(1+M) G_+(k/M(1-M^2)^{1/2}) G_-(-k/(1-M^2)^{1/2})}$$
$$- \sum_{n=1}^{\infty} (-)^n \frac{e^{-i\chi_{2n} x/(1-M^2)^{1/2}} e^{-ikMx/(1-M^2)}}{a\chi_{2n}(\chi_{2n} + k/(1-M^2)^{1/2})} \times \frac{\cos \alpha_{2n} y}{G_+(k/(1-M^2)^{1/2}) G_-(-\chi_{2n})}$$
$$- F \sum_{n=1}^{\infty} (-)^n \frac{e^{-i\chi_{2n} x/(1-M^2)^{1/2}} e^{-ikMx/(1-M^2)}}{a\chi_{2n}(\chi_{2n} + k/M(1-M^2)^{1/2})} \times \frac{\cos \alpha_{2n} y}{G_+(k/M(1-M^2)^{1/2}) G_-(-\chi_{2n})}, \tag{111}$$

where

$$\alpha_{2n} = \sqrt{\left(\frac{k^2}{(1-M^2)} - \chi_{2n}^2\right)} = \frac{n\pi}{a}; \tag{112}$$

region $P3(a \leqslant y \leqslant b, x < 0)$,

$$\phi(x, y) = -\sum_{n=0}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_n x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{\tilde{\alpha}_n (b-a)(\tilde{\alpha}_n + k/(1-M^2)^{1/2})}$$

$$\times \frac{\cos \delta_n (y-b)}{G_+(k/(1-M^2)^{1/2})G_-(-\tilde{\alpha}_n)}$$

$$-F \sum_{n=0}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{-\mathrm{i}\tilde{\alpha}_n x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{\tilde{\alpha}_n (b-a)(\tilde{\alpha}_n + k/M(1-M^2)^{1/2})}$$

$$\times \frac{\cos \delta_n (y-b)}{G_+(k/M(1-M^2)^{1/2})G_-(-\tilde{\alpha}_n)}, \tag{113}$$

region $Q3(a \leqslant y \leqslant b, x > 0)$,

$$\phi(x, y) = \frac{a}{b} \mathrm{e}^{\mathrm{i}kx/(1+M)} + F \frac{\mathrm{e}^{\mathrm{i}kx/M} \cos \bar{\kappa}(y-b) \sin \bar{\kappa} a}{\sin \bar{\kappa} b}$$

$$+ \sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{\mathrm{i}\sigma_{2n} x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n} - k/(1-M^2)^{1/2})}$$

$$\times \frac{\beta_{2n} \cos \beta_{2n}(y-b) \sin \beta_{2n} a G_+(\sigma_{2n})}{G_+(k/(1-M^2)^{1/2})}$$

$$+F \sum_{n=1}^{\infty} (-)^{n+1} \frac{\mathrm{e}^{\mathrm{i}\sigma_{2n} x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n} - k/M(1-M^2)^{1/2})}$$

$$\times \frac{\beta_{2n} \cos \beta_{2n}(y-b) \sin \beta_{2n} a G_+(\sigma_{2n})}{G_+(k/M(1-M^2)^{1/2})}, \tag{114}$$

where

$$\beta_{2n} = \sqrt{\left(\frac{k^2}{(1-M^2)} - \sigma_{2n}^2\right)} = \frac{n\pi}{b}, \qquad \bar{\kappa} = \frac{\mathrm{i}k}{M}; \tag{115, 116}$$

region $Q2(-a \leqslant y \leqslant a, x > 0)$,

$$\phi(x, y) = \frac{a}{b} \mathrm{e}^{\mathrm{i}kx/(1+M)} + F \frac{\mathrm{e}^{\mathrm{i}kx/M} \sin \bar{\kappa}(b-a) \cos \bar{\kappa} y}{\sin \bar{\kappa} b}$$

$$+ \sum_{n=1}^{\infty} (-)^{n} \frac{\mathrm{e}^{\mathrm{i}\sigma_{2n} x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n} - k/(1-M^2)^{1/2})}$$

$$\times \frac{\beta_{2n} \sin \beta_{2n}(b-a) \cos \beta_{2n} y G_+(\sigma_{2n})}{G_+(k/(1-M^2)^{1/2})}$$

$$+F \sum_{n=1}^{\infty} (-)^{n} \frac{\mathrm{e}^{\mathrm{i}\sigma_{2n} x/(1-M^2)^{1/2}} \mathrm{e}^{-\mathrm{i}kx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n} - k/M(1-M^2)^{1/2})}$$

$$\times \frac{\beta_{2n} \sin \beta_{2n}(b-a) \cos \beta_{2n} y G_+(\sigma_{2n})}{G_+(k/M(1-M^2)^{1/2})}; \tag{117}$$

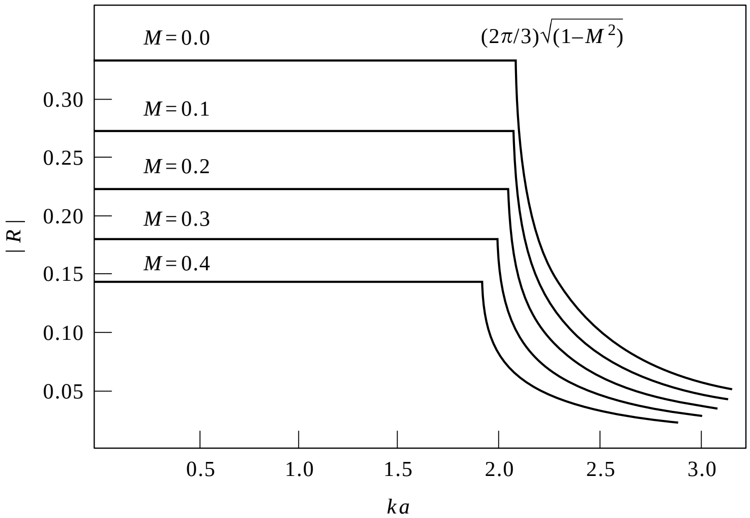

Figure 6. The absolute value of the modulus $|R|$ as a function of the wave number ka for $b = 3a/2$.

region $Q1(-b \leqslant y \leqslant -a, x > 0)$,

$$\phi(x, y) = \frac{a}{b}\,\mathrm{e}^{\mathrm{i}kx/(1+M)} + F\,\frac{\mathrm{e}^{\mathrm{i}kx/M}\cos\bar{\kappa}(y+b)\sin\bar{\kappa}b}{\sin\bar{\kappa}b}$$

$$+\sum_{n=1}^{\infty}(-)^{n+1}\frac{\mathrm{e}^{\mathrm{i}\sigma_{2n}x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kMx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n}-k/(1-M^2)^{1/2})}$$

$$\times\frac{\beta_{2n}\cos\beta_{2n}(y+b)\sin\beta_{2n}aG_+(\sigma_{2n})}{G_+(k/(1-M^2)^{1/2})}$$

$$+F\sum_{n=1}^{\infty}(-)^{n+1}\frac{\mathrm{e}^{\mathrm{i}\sigma_{2n}x/(1-M^2)^{1/2}}\,\mathrm{e}^{-\mathrm{i}kx/(1-M^2)}}{b\sigma_{2n}(\sigma_{2n}-k/M(1-M^2)^{1/2})}$$

$$\times\frac{\beta_{2n}\cos\beta_{2n}(y+b)\sin\beta_{2n}aG_+(\sigma_{2n})}{G_+(k/M(1-M^2)^{1/2})}. \tag{118}$$

9. NUMERICAL AND GRAPHICAL RESULTS

The reflexion coefficient in this case is given by

$$R = -\frac{(1-M^2)(\frac{1}{2}-M/(1+M))}{2ak^2G_+^2(k/(1-M^2)^{1/2})}. \tag{119}$$

After some algebra, the modulus of the reflection coefficient $|R|$ can be given in a simple analytic form suitable for numerical calculations. In particular two situations will be considered: (1) when $b = 3a/2$ and $0 < ka < \pi\sqrt{(1-M^2)}$,

$$|R| = \frac{8\pi^2(\frac{1}{2}-M/(1+M))}{3\left|\frac{3ka}{\sqrt{(1-M^2)}}+\sqrt{\left(\frac{9k^2a^2}{(1-M^2)}-4\pi^2\right)}\right|^2}; \tag{120}$$

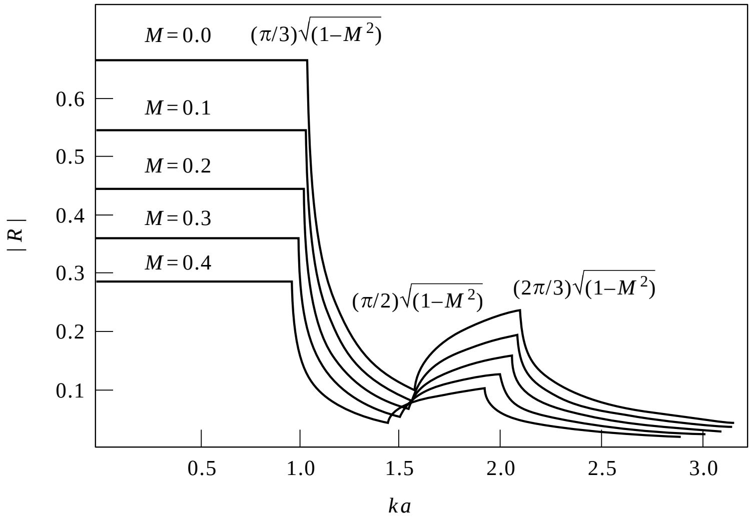

Figure 7. The absolute value of the modulus $|R|$ as a function of the wave number ka for $b = 3a$.

(2) when $b = 3a$ and $0 < ka < \pi\sqrt{(1 - M^2)}$,

$$|R| = \frac{64\pi^2\left|\dfrac{ka}{\sqrt{(1-M^2)}} + \sqrt{\left(\dfrac{k^2a^2}{(1-M^2)} - \dfrac{\pi^2}{4}\right)}\right|^2}{243\left|\dfrac{ka}{\sqrt{(1-M^2)}} + \sqrt{\left(\dfrac{k^2a^2}{(1-M^2)} - \dfrac{\pi^2}{9}\right)}\right|^2}$$

$$\times \frac{\left(\dfrac{1}{2} - \dfrac{M}{(1+M)}\right)}{\left|\dfrac{ka}{\sqrt{(1-M^2)}} + \sqrt{\left(\dfrac{k^2a^2}{(1-M^2)} - \dfrac{4\pi^2}{9}\right)}\right|^2}. \tag{121}$$

Figures 6 and 7 are for cases (1) and (2), respectively. $|R|$ is plotted for $M = 0$ (fluid is at rest/no wake), $M = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$, $0{\cdot}4$ in Figures 6 and 7. It is also noted that $F = 0$ if $M = 0$. The onset, of waves in the forward direction $x > 0$, $|y| < b$ is at $ka = (2\pi/3)\sqrt{(1 - M^2)}$ for $b = 3a/2$ (see Figure 6), and no mode propagates in the backward direction $x < 0$, $a < |y| < b$.

For $b = 3a$ (see Figure 7), there are two modes in the forward direction $(\pi/3)\sqrt{(1 - M^2)} < ka < (\pi/2)\sqrt{(1 - M^2)}$, and $(2\pi/3)\sqrt{(1 - M^2)} < ka < \pi\sqrt{(1 - M^2)}$,

$x > 0$, $|y| < b$. There is one mode in the backward direction $(\pi/2)\sqrt{(1 - M^2)} < ka < (2\pi/3)\sqrt{(1 - M^2)}$, $x < 0$, $a < |y| < b$.

10. CONCLUSIONS

Two new trifurcated waveguide problems with fluid flow have been solved by an application of the Wiener–Hopf technique. These results will be of use in acoustic waveguide problems.

It has been possible to evaluate numerically the reflection coefficients for the fundamental mode propagating into semi-infinite guide. This mode is of practical importance in applications since it carries most of the energy. R involves an infinite product derived from the representation of the split functions, which reduces to a simple algebraic expression for $|R|$.

For these expressions for $|R|$, graphs for $M = 0$, 0·1, 0·2, 0·3, 0·4 in Figures 4–7. It is found that higher energy levels radiate out of the semi-infinite guide as M increases from 0 to 0·4. Also higher radiated energy levels have been obtained for the second problem than those obtained for similar fluid flow regimes in the first problem.

REFERENCES

1. B. Nilsson and O. Brander 1980 *Journal of the Institute of Mathematics and its Applications* **26**, 269–298. The propagation of sound in cylindrical ducts with mean flow and bulk-reacting lining—I. Modes in an infinite duct.
2. B. Nilsson and O. Brander 1980 *Journal of the Institute of Mathematics and its Applications* **26**, 381–410. The propagation of sound in cylindrical ducts with mean flow and bulk-reacting lining—II. Bifurcated ducts.
3. A. D. Rawlins 1995 *Institute of Mathematics and its Applications Journal of Applied Mathematics* **54**, 59–81. A bifurcated waveguide problem.
4. A. D. Rawlins 1977 *Journal of Sound and Vibration* **50**, 553–569. The engine-over the wing noise problem.
5. D. S. Jones 1972 *Journal of the Institute of Mathematics and its Applications* **9**, 114–122. Aerodynamic sound due to a source near a half-plane.
6. R. M. Munt 1975 *Journal of the Institute of Mathematics and its Applications* **16**, 1–10. Acoustic radiation from a circular cylinder in a subsonic stream.
7. G. F. Butler 1974 *Journal of Sound and Vibration* **32**, 367–369. A note on improving the attenuation given by a noise barrier.
8. A. D. Rawlins 1975 *Journal of Sound and Vibration* **41**, 391–393. Diffraction by an absorbing wedge.
9. K. Fujiwara and T. Ohkubo 1995 *15th International Congress on Acoustics*, *Trondheim*, 26–30 *June*. Sound shield efficiency of a noise barrier with soft surface and soft round obstacles at the edge.
10. A. D. Rawlins 1976 *Proceedings of the Royal Society*, *Edinburgh* **75A**, 83–95. Acoustic diffraction by an absorbing semi-infinite plane in a moving fluid. II.
11. D. S. Jones 1964 *The Theory of Electromagnetism*. London: Pergamon Press.
12. A. S. Peters and J. J. Stoker 1954 *Communications in Pure and Applied Mathematics* **7**, 563–583.
13. A. D. Rawlins 1977 *Mathematical Proceedings of the Cambridge Philosophical Society* **121**, 555–573. Two trifurcated waveguide problem.
14. B. Noble 1958 *The Wiener–Hopf Technique*. London: Pergamon Press.

APPENDIX A

Here the explicit analytic factorization of $G(\alpha)$ and $W(\alpha)$ is carried out and their asymptotic behaviours as $|\alpha|\to\infty$ are shown. For $G(\alpha)$ given by equation (36) one has

$$G(\alpha) = \cos \kappa b/\kappa \sin \kappa a \cos \kappa(b - a).$$

The product factorization of $G(\alpha)$ thus depends on the factorization of $\cos \kappa b$, $\kappa \sin \kappa a$, and $\cos \kappa(b - a)$. This procedure is now fairly standard (see references [13, 14]). One writes

$$\cos \kappa b = \cos (kb/\sqrt{(1 - M^2)}) \times \prod_{n=1}^{\infty} (1 - \alpha/\sigma_m)(1 + \alpha/\sigma_m),$$

$$\cos \kappa(b - a) = \cos\left(\frac{k(b - a)}{\sqrt{(1 - M^2)}}\right) \prod_{n=1}^{\infty} (1 - \alpha/\tilde{\alpha}_m)(1 + \alpha/\tilde{\alpha}_m), \quad m = 2n - 1,$$

$$\kappa \sin \kappa a = \left(\frac{k}{\sqrt{(1 - M^2)}} + \alpha\right)\left(\frac{k}{\sqrt{(1 - M^2)}} - \alpha\right) \times \frac{\sqrt{(1 - M^2)} \sin (ka/\sqrt{(1 - M^2)})}{k} \times \prod_{n=1}^{\infty} (1 - \alpha/\chi_{2n})(1 + \alpha/\chi_{2n}).$$

Thus, $G(\alpha) = G_-(\alpha)G_+(\alpha)$, where

$$G_+(\alpha) = G_-(-\alpha) = \sqrt{\left(\frac{k \cos (kb/\sqrt{(1 - M^2)})}{\sqrt{(1 - M^2)} \sin (ka/\sqrt{(1 - M^2)}) \sin (k(b - a)/\sqrt{(1 - M^2)})}\right)} \times \frac{1}{(k/\sqrt{(1 - M^2)} + \alpha)} \times \prod_{n=1}^{\infty} \frac{(1 + \alpha/\sigma_m)\, \mathrm{e}^{-\alpha/\sigma_m}}{(1 + \alpha/\tilde{\alpha}_m)\, \mathrm{e}^{-\alpha/\tilde{\alpha}_m}(1 + \alpha/\chi_{2n})\, \mathrm{e}^{-\alpha/\chi_{2n}}}.$$

The exponential factors $e^{-\alpha/\sigma_m}$, $e^{-\alpha/\tilde{\alpha}_m}$, and $e^{-\alpha/\chi_{2n}}$ are inserted to ensure the convergence of the infinite product. Hence, one has

$$G_\pm(\alpha) = \sqrt{\left(\frac{k\cos(kb/\sqrt{(1-M^2)})}{\sqrt{(1-M^2)}\sin(ka/\sqrt{(1-M^2)})\cos(k(b-a)/\sqrt{(1-M^2)})}\right)}$$

$$\times \frac{1}{(k/\sqrt{(1-M^2)} \pm \alpha)}$$

$$\times \prod_{n=1}^{\infty} \frac{(1 \pm \alpha/\sigma_m)\,e^{\mp\alpha/\sigma_m}}{(1 \pm \alpha/\tilde{\alpha}_m)\,e^{\mp\alpha/\tilde{\alpha}_m}(1 \pm \alpha/\chi_{2n})\,e^{\mp\alpha/\chi_{2n}}}.$$

It can also be shown that

$$G_\pm(\alpha) \sim \sqrt{(2)}\mathrm{i}(|\alpha|^{-1/2}) \quad \text{as} \quad |\alpha| \to \infty$$

in the respective regions of analyticity. Similarly, for $W(\alpha)$ given by equation (37), one has

$$W(\alpha) = W_-(\alpha)W_+(\alpha),$$

where

$$W_\pm(\alpha) = \sqrt{\left(\frac{\sqrt{(1-M^2)}\sin(kb/\sqrt{(1-M^2)})}{k\cos(ka/\sqrt{(1-M^2)})\cos(k(b-a)/\sqrt{(1-M^2)})}\right)}$$

$$\times \prod_{n=1}^{\infty} \frac{(1 \pm \alpha/\tilde{\chi}_{2n})\,e^{\mp\alpha/\tilde{\chi}_{2n}}}{(1 \pm \alpha/\tilde{\sigma}_m)\,e^{\mp\alpha/\tilde{\sigma}_m}(1 \pm \alpha/\tilde{\alpha}_m)\,e^{\mp\alpha/\tilde{\alpha}_m}},$$

where $\tilde{\chi}_{2n} = \sqrt{((k_2/(1-M^2)) - (n^2\pi^2/b^2))}$ $(n = 1, 2, \ldots)$ are roots of $\sin\kappa b/\kappa = 0$, and $\tilde{\sigma}_m = \sqrt{((k^2/(1-M^2)) - (m^2\pi^2/4a^2))}$ are the roots of $\cos\kappa a = 0$ $(m = 2n-1, n = 1, 2, \ldots)$. The asymptotic behaviour of $W_\pm(\alpha) \sim \sqrt{(2)}(|\alpha|^{-1/2})$ as $|\alpha| \to \infty$ in the respective regions of analyticity.

APPENDIX B

Here the explicit analytic factorization of $G(\alpha)$, and $W(\alpha)$ is carried out and their asymptotic behaviours as $|\alpha| \to \infty$ are shown for the second problem. For $G(\alpha)$ given by (97) one can write, from Appendix A, that

$$G(\alpha) = G_-(\alpha)G_+(\alpha),$$

where

$$G_{\pm}(\alpha) = \sqrt{\left(\frac{k \sin(kb/\sqrt{(1-M^2)})}{\sqrt{(1-M^2)} \sin(ka/\sqrt{(1-M^2)}) \sin(k(b-a)/\sqrt{(1-M^2)})}\right)}$$

$$\times \frac{1}{(k/\sqrt{(1-M^2)} \pm \alpha)}$$

$$\times \prod_{n=1}^{\infty} \frac{(1 \pm \alpha/\sigma_{2n})\, e^{\mp\alpha/\sigma_{2n}}}{(1 \pm \alpha/\tilde{\alpha}_n)\, e^{\mp\alpha/\tilde{\alpha}_m}(1 \pm \alpha/\chi_{2n})\, e^{\mp\alpha/\chi_{2n}}}.$$

Since $G_+(-\alpha) = G_-(\alpha)$, it is not difficult to show that $G_{\pm}(\alpha) \sim \sqrt{(2)}\mathrm{i}|\alpha|^{-1/2}$ as $|\alpha| \to \infty$.

Similarly, for $W(\alpha)$ given by equation (98) one has

$$W(\alpha) = W_-(\alpha) W_+(\alpha),$$

$$W_{\pm}(\alpha) = \sqrt{\left(\frac{k \cos(kb/\sqrt{(1-M^2)})}{\sqrt{(1-M^2)} \cos(ka/\sqrt{(1-M^2)}) \sin(k(b-a)/\sqrt{(1-M^2)})}\right)}$$

$$\times \frac{1}{(k/\sqrt{(1-M^2)} \pm \alpha)}$$

$$\times \prod_{n=1}^{\infty} \frac{(1 \pm \alpha/\sigma_m)\, e^{\mp\alpha/\sigma_m}}{(1 \pm \alpha/\chi_m)\, e^{\mp\alpha/\chi_m}(1 \pm \alpha/\tilde{\alpha}_n)\, e^{\mp\alpha/\tilde{\alpha}_n}},$$

where $\chi_m = \sqrt{((k^2/(1-M^2)) - (m^2\pi^2/4a^2))}$ are the roots of $\cos \kappa a = 0$, $(m = 2n-1, n = 1, 2, \ldots)$. $\sigma_m = \sqrt{((k^2/(1-M^2)) - (m^2\pi^2/4b^2))}$ are the roots of $\cos \kappa b = 0$ $(n = 1, 2, \ldots)$. The asymptotic behaviour of $W_{\pm}(\alpha) \sim \sqrt{(2)}\mathrm{i}|\alpha|^{-1/2}$ as $|\alpha| \to \infty$ in the respective regions of analyticity.