Journal of Sound and Vibration (1998) **216**(5), 851–863
Article No. sv981726

A GENERALIZED BOLOTIN'S METHOD FOR STABILITY LIMIT DETERMINATION OF PARAMETRICALLY EXCITED SYSTEMS

Ö. Turhan

Faculty of Mechanical Engineering, *Istanbul Technical University*, 80191 *Istanbul*, *Turkey*

(*Received* 25 *March* 1997, *and in final form* 27 *April* 1998)

A boundary tracing method is presented for the construction of stability charts for non-canonical parametrically excited systems. The method is an extension, so as to cover the combination resonances, of the well known Bolotin's method, and reduces the boundary tracing problem into an eigenvalue analysis problem of some special matrices.

1. INTRODUCTION

Dynamical systems mathematically modelled by linear ordinary differential equations with periodic coefficients (or Mathieu–Hill equations) are referred to as parametrically excited systems. A stability investigation constitutes the focal point in the analysis of such systems and numerous methods have been devised for that purpose. These methods may be grouped, with respect to their origins, into three categories as Lyapunovian methods [1–4], perturbation methods [5–8] and Floquet methods. The results of stability investigations are generally and preferably presented in the form of stability charts reflecting stability's dependence on two selected system parameters. Another convenient classification of the stability analysis methods follows from the way they imply for the construction of these charts: they are either scatter plot methods or boundary tracing methods.

In order to state clearly the scope of this paper, a brief overview of the Floquet methods which as such either consider the Floquet multipliers or the Floquet exponents to provide stability information will be given.

The Floquet multiplier methods determine first the so-called monodromy matrix, by direct numerical integration of the system's equations [9] or resorting to an approximation technique [10–12], and calculate its eigenvalues which are the Floquet multipliers. Recently Weyh and Kostyra have given a general Floquet multipliers-boundary tracing method [13]. Its application [14] has shown that boundary tracing methods do not only reduce the process-time but also facilitate getting insight into some theoretically interesting peculiarities of multi-degree-of-freedom systems. A similar method has also been given [15, 16] for the special case of 2-degree-of-freedom canonical systems.

0022–460X/98/400851 + 13 $30.00/0

The Floquet exponent methods start with the Floquet solutions, proceed via Fourier series and harmonic balance and arrive at a vanishing, infinite determinant known as Hill's determinant. The Floquet exponents are calculated as the roots of the determinantal equation. A basic Floquet exponent method introduced by Hill [17] for the analysis of a single Hill's equation, and generalized by some authors to systems of Mathieu–Hill equations [18–21], consists of making use of the convergence properties of Hill's determinants to extract a closed form expression for the determinantal equation. Some other Floquet exponent methods confine the periodic coefficients to be of small order and resort to small parameter expansions to obtain approximations to the determinantal equation [22, 23] or to the stability boundaries [24]. Yet another Floquet exponent method, due to Bolotin [25], consists of approximating the infinite Hill's determinants by finite segments and using the resulting determinantal equations to calculate the stability boundaries rather than the Floquet exponents. This requires the values, the Floquet exponents would assume on the stability boundaires, to be *a priori* known. This indeed is the case for the boundaries of parametric resonance regions and the method turns out to be an efficient boundary tracing method for those kind of boundaries, but fails to deal with combination resonance boundaries as no information is available on the required values in that case. Some authors proposed generalizations of Bolotin's method [26–28], but doing this, they sacrificed its perhaps most valuable aspect of being a boundary tracing method. The only exception is the work of Szemplińska-Stupnicka [29] who devised an *ad hoc* approximate boundary tracing method for the primary instability regions of 2-degree-of-freedom systems.

Here a generalization of Bolotin's boundary tracing method, applicable to non-canonical parametrically excited systems, is presented. The essential feature of the method is to eliminate the unknown Floquet exponents by using some indirect knowledge on their aspect when passing from stability to instability and to formulate the resulting boundary tracing problem in the form of an eigenvalue analysis problem. A somewhat detailed analysis on the behaviour of the Floquet characteristics is also given, which provides the required knowledge. As it will be apparent in the sequel, the method turns out to be a computationally expensive one. Yet, in the author's opinion, it is not devoid of theoretical nor of practical interest.

2. FLOQUET THEORY AND STABILITY BOUNDARIES

Consider an n-degree-of-freedom dynamical system whose state–space representation leads to

$$\dot{\mathbf{u}} = \mathbf{A}(t)\mathbf{u}, \tag{1}$$

where $\mathbf{A}(t)$ is a $2n \times 2n$ T periodic matrix. Floquet's theory states that a fundamental matrix of the system (1) may be expressed as $\mathbf{\Phi}(t) = \mathbf{Q}(t)\,\mathrm{e}^{\mathbf{R}t}$ where $\mathbf{Q}(t)$ is a T periodic matrix and $\mathbf{R}$ is a constant matrix which is related to another constant matrix $\mathbf{S}$, referred to as a monodromy matrix, by $\mathbf{R} = 1/T \log \mathbf{S}$. If the fundamental matrix is normalized so that $\mathbf{\Phi}(t_0) = \mathbf{I}$, then $\mathbf{S} = \mathbf{\Phi}(t_0 + T)$. The eigenvalues σ_i of $\mathbf{S}$ are the Floquet multipliers while the eigenvalues ρ_i of $\mathbf{R}$ are the Floquet exponents. These two sets of generally complex characteristic numbers are interrelated by

$$\mathrm{Re}\,(\rho_i) = 1/T \log [\mathrm{mod}\,(\sigma_i)], \qquad \mathrm{Im}\,(\rho_i) = 1/T[\arg(\sigma_i) \pm 2k\pi], \qquad k = 0, 1, 2, .. \tag{2}$$

and either of them governs the stability of the system (1). Specifically the system is stable if and only if $\mathrm{mod}\,(\sigma_i) \leqslant 1$ i.e., $\mathrm{Re}\,(\rho_i) \leqslant 0$ for all i's, where the equality sign holds only when the multiplicity of the ith eigenvalue equals its nullity. It is clear that the equality sign, for a certain i, also constitutes a necessary condition for the passage from stability to instability or *vice versa*. But in view of the present goal, one needs some further information on the features of the Floquet exponents crossing a stability limit. Thus, there follows a brief discussion on the subject. It proves to be more convenient to pursue this discussion in terms of Floquet multipliers and then translate the results, by means of equation (2), into terms of Floquet exponents. To this end two basic theorems are first evoked: (i) If $\mathbf{A}(t)$ is real (the contrary case is discarded in this study), then complex Floquet multipliers can only occur in complex conjugate pairs ($\sigma_i, \sigma_j = \sigma_i^*; i \neq j$). (ii) If the dynamical system is canonical (i.e., if it is a conservative system with ideal holonomic constraints so that its dynamics may be described by Hamilton's canonical equations) then the Floquet multipliers occur in reciprocal pairs ($\sigma_i, \sigma_j = \sigma_i^{-1}, i \neq j$). As it will soon become apparent, the situation is quite different for canonical and non-canonical systems:

In canonical systems (Figure 1(a)), existence of one complex σ_i with $\mathrm{mod}\,(\sigma_i) \neq 1$ implies, by virtue of the above theorems, existence of four of them: $\sigma_i \rightarrow \sigma_j = \sigma_i^*,\ \sigma_k = \sigma_i^{-1} = \sigma_j/\mathrm{mod}^2(\sigma_i),\ \sigma_1 = \sigma_j^{-1} = \sigma_k^* = \sigma_i/\mathrm{mod}^2(\sigma_i);\ i \neq j \neq k \neq l$ and this obviously corresponds to instability as two of the σ's have necessarily moduli exceeding unity. But for a complex σ_i with $\mathrm{mod}\,(\sigma_i) = 1$, which corresponds to simple or limit stability (the best possible for a canonical system),

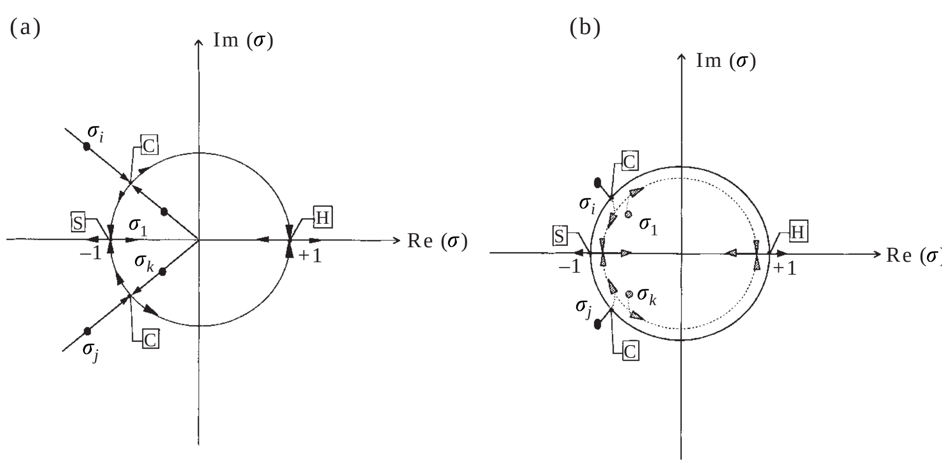

Figure 1. Behaviour of Floquet multipliers. (a) Canonical systems; (b) non-canonical systems. (H: Harmonic, S: subharmonic, C: combination resonance boundary.)

one has $\sigma_i^* = \sigma_i^{-1}$ and the implications of the two theorems collapse in one so that the existence of such a σ_i implies existence of only two of them: $\sigma_i \rightarrow \sigma_j = \sigma_i^* = \sigma^{-1}$; $i \neq j$. It then follows that the passage from instability to stability via a complex σ (boundary of combination resonance region) occurs when two pairs of complex σ's of unit modulus meet. Thus, on a combination resonance boundary one has $\sigma_i = \sigma_1 = \sigma$, $\sigma_j = \sigma_k = \sigma^* = \sigma^{-1}$; $i \neq j \neq k \neq l$. On the other hand, for a real σ_i we have by theorem (ii) $\sigma_i \rightarrow \sigma_j = \sigma_i^{-1}$; $i \neq j$ which obviously corresponds to instability unless $\mathrm{mod}\,(\sigma_i) = |\sigma_i| = 1$. It then follows that the passage from stability to instability via a real σ (boundary of parametric resonance region) occurs either when $\sigma_i = \sigma_j = +1$; $i \neq j$ (boundary of harmonic parametric resonance region) or when $\sigma_k = \sigma_1 = -1$; $k \neq l$ (boundary of subharmonic parametric resonance region). It results from the above discussion that in canonical systems a stability boundary, of whatever type, is characterized by the occurence of repeated Floquet multipliers: $\sigma_i/\sigma_j = 1$; $i \neq j$. It must however be remarked here that in some systems with perfect symmetry, which in their stationary state have repeated eigenfrequencies, existence of repeated Floquet multipliers will not be peculiar to stability boundaries. Under the restriction voiced by this remark we are now at a point to state in terms of Floquet exponents: in canonical systems, for a stability boundary (of any kind) to occur it is *necessary* (sufficiency does not hold) that

$$\rho_i - \rho_j = 0, \qquad i \neq j, \qquad 0 \leqslant |\mathrm{Im}\,(\rho_{i,j})| \leqslant \pi/T, \tag{3}$$

where the constraint on $\mathrm{Im}\,(\rho_{i,j})$ is introduced in order to remove the indefiniteness implied by equation (2) which does not single out the Floquet exponents but defines them as a member of a so-called congruent set.

In non-canonical systems (Figure 1(b)), existence of one complex σ_i implies, by virtue of theorem (i), existence of two of them: $\sigma_i \rightarrow \sigma_j = \sigma_i^*$; $i \neq j$. If $\mathrm{mod}\,(\sigma_i) \neq 1$ either instability or stability prevails depending on whether $\mathrm{mod}\,(\sigma_i)$ is greater or less than unity and the passage takes place when momentarily $\mathrm{mod}\,(\sigma_i) = 1$ (momentarily because permanent limit stability is exclusive to canonical systems). But then one has $\sigma_j = \sigma_i^* = \sigma_i^{-1}$. Thus, it is concluded that a boundary of combination resonance region is characterized by the occurrence of a pair of reciprocal Floquet multipliers: $\sigma_i \cdot \sigma_j = 1$; $i \neq j$. Similarly, a real σ_i may indicate stability or instability and the passage occurs when momentarily $\mathrm{mod}\,(\sigma_i) = |\sigma_i| = 1$. It then follows that a boundary of parametric resonance region is characterized either by $\sigma_i = +1$ or $\sigma_i = -1$. To summarize these results, we state in terms of Floquet exponents: in non-canonical systems, for a stability boundary to occur, it is *necessary* that either

$$\rho_i = 0(\pm 2k\pi \mathrm{i}/T), \qquad k = 1, 2, \ldots, \qquad \mathrm{i}^2 = -1, \tag{4}$$

$$\rho_i = \pi \mathrm{i}/T(\pm 2k\pi \mathrm{i}/T), \qquad k = 1, 2, \ldots, \tag{5}$$

where the terms in parentheses indicate possible congruents, or

$$\rho_i + \rho_j = 0, \qquad i \neq j, \qquad 0 < |\mathrm{Im}\,(\rho_{i,j})| < \pi/T, \tag{6}$$

and these correspond to stability limits of harmonic parametric, subharmonic parametric and combination resonance regions, respectively. [Note that equations

(4) and (5) also hold for canonical systems, with pairs of equal Floquet exponents instead of single ones.]

It is of interest to note that equations (5) and (6) may be combined in

$$\rho_i + \rho_j = 0, \qquad i \neq j, \qquad 0 < |\mathrm{Im}\,(\rho_{i,j})| \leqslant \pi/T, \tag{7}$$

where the congruents corresponding to $k = 1$ of equation (5) are allowed and equations (4)–(6) may be combined in

$$\rho_i + \rho_j = 0, \qquad i \neq \text{or} = j, \qquad 0 \leqslant |\mathrm{Im}\,(\rho_{i,j})| \leqslant \pi/T, \tag{8}$$

where summation with oneself is allowed. Equation (8) is the *necessary* condition for a stability boundary (of any kind) to occur in a non-canonical system.

In what follows, we concentrate on the study of non-canonical sytems and a boundary tracing method is presented on the basis of equations (4)–(8).

3. MATHEMATICAL FORMULATIONS

Consider an n-degree-of-freedom non-canonical system described by

$$\ddot{\mathbf{x}} + \mathbf{C}(t)\dot{\mathbf{x}} + \mathbf{K}(t)\mathbf{x} = \mathbf{0}, \tag{9}$$

where $\mathbf{C}(t)$ and $\mathbf{K}(t)$ are $n \times n$, T periodic matrices. [The phase–space representation of equation (1) is abandoned here for convenience.] Introducing the change of variable $\tau = \omega t$, $\omega = 2\pi/T$ which turns the period T to 2π, substituting for $\mathbf{x}(\tau)$ the Floquet solution

$$\mathbf{x}(\tau) = \mathrm{e}^{\rho\tau} \sum_{k=-\infty}^{\infty} \mathbf{D}_k\, \mathrm{e}^{\mathrm{i}k\tau}, \tag{10}$$

where $\mathbf{D}_k$'s are $n \times 1$ complex Fourier coefficients' matrices, and representing $\mathbf{C}(\tau)$ and $\mathbf{K}(\tau)$ by their Fourier series expansions up to the mth harmonic, equation (9) gives (see reference [21])

$$\omega^2 \sum_{k=-\infty}^{\infty} (\rho + \mathrm{i}k)^2 \mathbf{D}_k\, \mathrm{e}^{(\rho+\mathrm{i}k)\tau} + \sum_{p=-m}^{m} \sum_{k=-\infty}^{\infty} [\omega(\rho + \mathrm{i}k)\mathbf{C}_p + \mathbf{K}_p]\mathbf{D}_p\, \mathrm{e}^{[\rho+\mathrm{i}(k+p)]\tau} = \mathbf{0}, \tag{11}$$

where $\mathbf{C}_p$ and $\mathbf{K}_p$'s are $n \times n$ complex Fourier coefficients' matrices related to $\mathbf{C}(\tau)$ and $\mathbf{K}(\tau)$, respectively. Harmonic balance of equation (11) requires the following infinite set of algebraic equations to be satisfied

$$\omega^2(\rho + \mathrm{i}k)^2 \mathbf{D}_k + \sum_{p=-m}^{m} [\omega(\rho - \mathrm{i}q)\mathbf{C}_p + \mathbf{K}_p]\mathbf{D}_q = \mathbf{0},$$

$$k = \cdots -2, -1, 0, 1, 2, \ldots, q = k - p. \tag{12}$$

This set may be recast, with $\omega \neq 0$ into the form

$$[\rho^2\mathbf{I} + \rho[\mathbf{E}_0 + 1/\omega\mathbf{E}_1] + [\mathbf{F}_0 + 1/\omega\mathbf{F}_1 + 1/\omega^2\mathbf{F}_2]]\mathbf{D} = \mathbf{0}, \tag{13}$$

where $\mathbf{D}$ is an infinite column matrix defined as $\mathbf{D} = \{\ldots \mathbf{D}_{-2}^T, \mathbf{D}_{-1}^T, \mathbf{D}_0^T, \mathbf{D}_1^T, \mathbf{D}_2^T, \ldots\}^T$, $\mathbf{I}$ is the infinite dimensional unit matrix and $\mathbf{E}_i$, $\mathbf{F}_i$'s are infinite dimensional partitioned matrices made up of $n \times n$ submatrices given by

$$\mathbf{E}_0^{k,q} = 2k\mathrm{i}\mathbf{I}\delta_{kq}, \qquad \mathbf{E}_1^{k,q} = \mathbf{C}_p,$$

$$\mathbf{F}_0^{k,q} \doteq -k^2\mathbf{I}\delta_{kq}, \qquad \mathbf{F}_1^{k,q} = q\mathrm{i}\mathbf{C}_p, \qquad \mathbf{F}_2^{k,q} = \mathbf{K}_p, \tag{14}$$

where δ_{kq} is the Kronecker delta, $i^2 = -1$, and the superscripts k and q denote the row and column indices of the submatrix in question. We note that, except those which are obviously diagonal, these submatrices are banded matrices with bandwidth $h = 2n(m+1) - 1$. Existence of non-trivial solutions of equation (13) requires vanishing of the determinant of the coefficients' matrix which is a monic matrix polynomial of second degree in ρ. Linearizing, one may write

$$\det[[\mathbf{U}_0 + 1/\omega\mathbf{U}_1 + 1/\omega^2\mathbf{U}_2] - \rho\mathbf{I}] = \det[\bar{\mathbf{R}} - \rho\mathbf{I}] = 0, \tag{15}$$

where $\mathbf{U}_i$'s are doubly infinite matrices, whose definitions immediately follow from equations (13)–(15). Hence, the Floquet exponents are the eigenvalues of the matrix $\bar{\mathbf{R}}$ which is, therefore, an infinite dimensional substitute for the $2n \times 2n$ matrix $\mathbf{R}$. It is worth noting that the matrix pencil of equation (15) which is nothing but the Hill's matrix of the problem is i periodic in ρ, in accordance with the fact that the Floquet exponents are defined [equation (2)] up to an integral multiple of $2\pi\mathrm{i}/T$ (here $T = 2\pi$).

One may expect, following Bolotin [25], that finite dimensional, $2n(2K+1)$th order determinants taken from equation (15) by putting $-K \leqslant k \leqslant K$, $K = 1, 2, \ldots$ in equation (12) would give reasonable approximations to the Floquet exponents. Care, however, must be taken in such a calculation because, out of the resulting $2n(2K+1)$ eigenvalues, only $2n$ would be Kth order approximations to the Floquet exponents ρ_i; $|\mathrm{Im}(\rho_i)| \leqslant 1/2$ sought and the remaining ones would be $K - \kappa$th order approximations to the congruents $\rho_i \pm \kappa i$, $\kappa = 1, 2, \ldots, K$ (as noted in reference [30]). This fact, which may be utilized to construct a convergence criterion which let a so calculated ρ_i be accepted only if there are corresponding ρ_j's which are to a prescribed amount close to $\rho_i \pm i$, seems not to be fully recognized in some studies (e.g., reference [28]).

4. BOUNDARY TRACING METHOD

Let the frequency ω of the parametric excitation be one of the two components of the parameter space on which the stability chart is to be constructed. The second component, say λ, need not be specified beforehand. The boundary tracing method will, therefore, consist of equations yielding the ω values on the stability boundaries. Such equations can easily be given for parametric resonance boundaries as follows:

Substitution of $\rho = 0$ from equation (4) into equation (13) yields for harmonic resonance boundaries

$$\det [\mathbf{F}_0 + 1/\omega \mathbf{F}_1 + 1/\omega^2 \mathbf{F}_2] = 0 \tag{16}$$

and substitution of $\rho = \mathrm{i}/2$ ($\rho = -\mathrm{i}/2$ would equally do) from equation (5) with $T = 2\pi$, into equation (13) yields for subharmonic resonance boundaries

$$\det [[\mathbf{F}_0 + 1/2\mathrm{i}\mathbf{E}_0 - 1/4\mathbf{I}] + 1/\omega[\mathbf{F}_1 + 1/2\mathrm{i}\mathbf{E}_1] + 1/\omega^2 \mathbf{F}_2] = 0. \tag{17}$$

Equations (16) and (17) are nothing but the original Bolotin's method put in a different form and their various versions have long been successfully used in parametric stability analysis of various systems by many authors including the present one [31, 32].

In order to obtain similar equations for combination resonance boundaries, one has to express, according to equations (6)–(8), conditions for matrix $\bar{\mathbf{R}}$ of equation (15) to have eigenvalues with vanishing sum. This problem is mathematically equivalent to the problem of finding a matrix whose eigenvalues are the sums of the eigenvalues of $\bar{\mathbf{R}}$ taken in pairs, and putting its determinant to zero. Such matrices do indeed exist and are treated in detail by Fuller [33]. It turns out that, given an nth order matrix $\mathbf{M}$ with eigenvalues $\mu_i (i = 1, 2, \ldots, n)$ one may construct an $n(n-1)/2$-th order matrix $\mathbf{B}(\mathbf{M})$ called, in terms of Fuller, bialternate sum of $\mathbf{M}$ by itself, with eigenvalues $\mu_i + \mu_j (i = 2, 3, \ldots, n, j = 1, 2, \ldots, i-1)$, and an $n(n+1)/2$-th order matrix $\mathbf{L}(\mathbf{M})$, called Lyapunov matrix of $\mathbf{M}$ by Fuller, with eigenvalues $\mu_i + \mu_j (i = 1, 2, \ldots, n, j = 1, 2, \ldots, i)$. (See Appendix for the construction of these matrices.)

Hence, using equations (6) and (15), one may write for the combination resonance boundaries

$$\det [\mathbf{B}(\mathbf{U}_0) + 1/\omega \mathbf{B}(\mathbf{U}_1) + 1/\omega^2 \mathbf{B}(\mathbf{U}_2)] = 0; \qquad 0 < |\mathrm{Im}\,(\rho_{i,j})| < 1/2, \tag{18}$$

where $\mathbf{B}(\mathbf{U}_i)$ designates the bialternate sum of $\mathbf{U}_i$ by itself. Finally, by equations (8) and (15), one obtains for all the stability boundaries

$$\det [\mathbf{L}(\mathbf{U}_0) + 1/\omega \mathbf{L}(\mathbf{U}_1) + 1/\omega^2 \mathbf{L}(\mathbf{U}_2)] = 0; \qquad 0 \leqslant |\mathrm{Im}\,(\rho_{i,j}) \leqslant\ < 1/2, \tag{19}$$

where $\mathbf{L}(\mathbf{U}_i)$ designates the Lyapunov matrix of $\mathbf{U}_i$.

Equations (16)–(19) involve regular matrix polynomials of second degree in $1/\omega$. They are, however, neither monic nor comonic, nor is their leading or last coefficient matrix invertible [except for equations (16) and (17)]. Therefore first put $1/\omega = 1/\omega^* + \alpha$ where α is some scalar which does not equal an eigenvalue of the matrix polynomial in question, and then linearize in ω^* [34] to obtain

$$\det [\mathbf{G}(\lambda) - \omega^* \mathbf{I}] = 0, \qquad \omega^* = \omega/(1 - \alpha\omega), \tag{20}$$

where λ represents the second component of the parameter space and the structure of $\mathbf{G}(\lambda)$ depends, besides λ, on the boundary tracing problem [equations (16)–(19)] in question and on the scalar α used in linearization.

Equation (20) concludes the essence of the boundary tracing method: ω values corresponding to the stability boundaries will be calculated, for given values of λ, by means of the eigenvalues of the matrix $\mathbf{G}(\lambda)$. It should however be noted that all the so calculated ω values are not admissable and the method has to be supplemented with a selection procedure which: (1) rejects the ω values which are not real numbers [this follows from obvious physical considerations and applies to any of the problems (16)–(19)]; and (2) eliminates those ω values which correspond to $\rho_{i,j}$ pairs violating the accompanying inequality constraint and/or to unconverged ones [this applies to the problems (18) and (19)]. It is clear that though the implementation of (1) does not require an additional effort, that of (2) requires the corresponding Floquet exponents to be calculated through equation (15), and a carefully designed convergence criterion to be used. This somewhat disables the straightforward aspect of the method but as it is a consequence of the intrinsic redundancy of the matrix $\bar{\mathbf{R}}$, can apparently, not be circumvented within the conceptual framework adopted in this study. Another problem associated with the implementation of the method is the overdimensionality of the matrices involved. For an n-degree-of-freedom system, in order for the Kth order instability regions to be included in the calculations, the matrix $\mathbf{G}(\lambda)$ has to be of order $\eta_1 = 2n(2K+1)$ for equations (16) and (17), $\eta_2 = \eta_1(\eta_1 - 1)$ for equation (18), and $\eta_3 = \eta_1(\eta_1 + 1)$ for equation (19). These dimensions rapidly grow prohibitive in the analysis of high degree-of-freedom systems.

5. A NUMERICAL EXAMPLE

As an example, consider the 2-degree-of-freedom system

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0{\cdot}1 & 0 \\ 0 & 0{\cdot}1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \left[\begin{bmatrix} 0{\cdot}5 & 0 \\ 0 & 1{\cdot}5 \end{bmatrix} + \begin{bmatrix} 0{\cdot}4 & \lambda \\ \lambda & 0{\cdot}4 \end{bmatrix} \cos \omega t \right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (21)$$

which was first studied by Szemplińska-Stupnicka [29] and later revisited by Takahashi [28]. The calculations are performed by means of a special FORTRAN code developed for implementing the proposed method and the results are presented in Figures 2 and 3, where hatched zones represent unstable parameter regions. Figure 2 depicts the superposition of the results obtained by solving equations (16) and (17) separately, with $K = 12$ ($\eta_1 = 100$). Accordingly, only

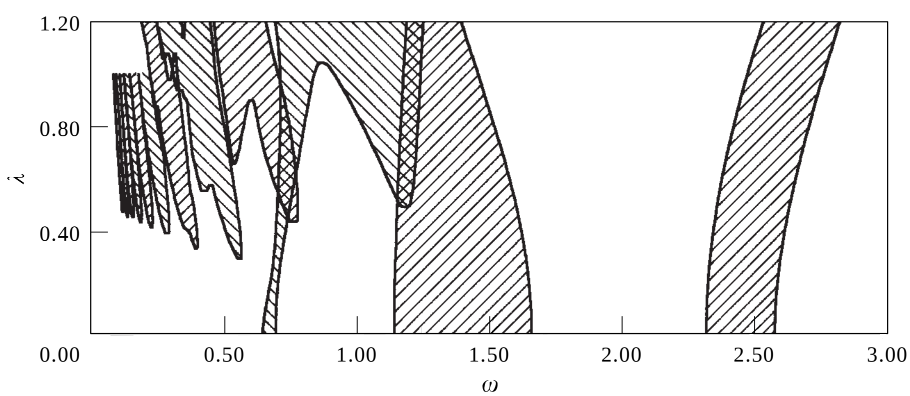

Figure 2. Parametric stability analysis. (\\\: Harmonic; ///: subharmonic resonance region.)

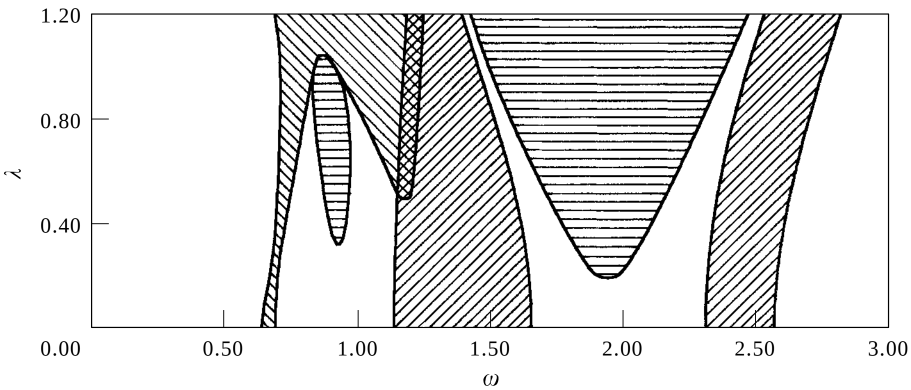

Figure 3. Complete stability analysis. (\\\: Harmonic; ///: subharmonic; ≡: combination resonance region.)

parametric resonances are accounted for and 12th order approximations are obtained for the boundaries of primary and secondary parametric resonance regions and diminishing order approximations to the boundaries of higher order ones. Low order boundaries are therefore highly reliable. We note that, of the calculated boundaries, some highest order ones are not shown on the figure. Figure 3 depicts the results of the complete stability analysis performed by solving equation (19) with $K = 3$ ($\eta_3 = 812$). One observes that besides the primary and secondary instability regions of Figure 2, which are accurately recovered, two combination resonance regions corresponding to $\omega \sim \omega_1 + \omega_2$ and $\omega \sim (\omega_1 + \omega_2)/2$ (where $\omega_1 = \sqrt{0{\cdot}5}$, $\omega_2 = \sqrt{1{\cdot}5}$ are the natural frequencies of the system) are also obtained. As a result of the fact that the "necessary" conditions used in determining the stability boundaries do not discern the difference between an area limit and an absolute limit of stability, both figures exhibit some regions where more than one kind of instability coexist. Though not of great practical significance, this feature may contribute to the understanding of some theoretically interesting peculiarities of high order systems, as discussed in reference [13].

Comparing the stability chart of Figure 3 to the previously published charts for the same system (Figure 1 of reference [29] and Figure 3(a) of reference [28]) one observes notable quantitative discrepancies. This is just natural with the former reference as it corresponds to a different level of approximation but is not justifiable with the latter. The numerical experiments lead one to conclude that the discrepancies with reference [28] should be traced back to the fact that no discrimination has been made in that study between converged and unconverged eigenvalues of the matrix $\bar{\mathbf{R}}$. But, one must say, this makes the calculated eigenvalues turn to a meaningless compilation of numbers corresponding to different levels of approximation.

Figure 4, which corresponds to a fragment of Figure 3, visualizes some rough results obtained by solving equation (19) and gives an idea about the selection procedure mentioned above. The strings of points constitute candidates for a stability boundary. Upon checking through equations (15) one finds out that a string labelled κ corresponds to Floquet exponents $\rho + \kappa i$. Thus, the strings

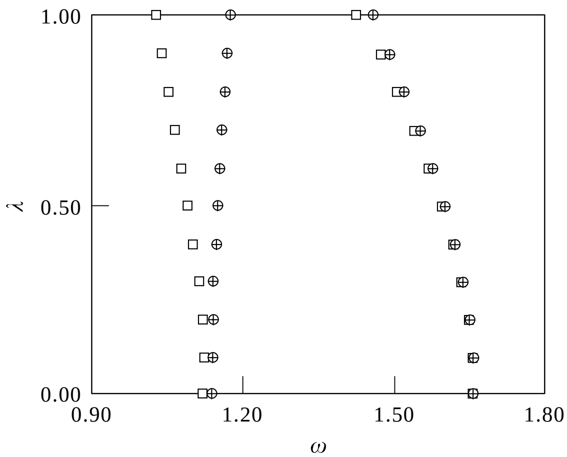

Figure 4. Some results in the rough. $+$, $\kappa = 0$; $\bigcirc$, $\kappa = 1$; $\square$, $\kappa = 2$.

labelled 1 and 2 are rejected and the 0 labelled ones are admitted after checking for convergence.

6. CONCLUSIONS

A method is presented for the stability limit calculations of parametrically excited systems. The method constitutes an extension of the well known Bolotin's method to the case of combination resonances. It is based on the recognition of the fact that the stability boundaries calculation problem of parametric systems can be viewed as two eigenvalue analysis problems (one for the parametric frequency and one for the Floquet exponents) nested inside one another (see equation (13)) and that the Floquet exponents may be eliminated from that problem by using some indirect information on their behaviour when crossing a stability boundary.

An analysis of the behaviour of the Floquet characteristics is also provided which makes it possible to derive the required information.

The applicability of the presented method is restricted to non-canonical systems. This is, however, not too severe a restriction because canonical models are generally used as approximations for actually non-canonical systems.

The method, on the other hand, severely suffers from the high-dimensionality of the matrices involved. But who knows how long the significance of this statement may persist in view of the prodigious developments in computer technologies?

REFERENCES

1. J. R. Dickerson 1970 *Journal of Applied Mechanics* **37**, 228–230. Stability of linear systems with parametric excitation.
2. T. H. Lee and C. S. Hsu 1972 *Journal of Applied Mechanics* **39**, 244–250. Lyapunov stability criteria for continuous systems under parametric excitation.

3. S. K. SHRIVASTAVA 1981 *Journal of Applied Mechanics* **48**, 174–177. Stability theorems for multidimensional linear systems with variable parameters.
4. R. K. YEDAVALLI and S. R. KOLLA 1988 *International Journal of Systems Science* **19**, 1853–1858. Stability analysis of linear time-varying systems.
5. C. S. HSU 1963 *Journal of Applied Mechanics* **30**, 267–372. On the parametric excitation of a dynamic system.
6. C. S. HSU 1965 *Journal of Applied Mechanics* **32**, 373–377. Further results on parametric excitation of a dynamic system.
7. F. C. L. FU and S. NEMAT-NASSER 1975 *Journal of Applied Mechanics* June, 458–463. Response and stability of linear dynamic systems with many degrees of freedom subjected to nonconservative and harmonic forces.
8. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: Wiley.
9. P. FRIEDMANN, C. E. HAMMOND and T. H. WOO 1977 *International Journal for Numerical Methods in Engineering* **11**, 1117–1136. Efficient numerical treatment of periodic systems with application to stability problems.
10. L. A. PIPES 1953 *Journal of Applied Physics* **24**, 902–910. Matrix solution of the equations of the Mathieu–Hill type.
11. C. S. HSU 1974 *Journal of Mathematical Analysis and Applications* **45**, 234–251. On approximating a general linear periodic system.
12. S. C. SINHA and D.-H. WU 1991 *Journal of Sound and Vibration* **151**, 91–117. An efficient computational scheme for the analysis of periodic systems.
13. B. WEYH and H. KOSTYRA 1991 *Mechanism and Machine Theory* **26**, 123–131. Direct Floquet method for stability limits determination—I: theory.
14. H. KOSTYRA and B. WEYH 1991 *Mechanism and Machine Theory* **26**, 133–144. Direct Floquet method for stability limits determination—II: application and phenomena.
15. D. A. STREIT, C. M. KRUSGRILL and A. K. BAJAJ 1986 *Journal of Dynamic Systems, Measurement and Control* **108**, 206–214. A preliminary investigation of the dynamic stability of flexible manipulators performing repetitive tasks.
16. R. I. ZADOKS and A. MIDHA 1987 *Journal of Mechanisms, Transmissions and Automation in Design* **109**, 216–223. Parametric stability of a two-degree-of-freedom machine system: part II—stability analysis.
17. G. W. HILL 1886 *Acta Mathematica* **8**, 1–36. On the part of the lunar perigee which is a function of the mean motions of the sun and moon.
18. K. G. VALEEV 1960 *Journal of Applied Mathematics and Mechanics* **24**, 585–602. On the solution and characteristic exponents of solutions of some systems of linear differential equations with periodic coefficients.
19. T. R. BEAL 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 486–494. Dynamic stability of a flexible missile under constant and pulsating trusts.
20. P. CRIMI 1970 *American Institute of Aeronautics and Astronautics Journal* **8**, 1760–1764. Stability of dynamic systems with periodically varying parameters.
21. S. T. NOAH and G. R. HOPKINS 1982 *Journal of Applied Mechanics* **49**, 217–223. A generalized Hill's method for the stability analysis of parametrically excited dynamic systems.
22. T. C. LEE 1976 *Journal of Applied Mechanics* **43**, 349–352. A study of coupled Mathieu equations by use of infinite determinants.
23. W.-T. WU, J. H. GRIFFIN and J. A. WICKERT 1995 *Journal of Sound and Vibration* **182**, 245–257. Perturbation method for the Floquet eigenvalues and stability boundary of periodic linear systems.
24. E. METTLER 1949 *Ingenieur Archiv* **17**, 418–449. Algemeine Theorie der Stabilitat erzwungener Schwingungen elastischer Körper.
25. V. V. BOLOTIN 1964 *The Dynamic Stability of Elastic Systems*, San Francisco, CA: Holden-Day.

26. K. G. LINDH and P. W. LIKINS 1970 *American Institute of Aeronautics and Astronautics Journal* **8**, 680–686. Infinite determinant methods for stability analysis of periodic-coefficient differential equations.
27. J. ZAJACKOWSKI and J. LIPIŃSKI 1979 *Journal of Sound and Vibration* **63**, 1–7. Vibrations of parametrically excited systems.
28. K. TAKAHASHI 1981 *Journal of Sound and Vibration* **8**, 519–529. An approach to investigate the instability of multiple-degree-of-freedom parametric dynamic systems.
29. W. SZEMPLIŃSKA-STUPNICKA 1978 *Journal of Sound and Vibration* **58**, 347–361. The generalized harmonic balance method for determining the combination resonance in the parametric dynamic systems.
30. J. ZAJACKOWSKI 1981 *Journal of Sound and Vibration* **79**, 581–588. An approximate method of analysis of parametric vibration.
31. Ö. TURHAN 1995 *Mechanism and Machine Theory* **30**, 871–882. Dynamic stability of four-bar and slider-crank mechanisms with elastic coupler.
32. Ö. TURHAN 1996 *Mechanism and Machine Theory* **31**, 77–89. Dynamic stability of four-bar and slider-crank mechanisms with viscoelastic (Kelvin–Voigt model) coupler.
33. A. T. FULLER 1968 *Journal of Mathematical Analysis and Applications* **23**, 71–98. Conditions for a matrix to have only characteristic roots with negative real parts.
34. I. GOHBERG, P. LANCASTER and L. RODMAN 1982 *Matrix Polynomials*. New York: Academic Press.

APPENDIX:
CONSTRUCTION OF BIALTERNATE SUM AND LYAPUNOV MATRICES

Let **A** be an $n \times n$ matrix with elements a_{ij}, **B** the bialternate sum of **A** by itself and **L** the Lyapunov matrix of **A**. The elements $b_{pq,rs}$ of **B**, where pq ($p = 2, 3, \ldots, n$, $q = 1, 2, \ldots, p-1$) labels the rows and rs ($r = 2, 3, \ldots, n$; $s = 1, 2, \ldots, r-1$) labels the columns of **B**, are defined as

$$b_{pq,rs} = \begin{cases} -a_{ps} & \text{if } r = q \\ a_{pr} & \text{if } r \neq p \text{ and } s = q \\ a_{pp} + a_{qq} & \text{if } r = p \text{ and } s = q \\ a_{qs} & \text{if } r = p \text{ and } s \neq q \\ -a_{qr} & \text{if } s = p \\ 0 & \text{otherwise} \end{cases}, \tag{A1}$$

and the elements $l_{pq,rs}$ of **L**, where pq ($p = 1, 2, \ldots, n$; $q = 1, 2, \ldots, p$) labels the rows and rs ($r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, r$) labels the columns of **L**, are defined for $p \neq q$ as

$$l_{pq,rs} = \begin{cases} a_{ps} & \text{if } r = q \text{ and } s \neq q \\ a_{pr} & \text{if } r \neq p \text{ and } s = q \\ a_{pp} + a_{qq} & \text{if } r = p \text{ and } s = q \\ a_{qs} & \text{if } r = p \text{ and } s \neq q \\ a_{qr} & \text{if } r \neq p \text{ and } s = p \\ 0 & \text{otherwise} \end{cases} \tag{A2}$$

and for $p = q$ as

$$l_{pq,rs} = \begin{cases} 2a_{ps} & \text{if } r = p \text{ and } s \neq p \\ 2a_{pp} & \text{if } r = p \text{ and } s = p \\ 2a_{pr} & \text{if } r \neq p \text{ and } s = p \\ 0 & \text{otherwise} \end{cases}. \tag{A3}$$

For details and proofs the reader is referred to the work of Fuller [33].