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BISPECTRAL ANALYSIS OF THE BILINEAR OSCILLATOR WITH APPLICATION TO THE DETECTION OF FATIGUE CRACKS

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Damage detection by means of non-destructive testing plays an important role in ensuring the integrity of machine elements and structures. Vibration testing is an effective means of detecting crack development in structures. In this paper the effect of crack closure on the dynamic behaviour of cantilever beams is studied. Since the crack opens or closes depending on the direction of the vibration, the beam exhibits bilinear characteristics. The aim of the paper is to analyze the system response by using bispectral analysis which forms a subset of higher order statistical analysis. The study has been conducted both on simulated and experimental vibration signals. Firstly, a simplified model is employed to simulate the bilinear behaviour of a beam with a closing crack. The model is an oscillator with a bilinear restoring forcing function. The analysis of the forced vibrations of the model is performed by means of the harmonic balance method showing the occurrence of harmonics in the response spectrum which are strictly related to the bilinear nature of the model. Moreover, an experimental test carried out on a straight beam with a fatigue crack is presented. The beam is excited with white noise by means of a vibration shaker. The bispectral analysis performed both on the model and the actual structure shows high sensitivity to the non-linear behaviour of the system: that is to say, to the presence of the fatigue crack in the structure. In particular, the interactions between the frequency components contained in the signal response are analyzed. The results provide a possibility of using the bispectral analysis technique to detect damages in structures.

1. INTRODUCTION

The development of structural integrity monitoring techniques has received increasing attention in recent years. The techniques employed are based on vibration measurements, which offer an effective and fast means of detecting

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fatigue cracks in structures; therefore, they play an important role in ensuring the integrity of machine elements and structures.

The fatigue cracks exhibit non-linear behaviour. In fact, if an insufficient static preload is present, they open and close depending on the vibration direction, causing the time-variation of the physical system parameters, of which the stiffness is the most important. A range of research has been conducted into the area of cracks which always keep open during vibration. In these works, crack location and severity may be identified from alterations in natural frequencies and modes of vibration as well as amplitude of forced vibrations. Cawley and Adams [1] developed an experimental technique to identify the cracks from changes in natural frequencies. Gudmunson [2] used a first order perturbation method to predict the change in resonance frequencies of a structure due to cracks, notches or other geometrical changes. Rizos *et al*. [3] estimated the location and depth of a fatigue crack from the vibration modes and an analytical solution of the dynamic response obtained with the open crack hypothesis. Al-Qaisia and Meneghetti [4] proposed a method based on the sensitivity analysis for crack localization.

On the other hand, the modal parameters may change due to not only structural damage, but also other factors, such as temperature. Besides, many studies have illustrated that it is important to consider the crack closure. In the field of rotating machinery Dimarogonas and Papadopoulos [5] have obtained analytical solutions for the closing crack under the assumption of large static deflections. In reference [6] Gudmunson pointed out that the alteration in natural frequencies due to a real fatigue crack is much lower than the drop caused by the narrow notch. Ibrahim *et al*. [7] confirmed that result; they employed a bondgraph model to simulate the dynamic behaviour of a cantilever beam including a non-linear fatigue crack. Qian *et al*. [8] observed that the difference of the amplitude response between the integral and cracked beam is reduced if a closing crack model is considered. In reference [9] Shen and Chu showed the feasibility of using the spectrum pattern to identify the existence of fatigue cracks in structures. They derived a bilinear equation of motion for each vibration mode of a simply supported beam and applied the Galerkin procedure to simulate the behaviour of the cracked beam. A single-degree-of-freedom (SDOF) non-linear model of a cracked beam was proposed by Friswell and Penny [10]. Abraham and Brandon [11] presented a substructure model to predict the vibration properties of a beam with a breathing transverse crack. A mathematical model of a beam with a closing crack was developed by Krawczuk and Ostachowicz [12]. In their work the authors examined the possibility of crack detection on the basis of changes of the high harmonics in the frequency spectrum.

The aim of this paper is to inspect the vibration response of a beam with a fatigue crack by means of bispectral analysis, in order to detect the presence of the structural damage. According to the authors' knowledge, studies concerning the bicoherence approach to the crack detection have not been reported in the literature.

The structure of the paper is as follows. Firstly, to gain some insight into the response of a cracked beam, a simple SDOF model of a cracked beam is developed in section 2. The model is an oscillator with bilinear stiffness characteristics. Due to their simplicity, bilinear oscillators have often been used to model the dynamic behaviour of non-linear mechanical systems [13–15]. Their use could be extended to simulate the behaviour of structures with a fatigue crack. The basic concept is that the crack alternately opens and closes depending on the direction of the vibration. When the crack is closed the beam acts, approximately, as a homogeneous beam with no crack, while, when the crack is open, a local reduction of flexural rigidity occurs, that is to say, there is a reduced stiffness in the SDOF model [7, 8, 10, 12, 13, 16].

The non-linear equation of motion of the SDOF model can be transformed into a linear periodically time-variant equation by assuming a square wave function in order to model the beam stiffness [12]. The validity of this assumption is limited to low frequency excitation, but that is enough for the purpose of the present paper. The closed form steady state solution of the forced response of the bilinear oscillator is obtained by means of the harmonic balance method.

In section 3, some theoretical background of bispectral analysis, which forms a subset of Higher Order Statistical Analysis (HOSA), is summarized. Most traditional signal processing techniques are based on the second-order properties of the signal, such as power spectrum, variance and autocorrelation function. Since these measures are easy to implement, they are in widespread use. However, many real signals found in several fields contain information which is not visible in the second-order measures, such as seismic measurements, chaotic signals, and vibrational signals for the diagnosis of the rotating machine components, e.g., rolling bearings and gears. Therefore, the HOSA is an area of signal processing that is gaining importance and has a wide variety of practical applications. In particular, the third-order spectral analysis, namely bispectral analysis, has been applied in fields as different as vibration analysis, underwater acoustics, speech processing, chaos and condition monitoring [17–21]. Moreover, techniques based on HOSA are developed to study transient signals, to suppress Gaussian noise or to detect non-linearities. The bispectrum, that is the third-order spectrum, is the decomposition of the third moment (skewness) of a signal over frequency and as such can detect non-symmetric non-linearities. The bispectrum computation can be performed from a single signal measurement.

Finally, section 4 is focused on the application of the bispectral analysis. This signal processing technique is applied to the forced response of the bilinear model of a cracked beam: the bilinear oscillator is a system with non-symmetric non-linearity; therefore the bispectral technique is a proper method to analyze the model behaviour. The results show that the bispectral analysis technique has great sensitivity to the model non-linearity. Moreover, experimental tests are carried out on a straight beam. Two different conditions are observed: the integral beam and the beam with a fatigue crack; two depth values are inspected for the latter case. The beam is excited by means of a vibration shaker fed with white noise and an accelerometer is used to measure the beam response. The bispectral analysis is applied to the measured signal in order to detect the presence of a fatigue crack. From the results it is possible to say whether non-linearities occur in the system or, in other words, if the beam has structural damage or not. Since such detection criterion is not affected by environmental factors, the procedure may be used to improve the reliability and effectiveness of damage diagnostic techniques.

2. ANALYSIS OF A BILINEAR MODEL OF A CRACKED BEAM

2.1. FORMULATION AND SOLUTION

In this part of the work, a SDOF model of a cracked beam is described and studied. The aim is to analyze and understand the dynamic behaviour of a cracked beam based on a very simple model. The fundamental assumption is that the beam vibrates mainly in its first mode. The exactness of this assumption depends on the position and the kind of force exciting the structure. With this assumption, the equivalent stiffness of a beam in pure bending is given by [4]

$$k_{eq} = \int_0^L EI\left(\frac{\mathrm{d}^2\Phi(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x, \tag{1}$$

where L is the overall length, E is Young's modulus for beam material, I is the area moment of inertia of beam cross-section, x is the beam axial co-ordinate, and $\Phi(x)$ is the beam mode shape of the first eigenmode. In particular, from equation (1) one notes that the equivalent stiffness depends on the product EI, which is known as the beam bending stiffness.

Henceforth, one assumes that the crack does not affect the beam mass. For a crack in one edge of the beam, under the action of the excitation force, crack opening and closing will alternate. When the crack is in compression (closed) the equivalent stiffness may be regarded as that of a beam without a crack, due to the contact of the crack walls. On the other hand, when the crack is in tension (crack open) the beam stiffness will reduce near the crack. For the purpose of this work, the actual function of the beam bending stiffness along the beam, EI, is of no importance. The major remark is that the equivalent stiffness takes two different values depending on whether or not the crack is open. This depends on the sign of the curvature of the beam, that is to say, on the sign of the beam displacement (the beam vibrates only in one mode). The equivalent damping of the beam is assumed to be constant although the system has increasing damping due to the friction forces at the point of the crack closure.

In conclusion, the simple SDOF model of a cracked beam may be described by the equation of motion of the bilinear oscillator,

$$m\ddot{u} + c\dot{u} + \tilde{k}(u)u = f(t), \tag{2}$$

where $u(t)$ is the displacement of the oscillator, m is the mass, c the damping coefficient and $\tilde{k}(u)$ is a piecewise stiffness function defined as

$$\tilde{k}(u) = \left\{\begin{matrix} k & \text{if} & u < 0 \\ k\alpha & \text{if} & u > 0 \end{matrix}\right\}, \tag{3}$$

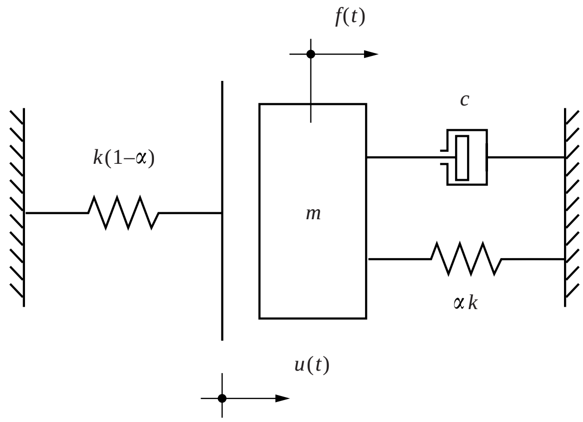


Figure 1. The bilinear model.

with $0 \leqslant \alpha \leqslant 1$. The parameter α is called the stiffness ratio. When the stiffness ratio is equal to one the system is linear. The force exciting the bilinear model is denoted by $f(t)$. A scheme of the bilinear oscillator is illustrated in Figure 1; the relationship between the restoring force and the displacement of the bilinear system is depicted in Figure 2. For the free undamped vibration of the oscillator, the period of the system is composed of two half sine waves, so that one can define the bilinear radian frequency ω_0, as [22]

$$\omega_0 = 2\omega_1\omega_2/(\omega_1 + \omega_2), \qquad \omega_1 = \sqrt{k/m}, \qquad \omega_2 = \sqrt{\alpha k/m}. \tag{4}$$

If the model is linear, that is, α is equal to one, a sinusoidal excitation will produce a sinusoidal response at the excitation frequency. On the contrary, the response is expected to contain several harmonics of the excitation frequency. In order to relate these harmonics to the stiffness ratio α, a solution for forced vibration is needed. Obtaining the closed form solution of equation (4) is quite difficult. Krawczuk and Ostachowicz [12] found the solution for a harmonic excitation force; they used a square-wave function $s(t)$, having the fundamental

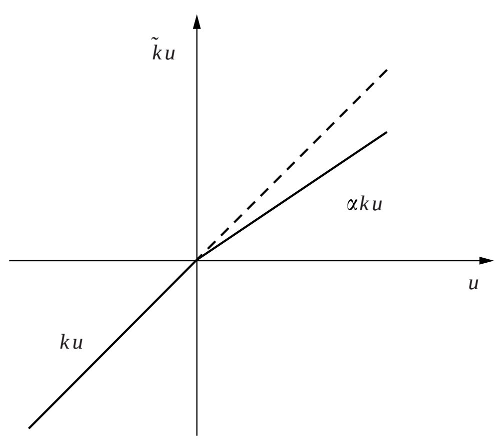


Figure 2. Piecewise restoring force.

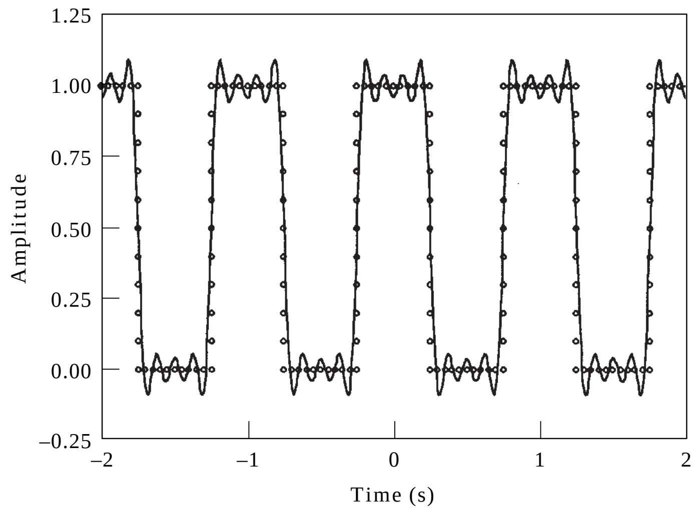


Figure 3. The square-wave function (oooo) and its Fourier series approximation with $n = -8, \ldots, 8$ (——).

frequency equal to the sinusoidal forcing frequency, in order to transform the non-linear equation (4) into a linear periodically time-variant equation:

$$m\ddot{u} + c\dot{u} + k[1 - (1 - \alpha)s(t)]u = f(t). \tag{5}$$

The square-wave function $s(t)$ changes between 0 and 1, and is represented in Figure 3; it can be approximated by the complex Fourier series

$$s(t) = \sum_{n=-\infty}^{\infty} q_n\, \mathrm{e}^{\mathrm{i}n\omega t}, \tag{6}$$

where i is the imaginary unit, ω is the fundamental frequency and q_n are appropriate constants defined by

$$q_n = \frac{(-1)}{n\pi}\cos\left[\frac{\pi}{2}(n+1)\right] \quad \text{if} \quad n \neq 0, \qquad q_n = \frac{1}{2} \quad \text{if} \quad n = 0. \tag{7}$$

In the case of a sinusoidal input force of amplitude f_0, if the square-wave function $s(t)$ has the fundamental frequency equal to the forcing frequency, the differential equation can be written in the form

$$m\ddot{u} + c\dot{u} + k\left[1 - (1-\alpha)\sum_{n=-\infty}^{\infty} q_n\, \mathrm{e}^{n\mathrm{i}\omega t}\right]u = \frac{f_0}{2}\mathrm{e}^{-\mathrm{i}\omega t} + \frac{f_0}{2}\mathrm{e}^{\mathrm{i}\omega t}. \tag{8}$$

Actually, to model the stiffness variation by using a square-wave function which has the fundamental frequency equal to that of the input force is not strictly exact. In fact, the non-linear behaviour of the system may cause distortion in the response due to the high harmonic components with respect to ω. When the amplitude of these components is significant, they should influence the stiffness variation; on the other hand, the square-wave function defined in equation (6) imposes that the stiffness changes once per fundamental period $2\pi/\omega$. Therefore, the validity of this solution method is limited to low frequency excitation. As a matter of fact, in this case, one can neglect the effect of the high harmonics that are contained in the response. In particular, the frequency of the excitation force should be much lower than the bilinear frequency ω_0, as demonstrated in reference [13] by the authors who handle the undamped forced vibrations.

For the purpose of this paper, it is interesting to study the response of the bilinear oscillator, despite the above limitation. A particular solution of equation (8) is periodic with principal period $2\pi/\omega$ and is given by the complex Fourier series:

$$u(t) = \sum_{n=-\infty}^{\infty} b_n \, \mathrm{e}^{\mathrm{i}n\omega t}, \tag{9}$$

where the coefficients b_n satisfy the condition $\overline{b}_n = b_{-n}$ ($\overline{b}_n$ denotes the complex conjugate of b_n). Equation (8) can then be written as

$$\sum_{n=-\infty}^{\infty} \left[b_n(-mn^2\omega^2 + \mathrm{i}cn\omega + k) - k(1-\alpha) \sum_{j=-\infty}^{\infty} q_{n-j} b_j \right] \mathrm{e}^{\mathrm{i}n\omega t} = \frac{f_0}{2}\,\mathrm{e}^{-\mathrm{i}\omega t} + \frac{f_0}{2}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{10}$$

By applying the harmonic balance approach or, in other words, equating the coefficients of the term $\mathrm{e}^{\mathrm{i}n\omega t}$, equation (10) can be transformed into the set of linear algebraic equations

$$\mathbf{Ab} = \mathbf{F}, \tag{11}$$

where

$$A_{n,j} = (-mn^2\omega^2 + \mathrm{i}cn\omega + k)\delta_{n,j} - k(1-\alpha)q_{n-j}, \tag{12}$$

$$\mathbf{b} = \begin{Bmatrix} b_{-N} \\ \vdots \\ b_{-2} \\ b_{-1} \\ b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}, \qquad \mathbf{F} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ f_0/2 \\ 0 \\ f_0/2 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}. \tag{13}$$

Here $\delta_{n,j}$ is the Kronecker delta and N is the value to which all the infinite sums are truncated: that is to say, the number of harmonics which are considered in the

TABLE 1

Constant parameter of the bilinear model

Mass	$m = 1$ kg
Damping coefficient	$c = 2{\cdot}75$ Ns/m
Stiffness	$k = 5 \times 10^4$ N/m
Force amplitude	$f_0 = 1$ N
Fundamental radian frequency	$\omega = 10\pi$ rad/s

computation. An analytical or numerical solution of the system (11) can be easily obtained by means of a computer program.

2.2. NUMERICAL RESULTS

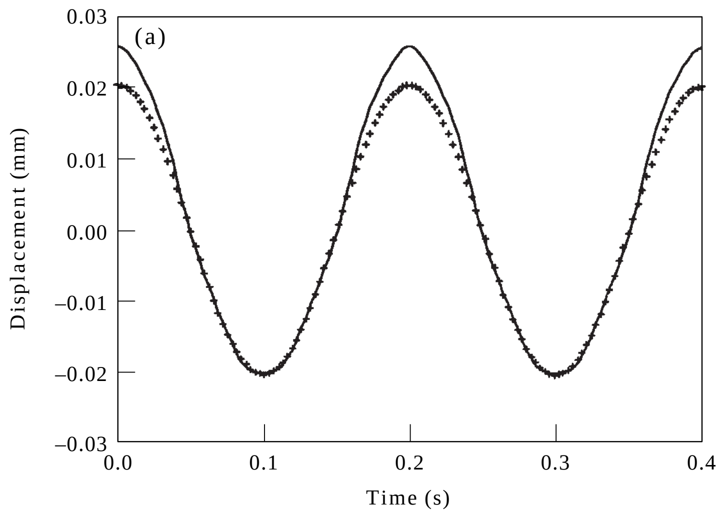
It is interesting to study the response of the bilinear model in order to analyze the effect of the stiffness ratio α on the system behaviour, or, in other words, the effect of the presence of a fatigue crack on the beam response. For this purpose, one can consider the following numerical example where the stiffness ratio range is between 1·0 and 0·8, and the constant parameters are summarized in Table 1. Values of the parameter α lower than 0·8 are not taken into account because they are related to a strong non-linearity of the system: that is to say, an exaggerated structural damage would injure the beam. The input force is sinusoidal. The bilinear radian frequency, ω_0, of the oscillator changes from 223·6 rad/s ($\alpha = 1{\cdot}0$) to 211·4 rad/s ($\alpha = 0{\cdot}8$).

Table 2 shows the modulus of the complex coefficients b_n ($n = 1, \ldots, 6$) for different values of the stiffness ratio α. Obviously, when $\alpha = 1{\cdot}0$, only b_1 has a non-zero value, whilst the others have values different from zero only if the stiffness ratio is lower than one. Besides, one notes that, excepting the case $n = 1$, the coefficients b_n have significant magnitude when n is even. Moreover, the modulus of the constants b_n increases when the stiffness ratio decreases, showing that the superharmonic terms ($n > 1$) are strongly connected to the value of α.

TABLE 2

Coefficients b_n ($n = 1, \ldots, 6$) of the complex Fourier series of the bilinear model's response for several values of the stiffness ratio

Stiffness ratio α	$\lvert b_1\rvert \times 10^6$ (m)	$\lvert b_2\rvert \times 10^6$ (m)	$\lvert b_3\rvert \times 10^9$ (m)	$\lvert b_4\rvert \times 10^6$ (m)	$\lvert b_5\rvert \times 10^9$ (m)	$\lvert b_6\rvert \times 10^6$ (m)
1·000	10·2014	0·0000	0·0000	0·0000	0·0000	0·0000
0·975	10·3348	0·06035	0·009092	0·01633	0·03100	0·01696
0·950	10·4755	0·1240	0·04343	0·03372	0·1569	0·03602
0·925	10·6240	0·1914	0·1164	0·05228	0·4433	0·05753
0·900	10·7809	0·2626	0·2463	0·07210	0·9860	0·08197
0·875	10·9469	0·3382	0·4581	0·09331	1·9252	0·1099
0·850	11·1230	0·4183	0·7862	0·1160	3·4681	0·1421
0·825	11·3101	0·5036	1·2785	0·1403	5·9256	0·1796
0·800	11·5092	0·5944	2·003150	0·1663	9·7736	0·2238



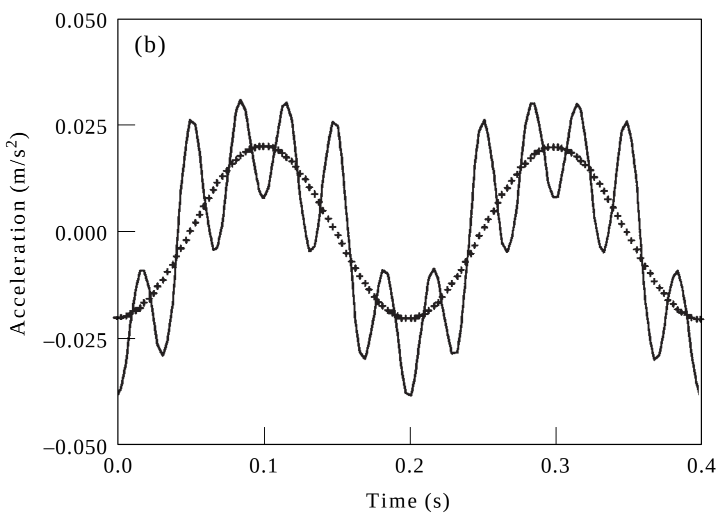


Figure 4. Comparison between time-histories of the bilinear's oscillator response in the linear case (+ + + +) and when $\alpha = 0{\cdot}8$ (——): (a) displacement; (b) acceleration.

Figure 4 shows the oscillator response for the linear case ($\alpha = 1{\cdot}0$) and when there is a stiffness drop corresponding to $\alpha = 0{\cdot}8$; both displacement and acceleration are presented.

Clearly, each constant b_n represents the corresponding term in the spectrum of the response of the bilinear model. Since the constants b_n are related to the stiffness ratio, it seems reasonable to employ the spectrum of the forced response in order to detect the presence of the non-linearity. In this sense, when considering the actual case, the spectrum may be used to detect the structural damage in a beam. In reference [13] the authors asserted that such a procedure may be used to diagnose damaged structures. However, from Table 2, one can see that the magnitude of the superharmonic components is very small when the stiffness ratio is close to one. Consequently, in the case of a beam, the spectrum of the experimental response could not be able to detect a crack at the early stage of development, because the harmonic components might be obscured, for example, by the noise. Therefore, in order to acquire more sensitivity to the system non-linearity, a more sophisticated signal processing technique is needed. The following sections will deal with this subject.

3. BISPECTRAL ANALYSIS

In the case of a stationary signal with a Gaussian probability density function (PDF), the statistical properties of the signal are completely described by the signal's second order statistics, namely its mean, μ, and correlation function $R(\tau)$ [23]. On the other hand, if the signal is non-Gaussian then higher order moments are needed to describe its PDF completely. In other words, higher order measures, such as skewness (third-order) and kurtosis coefficients (fourth-order), may provide details about the signal which the conventional second-order statistics cannot. Moreover, since the signals encountered in practice are often contaminated by Gaussian measurement noise, to which higher order measures are theoretically insensitive, HOSA are potentially powerful tools for analyzing real-life signals.

In the frequency domain, the decomposition over frequency of the signal power (which is obviously related to the signal variance σ^2) is the power spectrum. In the field of HOSA, the concept of the power spectrum is extended to higher orders. The extensions are called polyspectra; in particular, the third-order polyspectrum is termed the bispectrum, the fourth-order is the trispectrum, and so on. This paper deals only with the third-order polyspectrum, more specifically, the bispectrum, which can be viewed as a decomposition of the skewness of a signal over frequency, so that it may be used to detect non-symmetric non-linearities [24]. The bispectrum provides supplementary information to the power spectrum which is of limited value in studying vibrations where non-linearities are involved. For a discrete time series, the discrete bispectrum $B(k, l)$ can be defined in terms of the signal's Discrete Fourier Transform (DFT) $X(k)$ as [18]

$$B(k, l) = E[X(k)X(l)X^*(k + l)], \tag{14}$$

where $E[\,]$ denotes the expectation operator [23]. It should be noted that the bispectrum is complex-valued (it contains phase information) and is a function of two independent frequencies, k and l. Furthermore, it is not necessary to compute $B(k, l)$ for all (k, l) pairs, due to several symmetries existing in the bifrequency plane (k, l). In particular there exists a non-redundant region called the Principal Domain which is defined as

$$\{k, l\}: 0 \leqslant k \leqslant f_s/2, \qquad l \leqslant k, \qquad 2k + l \leqslant f_s, \tag{15}$$

where f_s is the sampling frequency [17, 18]. Such a region is analogous to the domain of the power spectrum which is the frequency range between 0 and $f_s/2$.

Since the bispectrum is closely related to the third-order moment of a signal, if a signal has a symmetric PDF (the skewness is zero) then it has zero bispectrum. Moreover, equation (14) asserts that the bispectrum at the bifrequency (k, l) is made up from the DFT at the three frequencies k, l, and $k + l$. Therefore, a peak in the bispectrum measures the amount of coupling between these three spectral components. When the frequency components couple in this way, it means that there exists a quadratic non-linearity in the signal [20]. From this point of view, one can say that the bispectrum is sensitive to the signal skewness and may be employed as a detector of a particular type of non-linearity. Bispectral analysis often does not deal with the bispectrum, as given by equation (14), but with a normalized form of the bispectrum, e.g., the bicoherence, $b^2(k, l)$, which is defined by [20]

$$b^2(k, l) = \frac{|E[X(k)X(l)X^*(k+1)]|^2}{E[|X(k)X(l)|^2]E[|X(k+1)|^2]}. \tag{16}$$

The reason for normalizing the bispectrum is due to the fact that this estimator has a variance which is proportional to the triple product of the power spectra, which can result in the second-order properties of the signal dominating the estimate [20]. The advantage of normalization is to make the variance of the estimator approximately flat across all bifrequencies. An important feature of the bicoherence is that it is always restricted to vary between 0 and 1 [17, 18]. There are other methods to normalize the bispectrum which have a variance flatter than that of the bicoherence defined by equation (16), but they are not bounded between 0 and 1 [17, 24]:

The bicoherence, as well as the bispectrum, can be computed from a signal by dividing it into K segments, applying an appropriate window to each segment to reduce leakage, computing the quantities in equation (14) and (16) for each segment by using the DFT, and then obtaining the statistical estimate by averaging over all segments. For stochastic processes, segment averaging is used in order to achieve a consistent estimate. Averaging procedures are also advisable in the case of deterministic signals, because they reduce the influence of the background noise. Accordingly, the bispectrum and the bicoherence can be calculated by using the following estimators [18]:

$$\hat{B}(k, l) = \sum_{j=1}^{K} X_j(k)X_j(l)X_j^*(k+l), \tag{17}$$

$$\hat{b}^2(k, l) = \frac{\left|\sum_{j=1}^{K} X_j(k)X_j(l)X_j^*(k+1)\right|^2}{\sum_{j=1}^{K} |X_j(k)X_j(l)|^2 \sum_{j=1}^{K} |X_j(k+l)|^2}. \tag{18}$$

The bicoherence's main disadvantage is the data length needed for the computation. In fact, it is recommended that the segment length is the square root of the total size of the signal [25]. The bicoherence quantitatively measures the fraction of the power of the signal due to the quadratic interaction between frequency components and contains useful information about the skewness of the signal, or, in other words, it is sensitive to the non-symmetric non-linearities of the signal. It is important to observe that the bicoherence is only a normalization of the bispectrum and it should not be confused with the second-order coherence function. In addition, it is noteworthy that the second-order coherence function is also employed to detect deviations from linearity of a system, but its computation requires two measurement sensors; on the contrary, the bicoherence can be computed from a single sensor measurement.

4. USE OF THE BISPECTRAL ANALYSIS FOR CRACK DETECTION

In section 2 it was pointed out that proper signal processing techniques are needed in order to detect system non-linearities. The required techniques should be sensitive to the non-linear behaviour of the system. When handling an unknown system, it is possible to say whether or not the system is linear by studying its response to an input excitation. As a matter of fact (see section 2), the non-linearities make a significant contribution to the output signal. Therefore, it is possible to identify a non-linear system by looking for specific relations between the frequency components of the output signal of the system, or, in addition, by inspecting the higher order moments of the response signal such as skewness and kurtosis coefficients.

In section 3 it was shown that the bispectrum analysis is sensitive to non-symmetric non-linearities and to the coupling of the spectral components. Consequently, it seems to be useful to gain some insight into the dynamic behaviour of the bilinear system treated in section 2. If the non-linearity of that SDOF model can be found by means of the bispectral analysis, the detection of cracks in structures by this signal processing technique appears to be possible.

4.1. BISPECTRAL ANALYSIS OF THE BILINEAR OSCILLATOR

With reference to the numerical example of section 2, the forced response of the model is evaluated for three different values of the stiffness ratio α ($\alpha = 1{\cdot}0$, $\alpha = 0{\cdot}95$ and $\alpha = 0{\cdot}9$). The acceleration is used because it has higher amplitude for the high frequency components. For each of the three values of the parameter α, a discrete time series $\ddot{u}_j$ is generated by employing the expression

$$\ddot{u}_j = \sum_{n=-N}^{N} (-n\omega^2) b_n \, \mathrm{e}^{\mathrm{i}n\omega j \Delta t}, \qquad j = 0, \ldots, P-1, \tag{19}$$

where P is the number of points of the series and Δt is the time step ($P = 4096$ and $\Delta t = 0{\cdot}0125$ s are used in the simulation). The coefficients b_n are computed by solving equation (11); only six harmonic components are considered: i.e., $N = 6$. The three time-histories of the bilinear's oscillator response are compared in Figure 5.

In order to simulate a real situation, random noise is added to the signals of Figure 5 and the power spectrum from 32 averages is computed for each signal. The signal to noise ratio (SNR $= 20 \log_{10} \sigma_s^2/\sigma_N^2$) is equal to 15 dB. Figure 6 shows the power spectral changes for decreasing values of α; in particular, as already

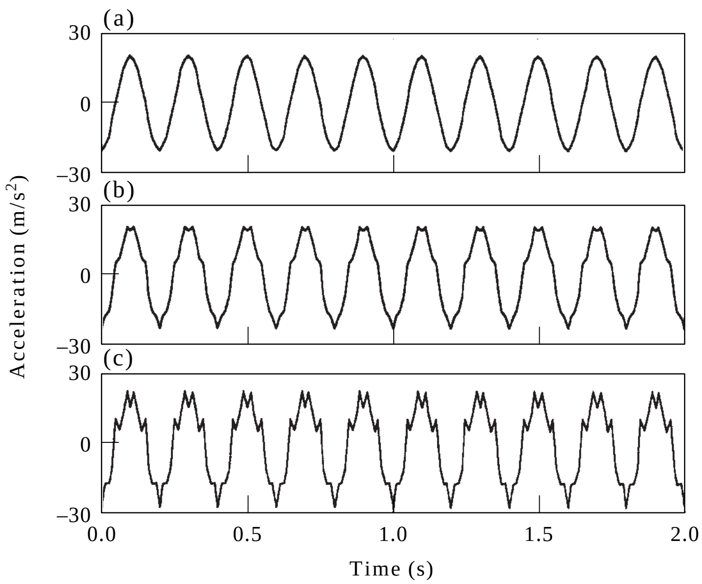


Figure 5. Comparison of the acceleration response of the bilinear oscillator for different values of stiffness ratio: (a) $\alpha = 1{\cdot}0$; (b) $\alpha = 0{\cdot}95$; (c) $\alpha = 0{\cdot}9$.

discussed in section 2, harmonic components of the frequency excitation rise for low values of the stiffness ratio.

Since bispectral analysis perceives the interactions between the spectral components, it is also expected to obtain interesting information from the bispectrum of the simulated signals. The normalized version of the bispectrum, that is the bicoherence defined by equation (16), is computed and the results are

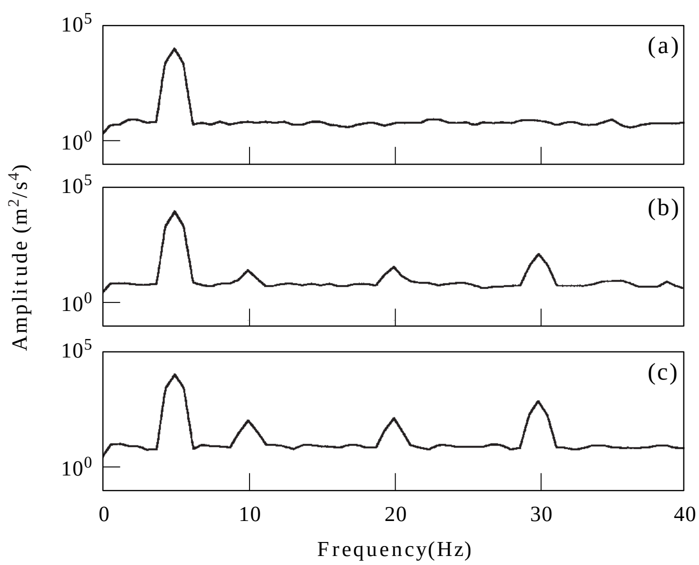


Figure 6. Power spectra of the signals of Figure 5 with noise (SNR = 15 dB): (a) $\alpha = 1{\cdot}0$; (b) $\alpha = 0{\cdot}95$; (c) $\alpha = 0{\cdot}9$.

in Figure 7. Only a small region of the bifrequency plane is considered because of symmetry relations. The bicoherence is obtained from 64 averages and a Hamming window is applied in the bicoherence computation since experience suggests this gives the best results [18].

When the system is linear ($\alpha = 1{\cdot}0$) the bicoherence is approximately flat across the bifrequency plane; the maximum of the bicoherence is about 0·06. On the other hand, as soon as the stiffness ratio decreases, that is to say, the system becomes non-linear, the bicoherence shows drastic deviations from the even shape. In the cases of $\alpha = 0{\cdot}95$ and $\alpha = 0{\cdot}9$ the bicoherence has a maximum of 0·67 and 0·84, respectively. Therefore, the bicoherence appears to be very sensitive to the presence of non-linearities.

Moreover, Figure 8 presents the contour plot of the bicoherence in the case of $\alpha = 0{\cdot}9$, in order to underline the dependencies between the spectral components. In particular, the bicoherence peak at the frequency pair (5, 5) Hz indicates a coupling between frequency components at the triplet (5, 5, 10) Hz. Analogously, there is a significant peak at the frequency pair (20, 10) Hz showing an interaction between the components at 20, 10 and 30 Hz. As a matter of fact, since the system response shows several components as having frequency which is a multiple of the forcing frequency, the above occurrence is particularly evident in the bicoherence which is sensitive to the coupling of the components.

4.2. EXPERIMENTAL CRACK DETECTION BY BISPECTRAL ANALYSIS

The simple SDOF bilinear model presented in section 2 simulates the dynamic behaviour of a beam with a closing crack. Since the bispectral analysis is able to identify non-linearities of the SDOF model, the expectation is that such signal processing technique will show the same capability for the diagnostics of fatigue cracks in beams.

An experimental test was carried out on a beam with structural damage. The system under consideration is a 450-mm long straight beam with a uniform 10×10 mm cross-section. It is made of cold-drawn B8 UNI 2527 bronze. The beam is supported and excited by a vibration shaker (see Figure 9). The acceleration at the free end of the beam (point A in Figure 9) is measured by means of a Brüel & Kjaer (B&K) 4393 accelerometer and a charge preamplifier B&K 2635; the signal is converted to a digital time-series using a DIFA Scadas unit which has a 12-bit ADC with variable sampling frequency and is driven by the LMS-Cada-X software. The analysis is limited to the frequency range 0–750 Hz where the beam has four modes measured at about 23, 170, 420 and 503 Hz. Actually, within the examined frequency range, the clamped beam should have only three resonance frequencies, but the structure consists of a beam and supporting fixture which introduces another mode of vibration.

The tests were performed in two different conditions: when the beam is integral and with a fatigue crack at 300 mm from the free end A. The crack was initiated by denting a V 0·5-mm deep notch. It was propagated on a RUMUL MIKROTRON 654 high frequency fatigue testing machine. The beam was mounted in a three-point bending fixture and was loaded by applying a static force, $Q_{st} = 1100$ N, and a dynamic force of amplitude $Q_{dyn} = 800$ N; the distance

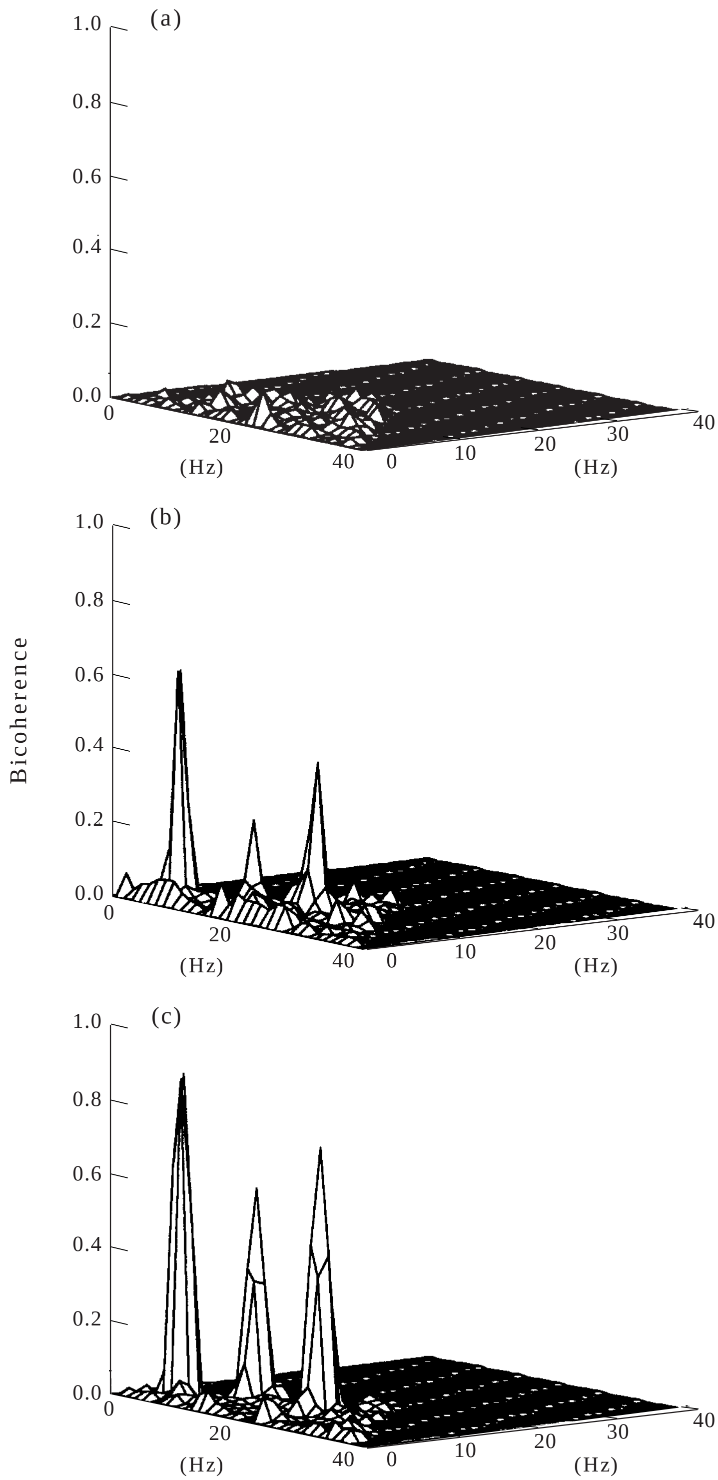


Figure 7. Bicoherence of the signals of Figure 5 (noise is present with SNR = 15 dB): (a) $\alpha = 1{\cdot}0$; (b) $\alpha = 0{\cdot}95$; (c) $\alpha = 0{\cdot}9$.

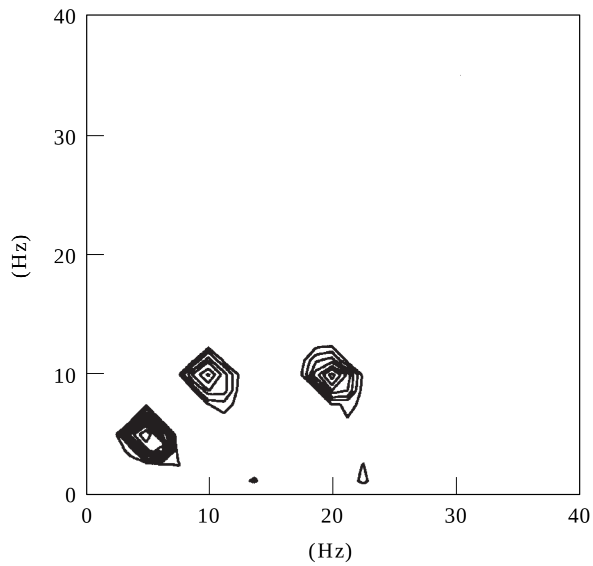


Figure 8. Contour plot of the bicoherence of Figure 7(c); stiffness ratio $\alpha = 0{\cdot}9$.

between the two rollers of the bending fixture was 60 mm (see the scheme of Figure 10). With reference to Figure 9, the severity of the crack is expressed in terms of the ratio between its depth, a, and the height of the beam, h. For a cracked beam, the vibration measurements were conducted on the beam having a crack with severities $a/h = 0{\cdot}0$, that is no crack is present, $a/h = 0{\cdot}4$, and $a/h = 0{\cdot}6$.

In order to detect the presence of the crack, the beam was excited with Gaussian white noise and the bicoherence of the measured acceleration response was computed. As a matter of fact, if a signal with a Gaussian PDF operates on a linear system, the resulting output will be Gaussian. On the contrary, the output is non-Gaussian for a non-linear system. The analysis of the SDOF model of a cracked beam has shown that such a system has a non-linearity which is of

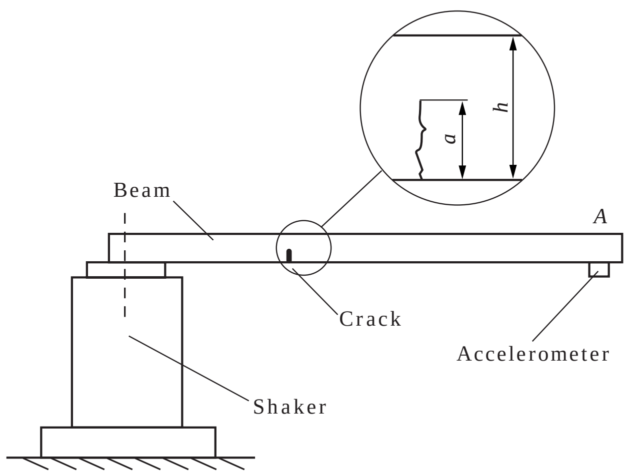


Figure 9. Set-up of the experimental apparatus.

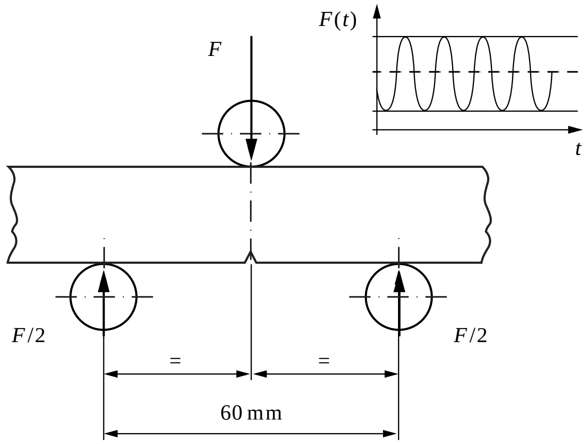


Figure 10. Scheme of the three-point bending fixture.

quadratic form, that is, non-symmetric. Therefore, the bicoherence, that takes non-zero values only in the case of signals which have a non-symmetric PDF, should be able to extract information regarding the presence of a fatigue crack in the beam. In addition, experience suggests that random excitation helps the opening and closing of the crack.

Figure 11 shows the bicoherence of the vibration signal of the beam for the three different conditions mentioned above; only the principal domain is plotted. The bicoherence is computed from $K = 128$ averages (the total signal length is 16 384); a Hamming window was applied. It can be seen that the bicoherence is very sensitive to the presence of the damage: the bicoherence rises in the case of a cracked beam, thus showing a significant coupling between frequencies due to the system non-linearities.

Since the non-redundant region of the bicoherence computed with a 128-point DFT contains over 1000 points, it is sensible to process the bicoherence further to reduce the number of data for the diagnostic task. For this purpose, some features are extracted from the principal domain of the bicoherence and are summarized in Table 3. In particular, Table 3 indicates that the maximum and the kurtosis coefficients [23] of the bicoherence are very sensitive to the presence of the structural damage. For example, the ratio of the kurtosis value in the case of crack severity $a/h = 0{\cdot}4$ to the corresponding value for the integral beam is about 3·7. Moreover, these features provide a measure of the crack depth.

Some statistical parameters are also evaluated for the acceleration signals in the time domain (see Table 4); in this case they barely change and, consequently, they do not give clear information about the beam condition. For instance, the trend of the skewness coefficient may cause an ambiguous interpretation of the results. Therefore, one can assert that the bicoherence could be employed to implement an automatic fault detection procedure.

To compare the results of the bispectral analysis to those of the traditional linear spectral analysis, the power spectrum of the beam acceleration response was computed as well. The power spectrum was obtained by dividing the signals into

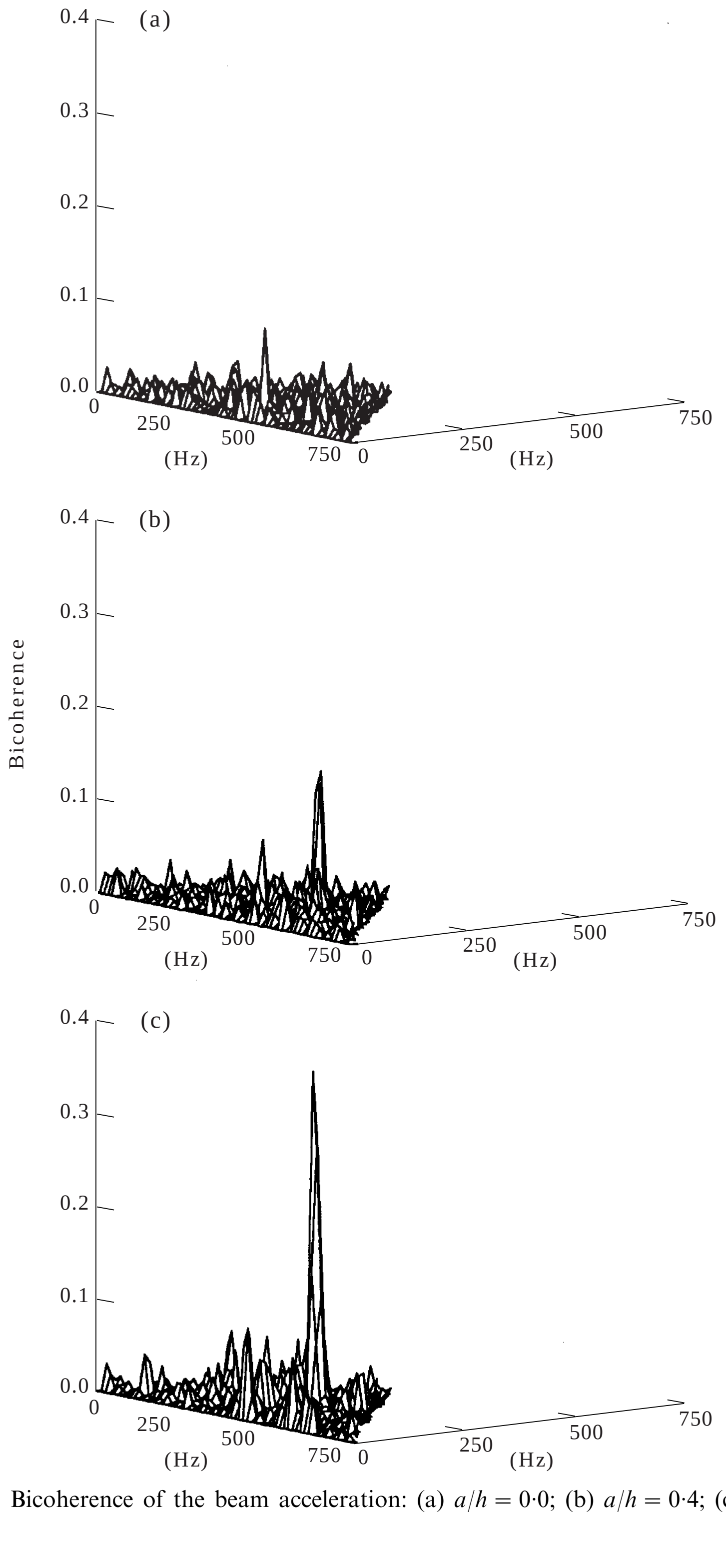


Figure 11. Bicoherence of the beam acceleration: (a) $a/h = 0{\cdot}0$; (b) $a/h = 0{\cdot}4$; (c) $a/h = 0{\cdot}6$.

TABLE 3

Results from the experimental test: bicoherence features computed over the non-redundant region

Crack severity a/h	Bicoherence	
	Maximum	Kurtosis
0·0	0·09816	17·1361
0·4	0·1524	62·7457
0·6	0·3644	114·8211

16 segments which have 1024 points each. Figure 12 shows that the presence of the crack causes only small changes in the power spectrum, but it does not provide much information about the nature of the system. In particular, it can be seen that resonant frequencies decrease due to the presence of the structural damage; the decrement is especially evident for the third and fourth beam's resonances.

In the case of crack severity $a/h = 0{\cdot}6$, by inspecting the power spectra carefully, it is possible to better interpret the results achieved by means of the bispectral analysis. In fact, the power spectrum of Figure 12(c) shows a weak peak at about 570–575 Hz which is not visible in the case of the integral beam. Such a peak is due to the non-linear behaviour of the cracked beam; in other words, the system resonances interact causing the birth of new components. The phenomenon scarcely appears in the power spectrum, but it is clearly evident in the bicoherence. As a matter of fact, in the bicoherence's contour plot shown in Figure 13, there exists a peak at the frequency pair (167, 406) Hz that indicates a coupling between frequency components at the triplet (167, 406, 573) Hz, that is, between the second (167 Hz) and the third (406 Hz) beam's resonance, and the small peak of the power spectrum at about 573 Hz. Figure 13 shows other couplings between spectral components, for example the peak at the pair (167, 167) Hz means that there is a dependency between the three frequency components 167, 167 and 334 Hz. The latter occurrence is not visible in the power spectrum of Figure 12(c). In

TABLE 4

Results from the experimental test: statistical parameters of the beam acceleration response

Crack severity a/h	Acceleration		
	Skewness	Kurtosis	Standard deviation (m/s^2)
0·0	−0·01306	3·2292	265·1712
0·4	0·0007689	3·1860	264·2556
0·6	0·04865	3·06775	254·7773

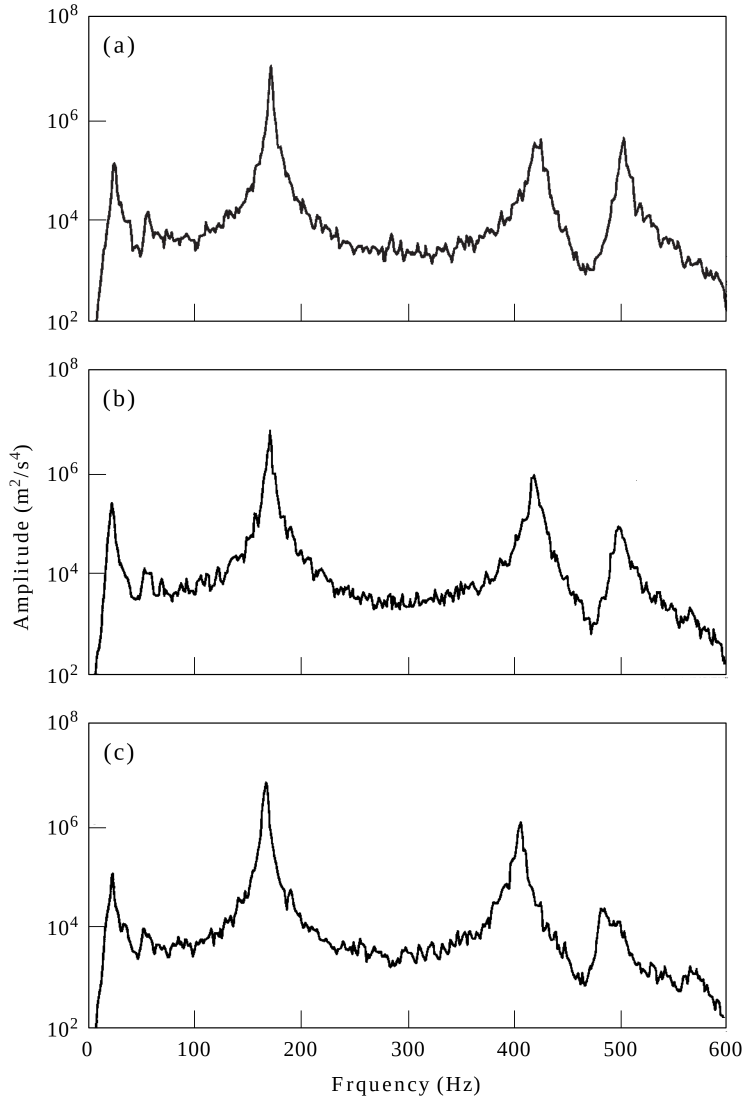


Figure 12. Power spectrum of the beam acceleration: (a) $a/h = 0{\cdot}0$; (b) $a/h = 0{\cdot}4$; (c) $a/h = 0{\cdot}6$.

conclusion, although it is not very easy to interpret the bicoherence data, this higher order signal processing technique gives more information than the linear spectral analysis.

5. CONCLUSIONS

In order to analyze the dynamic behaviour of a beam with a closing crack, a simple SDOF model of such a system has been studied. The model forced response was obtained in closed form by the harmonic balance method for the low excitation frequency range. The bispectral analysis, which is a new signal processing technique, was employed to inspect the model's non-linearity.

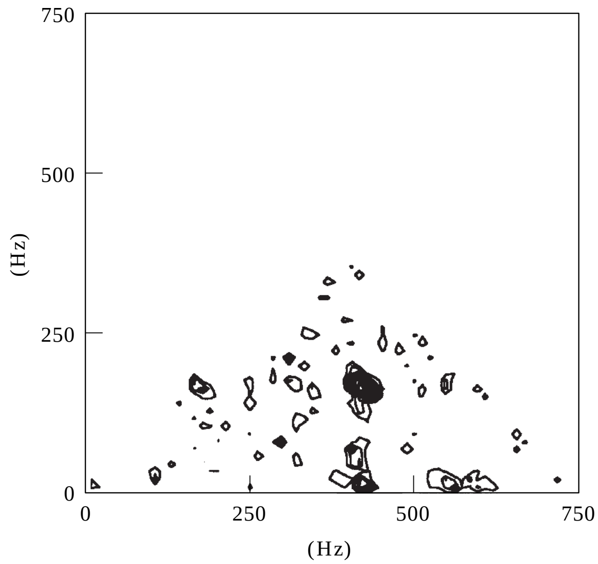


Figure 13. Contour plot of the bicoherence of Figure 11(c); crack severity $a/h = 0{\cdot}6$.

In particular, the normalized version of the bispectrum, namely the bicoherence, shows high sensitivity to the bilinear nature of the model. Moreover, the bicoherence measure has been adopted as a tool for detection of a fatigue crack in a straight beam. The bicoherence measure can be evaluated from a single sensor measurement and can be estimated in quite an easy way. From the bicoherence it is possible to say whether or not non-linearities occur in the system or, in other words, this signal processing methodology is able to detect the presence of structural damage in beams. In addition, the discussed detection criterion is not affected by environmental factors, therefore this procedure may be used to improve the reliability and effectiveness of damage diagnostic techniques. Although there are still some difficulties in interpreting information contained in the bicoherence, the presented study shows that there is scope for using the bicoherence as a condition monitoring tool.

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