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ON THE EIGENCHARACTERISTICS OF A LONGITUDINALLY VIBRATING ROD RESTRAINED BY A LINEAR SPRING IN-SPAN

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1. INTRODUCTION

Recently, in the context of a project study, a need was felt to gain insight into how the sensitivity of the eigenfrequencies of longitudinally vibrating elastic rods is restrained by a linear spring in-span with respect to small changes of the spring attachment point. As the corresponding frequency equation could not be found in the technical literature, except for the case when the spring acts at the tip [1], it was necessary to derive the frequency equation first. Afterwards, the mode shapes are given and finally a sensitivity formula is established. It is not claimed that the results presented in this letter have extreme originality. But, the authors have the opinion that the expressions derived here can be useful for a design engineer who is interested in the eigencharacteristics of similar systems and their sensitivity.

2. THEORY

The problem to be investigated in the present note is the natural vibration problem of the system shown in Figure 1. It consists of a fixed–free axially vibrating elastic rod, which is restrained by a linear spring in-span. The length, mass per unit length, location of the spring attachment point, axial rigidity and the stiffness coefficient of the spring are L, m, ηL, EA and k, respectively. The equations of the longitudinal motion of the two-rod portions are the well-known partial differential equations

$$EA\, \partial^2 u_i(x, t)/\partial x^2 = m\, \partial^2 u_i(x, t)/\partial t^2, \qquad (i = 1, 2), \tag{1}$$

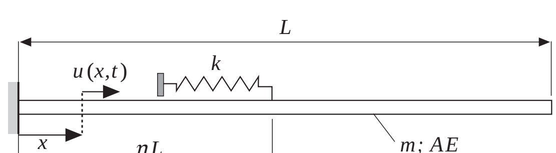

Figure 1. Longitudinally vibrating elastic rod restrained by a linear spring in-span.

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where $u_1(x, t)$ and $u_2(x, t)$ denote the axial displacements of the rod portions to the left and right of the spring attachment at x and time t. The corresponding boundary and matching conditions are:

$$u_1(0, t) = 0, \qquad u_1(\eta L, t) = u_2(\eta L, t),$$

$$EAu_1'(\eta L, t) - EAu_2'(\eta L, t) + ku_1(\eta L, t) = 0, \qquad u_2'(L, t) = 0. \tag{2}$$

Here, prime denotes the partial derivatives with respect to the position co-ordinate x. Using the standard method of separation of variables one assumes

$$u_i(x, t) = U_i(x) \cos \omega t, \qquad (i = 1, 2), \tag{3}$$

where $U_i(x)$ are the corresponding amplitude functions of the rods and ω is the unknown eigenfrequency of the vibrating system. Substituting these into equations (1) results in the following ordinary differential equations

$$U_i''(x) + \beta^2 U_i(x) = 0, \qquad (i = 1, 2), \tag{4}$$

where

$$\beta^2 = m\omega^2/EA \tag{5}$$

is introduced.

Substitution of (3) into (2) yields the corresponding boundary and matching conditions for the amplitude function $U_i(x)$:

$$U_1(0) = 0, \qquad U_1(\eta L) = U_2(\eta L),$$

$$EAU_1'(\eta L) - EAU_2'(\eta L) + kU_1(\eta L) = 0, \qquad U_2'(L) = 0. \tag{6}$$

The general solutions of the ordinary differential equations (4) are simply

$$U_1(x) = c_1 \sin \beta x + c_2 \cos \beta x, \qquad U_2(x) = c_3 \sin \beta x + c_4 \cos \beta x, \tag{7}$$

where c_1–c_4 are four integration constants to be evaluated via conditions (6). The application of these conditions to the solutions (7) yields a set of four homogeneous equations for the four unknown constants $c_1, \ldots, c_4$.

A non-trivial solution of this set of equations is possible only if the characteristic determinant of the coefficients vanishes. This condition leads to the equation

$$\overline{\beta} \cos \overline{\beta} + \alpha_k \sin \eta\overline{\beta} \cos [(1 - \eta)\overline{\beta}] = 0, \tag{8}$$

where

$$\overline{\beta} = \beta L, \qquad \alpha_k = k/EA/L. \tag{9}$$

The equation (8) is the frequency equation of the system shown in Figure 1. Its numerical solution yields the dimensionless eigenfrequency parameters $\overline{\beta}$, which then given via (5) the unknown eigenfrequencies ω of the system.

It is interesting to consider the special case $\eta = 1$, which corresponds physically to the case that the restraining spring acts at the free end of the rod. In this case, the frequency equation above reduces to the simple form

$$\overline{\beta} \cos \overline{\beta} + \alpha_k \sin \overline{\beta} = 0, \tag{10}$$

which is also given in reference [1] in a different notation.

The interest here lies not only in obtaining the eigenfrequencies of the system, but also in the mode shapes of the system. Considering that c_2 vanishes, and choosing $c_1 = 1$ arbitrarily the set of homogeneous equations mentioned above yields

$$c_3 = 1 + (\alpha_k/2\overline{\beta}) \sin 2\eta\overline{\beta}, \qquad c_4 = -(\alpha_k/\overline{\beta}) \sin^2 \eta\overline{\beta}, \tag{11}$$

such that the mode shapes are obtained from equations (7) in the forms

$$U_1(\bar{x}) = \sin \overline{\beta}\bar{x}, \qquad U_2(\bar{x}) = \left(1 + \frac{\alpha_k}{2\overline{\beta}} \sin 2\eta\overline{\beta}\right) \sin \overline{\beta}\bar{x} - \frac{\alpha_k}{\overline{\beta}} \sin^2 \eta\overline{\beta} \cos \overline{\beta}\bar{x} \tag{12}$$

where the non-dimensional position co-ordinate $\bar{x} = x/L$ is introduced.

The final aim is to give in the following a formula for the sensitivity of the eigenfrequencies of the system in Figure 1 with respect to small changes in the location of the in-span spring attachment point around its nominal position, i.e., the rate of change of the eigenfrequencies with respect to the location parameter η. To this end, the frequency equation (8) has to be differentiated partially with respect to η. This operation yields after some rearrangements

$$\partial\overline{\beta}/\partial\eta = -\alpha_k\overline{\beta}(p/q), \tag{13}$$

where the following abbreviations are introduced.

$$p = \cos [(1 - 2\eta)\overline{\beta}],$$
$$q = \cos \overline{\beta} - \overline{\beta} \sin \overline{\beta} + \alpha_k\{\eta \cos [(1 - 2\eta)\overline{\beta}] - \sin \eta\overline{\beta} \sin [(1 - 2\eta)\overline{\beta}]\}. \tag{14}$$

It is now possible to give an approximate formula for the modified value $\overline{\beta}_{mod}$ of a non-dimensionalized eigenfrequency if the location of the spring attachment point is changed slightly by an amount $\Delta\eta$

$$\overline{\beta}_{mod} \approx \overline{\beta}(\eta) + (\partial\overline{\beta}/\partial\eta)\Delta\eta. \tag{15}$$

3. NUMERICAL APPLICATIONS

This section is devoted to the numerical evaluation of the formulae established in the proceeding section. The numerical solution of the frequency equation and production of the mode shapes were carried out using MATLAB. The first two dimensionless eigenfrequency parameters $\overline{\beta}_1$ and $\overline{\beta}_2$ are given in Table 1 for various values of the location and stiffness parameters η and α_k, respectively, which include a sufficiently great range of practical applications. For further numerical applications, the following values are chosen for the non-dimensional data of the mechanical system in Figure 1: $\eta = 0{\cdot}5$ and $\alpha_k = 1$. The selected data means that

TABLE 1

The first two dimensionless eigenfrequency parameters of the system in Figure 1 for variable values of location and stiffness parameters η and α_k, respectively

	η									
α_k	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·10	1·572341	1·576785	1·583677	1·592359	1·601997	1·611634	1·620267	1·626967	1·631011	1·631995
	4·716729	4·726154	4·732972	4·731529	4·722975	4·714412	4·712909	4·719753	4·729290	4·733512
0·25	1·574607	1·585441	1·602137	1·623162	1·646567	1·669980	1·690759	1·706362	1·714885	1·715507
	4·723114	4·746343	4·763405	4·760029	4·738761	4·717434	4·713691	4·730923	4·754833	4·764809
0·50	1·578255	1·599056	1·630798	1·670772	1·715507	1·760374	1·799666	1·827610	1·840193	1·836597
	4·733435	4·778808	4·812955	4·806951	4·764809	4·722434	4·715001	4·749859	4·797845	4·815842
0·75	1·581752	1·611745	1·657105	1·714255	1·778594	1·843390	1·899512	1·937306	1·950641	1·939741
	4·743374	4·809856	4·861051	4·853125	4·790509	4·727391	4·716319	4·769187	4·841300	4·865337
1·00	1·585107	1·623598	1·681329	1·754126	1·836597	1·920095	1·991744	2·037393	2·048930	2·028758
	4·752949	4·839556	4·907713	4·898524	4·815842	4·732301	4·717644	4·788893	4·885067	4·913180
1·50	1·591422	1·645100	1·724433	1·824678	1·939741	2·057730	2·157497	2·214123	2·216464	2·174626
	4·771083	4·895177	4·996833	4·986925	4·865337	4·741982	4·720317	4·829381	4·973033	5·003645
2·00	1·597260	1·664087	1·761604	1·885128	2·028758	2·178179	2·303161	2·365797	2·353814	2·288930
	4·787972	4·946169	5·080536	5·072058	4·913180	4·751471	4·723020	4·871186	5·060764	5·086985
2·50	1·602674	1·680970	1·793965	1·937457	2·106376	2·284792	2·432833	2·497566	2·468097	2·380644
	4·803733	4·992988	5·159073	5·153894	4·959298	4·760763	4·725752	4·914148	5·147384	5·163306
3·00	1·607708	1·696078	1·822375	1·983157	2·174626	2·379987	2·549388	2·613029	2·564286	2·455644
	4·818472	5·036045	5·232714	5·232452	5·003645	4·769853	4·728513	4·958093	5·232144	5·232938
4·00	1·616785	1·721969	1·869876	2·059023	2·288930	2·543007	2·751129	2·805095	2·716086	2·570432
	4·845239	5·112311	5·366411	5·379986	5·086985	4·787422	4·734118	5·048169	5·393809	5·354032
5·00	1·624742	1·743339	1·907960	2·119308	2·380644	2·677605	2·920212	2·957131	2·829218	2·653662
	4·868896	5·177490	5·483805	5·515368	5·163306	4·804168	4·739828	5·139881	5·542558	5·454354
7·50	1·640915	1·783318	1·976527	2·226416	2·545278	2·929306	3·244141	3·220793	3·012572	2·785931
	4·917455	5·304177	5·718962	5·806712	5·325868	4·842479	4·754518	5·367324	5·851873	5·638531
10·0	1·653271	1·811079	2·022108	2·296418	2·653662	3·102970	3·474650	3·383350	3·119681	2·862773
	4·954918	5·395004	5·891230	6·042347	5·454354	4·875991	4·769693	5·580311	6·079568	5·760558

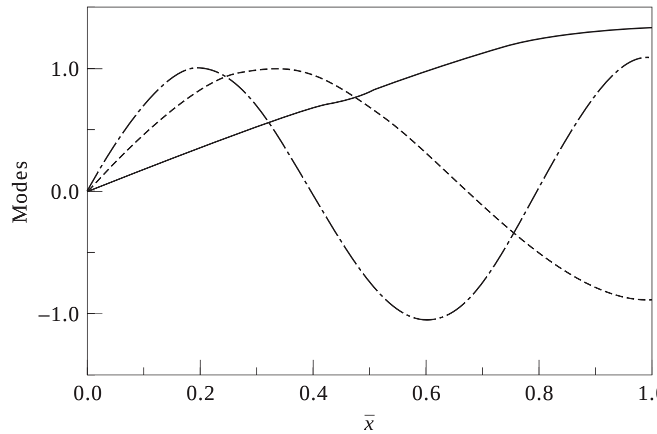

Figure 2. Mode shapes of the system in Figure 1 for $\eta = 0{\cdot}5$, $\alpha_k = 1{\cdot}0$: ——, first mode; ---, second mode; —·—·—, third mode.

the spring acts at the mid-point of the rod and the stiffness of the linear spring is equal to the tip stiffness of the rod. The numerical solution of the frequency equation (8) yields the first three non-dimensional eigenfrequencies of the system as $\bar{\beta}_1 = 1{\cdot}836597$, $\bar{\beta}_2 = 4{\cdot}815842$ and $\bar{\beta}_3 = 7{\cdot}917053$. The first two of these values are also included in Table 1. The corresponding mode shapes are given in the Figure 2. It is worth noting that similarity with the results reported in reference [2]; the mode shape curves have a small break at the point of spring attachment,

TABLE 2

Comparison of the dimensionless fundamental frequency parameter $\bar{\beta}_1$ if the spring attachment point is moved slightly from the nominal value $\eta = 0{\cdot}5$, $\alpha_k = 0{\cdot}5$ is taken

	$\bar{\beta}_1$ of the modified system				
η	from eq. (8)	from eq. (15)	η	from eq. (8)	from eq. (15)
0·425	1·681669	1·681249	0·500	1·715507	1·715507
0·430	1·683879	1·683533	0·505	1·717791	1·717791
0·435	1·686098	1·685817	0·510	1·720075	1·720075
0·440	1·688326	1·688100	0·515	1·722358	1·722359
0·445	1·690561	1·690384	0·520	1·724640	1·724643
0·450	1·692804	1·692668	0·525	1·726920	1·726927
0·455	1·695054	1·694952	0·530	1·729197	1·729210
0·460	1·697310	1·697236	0·535	1·731470	1·731494
0·465	1·699571	1·699520	0·540	1·733739	1·733778
0·470	1·701838	1·701804	0·545	1·736004	1·736062
0·475	1·704109	1·704088	0·550	1·738262	1·738346
0·480	1·706383	1·706372	0·555	1·740514	1·740630
0·485	1·708661	1·708605	0·560	1·742759	1·742914
0·490	1·710941	1·710939	0·565	1·744996	1·745198
0·495	1·713224	1·713223	0·570	1·747225	1·747482
			0·575	1·749444	1·749766

i.e., at $x = 0{\cdot}5L$. This can be explained best via the matching condition in (6), which is rearranged as

$$(U_2' - U_1')|_{\bar{x}=\eta} = \alpha_k (U_1)|_{\bar{x}=\eta}, \tag{16}$$

where a prime denotes partial differentiation with respect to the non-dimensional position co-ordinate $\bar{x}$. The left side of the above expression corresponds to the difference of the slopes of the curves $U_2(\bar{x})$ and $U_1(\bar{x})$ at the point of the spring attachment point. According to the right side, this difference is proportional to the dimensionless spring constant α_k. Hence, the breaks in Figure 2 would be much more pronounced if α_k has a greater value than 1·0.

Table 2 gives an indication of the accuracy of the sensitivity-related formula (15). As an application, the fundamental frequency parameters from the numerical solution of the frequency equation (8) are given together with approximate values obtained from the formula (15), assuming that slight changes occurred around the nominal spring attachment point $\eta = 0{\cdot}5$ (α_k is taken as 0·5). For the data chosen, the formula (13) results in $\partial\overline{\beta}/\partial\eta = 0{\cdot}456778188$. The values in the second column are fundamental frequency parameters calculated from the equation (8), in other words, these are the "exact" values. The values collected in the third column originate from the sensitivity-based formula (15). The comparison of the values in the second and third columns indicates clearly that the formula (15) gives very accurate approximations to the dimensionless eigenfrequencies of the modified system without having to solve the frequency equation (8) for the parameters of the modified system.

4. CONCLUSIONS

This note is concerned with the natural vibration problem of a mechanical system, consisting of a fixed–free axially vibrating elastic rod which is restrained by a linear spring in-span. The frequency equation of the system is derived first. Then the mode shapes are given and finally a sensitivity formula is established. Numerical results are given in form of two tables.

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