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ANTISYMMETRIC MODES OF VIBRATIONS OF COMPOSITE, DOUBLY-CONNECTED MEMBRANES

C. A. Rossit, S. La Malfa and P. A. A. Laura
Department of Engineering, Universidad Nacional del Sur and Institute of Applied Mechanics (CONICET), 8000—Bahía Blanca, Argentina

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1. INTRODUCTION

Several recent publications deal with axisymmetric modes of transverse vibration of composite doubly-connected membranes [1, 2]. However, no studies seem to be available on antisymmetric modes of simply- and doubly-connected membranes [1–3].

The present study deals with the general formulation of the problem for the case of m-discontinuous variations of the density ρ_j (see Figure 1). Numerical results of the frequency coefficients are presented for $m = 2$ and several combinations of the geometric and mechanical parameters.

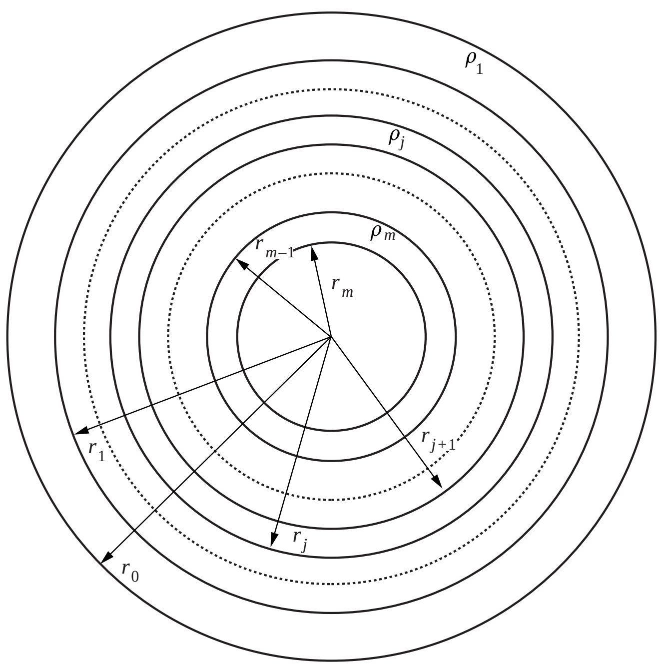

Figure 1. Vibrating system under study.

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TABLE 1

Frequency coefficients Ω_{1i} for the configuration shown in Figure 1 ($m = 2$)

ρ_2/ρ_1	Ω_{11}	Ω_{12}	Ω_{13}	Ω_{14}	Ω_{15}
		(a) $r_1/r_0 = 0{\cdot}10$, $r_2/r_0 = 0{\cdot}50$			
0·10	4·52845	9·893	15·6809	21·3438	25·8914
0·50	4·2767	8·63894	12·3322	16·2242	20·5323
0·90	4·00791	7·53211	10·9727	14·5517	18·0364
1·50	3·62267	6·62817	9·78828	12·862	16·0586
2	3·34488	6·21329	8·99287	12·0664	14·7621
5	2·36923	4·99348	6·7597	9·04689	11·6432
10	1·7331	3·82894	5·74845	6·95202	8·85615
		(b) $r_1/r_0 = 0{\cdot}20$, $r_2/r_0 = 0{\cdot}50$			
0·10	4·67208	10·0453	15·8795	21·7857	27·5109
0·50	4·48687	9·16327	13·6008	17·5666	22·1416
0·90	4·28686	8·24847	12·1476	16·1210	20·1476
1·50	3·98186	7·32023	11·1106	14·4770	18·2633
2	3·74227	6·85956	10·4524	13·523	17·2495
5	2·76733	5·77031	7·81218	11·120	13·2118
10	2·05395	4·76427	6·49793	8·50204	11·2443
		(c) $r_1/r_0 = 0{\cdot}30$, $r_2/r_0 = 0{\cdot}50$			
0·10	4·94931	10·3363	16·1738	22·1651	28·1972
0·50	4·84631	9·83468	15·0099	19·8791	24·3902
0·90	4·73481	9·25069	13·7912	18·2672	22·8029
1·50	4·55532	8·4364	12·7177	17·0577	21·0270
2	4·39952	7·91428	12·2183	16·1897	19·9070
5	3·55881	6·59088	10·2176	13·0570	16·9390
10	2·74048	5·97725	7·9547	11·6227	13·4338

2. FORMULATION AND SOLUTION OF THE PROBLEM

For each concentric portion of the composite membrane the governing partial differential equation is

$$S\nabla^2 w_j(r, \theta, t) = \rho_j \frac{\partial^2 w_j}{\partial t^2}(r, \theta, t), \qquad j = 1, 2, \ldots, m, \tag{1}$$

while the boundary and compatibility conditions are ($j = 1, 2, \ldots, m-1$)

$$w_1(r_0, \theta, t) = 0, \qquad w_j(r_j, \theta, t) = w_{j+1}(r_j, \theta, t),$$

$$\frac{\partial w_j}{\partial r}(r_j, \theta, t) = \frac{\partial w_{j+1}}{\partial r}(r_j, \theta, t), \qquad w_m(r_m, \theta, t) = 0. \tag{2}$$

Making use of the classical method of separation of variables one writes

$$w(r, \theta, t) = W_j(r)\Theta(\theta)\tau(t) \tag{3}$$

and substituting in equation (1) one obtains

$$\tau(t) = C_1\, e^{i\omega t}, \qquad \Theta(\theta) = C_2\, e^{in\theta}, \qquad n = 1, 2, 3, \ldots, \tag{4a, b}$$

where ω is the circular frequency, and

$$W_j(r) = A_{jn}J_n\left(\sqrt{\frac{\rho_j}{S}}\,\omega r\right) + B_{jn}Y_n\left(\sqrt{\frac{\rho_j}{S}}\,\omega r\right), \qquad j = 1, 2, \ldots, m. \tag{4c}$$

In terms of $W_j(r)$ the boundary and compatibility conditions become $j = 1, 2, \ldots, m-1$

$$W_1(r_0) = 0, \qquad W_j(r_j) = W_{j+1}(r_j),$$

$$\frac{\mathrm{d}W_j}{\mathrm{d}r}(r_j) = \frac{\mathrm{d}W_{j+1}}{\mathrm{d}r}(r_j), \qquad W_m(r_m) = 0. \tag{5}$$

Conditions (5) yield a system of $(2m)$ linear, homogeneous equations in the constants $(A_{1n}, A_{2n} \ldots A_{mn})$ and $(B_{1n}, B_{2n} \ldots B_{nm})$. Finally, a determinantal equation in the natural frequencies of the antisymmetric modes is obtained from the non-triviality condition.

TABLE 2

Frequency coefficients Ω_{2i} for the configuration shown in Figure 1 ($m = 2$)

ρ_2/ρ_1	Ω_{21}	Ω_{22}	Ω_{23}	Ω_{24}	Ω_{25}
		(a) $r_1/r_0 = 0{\cdot}10$, $r_2/r_0 = 0{\cdot}50$			
0·10	5·54052	10·5726	16·2519	22·0943	27·7710
0·50	5·38461	9·73061	13·7423	17·1860	21·4306
0·90	5·19475	8·68795	12·0003	15·4253	18·8217
1·50	4·8597	7·58628	10·7184	13·5574	16·7347
2	4·56572	7·06259	9·87195	12·6722	15·3787
5	3·30357	5·75418	7·28680	9·57191	12·0357
10	2·41672	4·42136	6·22874	7·31905	9·16972
		(b) $r_1/r_0 = 0{\cdot}20$, $r_2/r_0 = 0{\cdot}50$			
0·10	5·56868	10·6138	16·3043	22·1872	28·0501
0·50	5·43077	9·89776	14·3733	18·1940	22·5375
0·90	5·26667	9·01181	12·7460	16·5760	20·5303
1·50	4·9793	7·96769	11·6104	14·8757	18·5670
2	4·72169	7·43044	10·9344	13·8622	17·5444
5	3·51115	6·26555	8·14823	11·3951	13·4251
10	2·5915	5·15092	6·8394	8·69456	11·4191
		(c) $r_1/r_0 = 0{\cdot}30$, $r_2/r_0 = 0{\cdot}50$,			
0·10	5·68936	10·7813	16·4821	22·4088	28·4131
0·50	5·59972	10·3214	15·3972	20·2460	24·6930
0·90	5·4977	9·74982	14·1568	18·5491	23·0252
1·50	5·32186	8·89487	13·0221	17·3066	21·2271
2	5·15785	8·32904	12·5058	16·4282	20·0845
5	4·16322	6·95487	10·4548	13·2503	17·0857
10	3·16996	6·36034	8·13197	11·7953	13·5732

TABLE 3

Frequency coefficients Ω_{3i} for the configuration shown in Figure 1 ($m = 2$)

ρ_2/ρ_1	Ω_{31}	Ω_{32}	Ω_{33}	Ω_{34}	Ω_{35}
			(a) $r_1/r_0 = 0{\cdot}10$, $r_2/r_0 = 0{\cdot}50$		
0·10	6·62502	11·3662	16·8613	22·6643	28·5526
0·50	6·53351	10·8159	15·2544	18·7280	22·5755
0·90	6·41514	9·98804	13·3462	16·6536	20·0153
1·50	6·17215	8·77994	11·9328	14·6172	17·7573
2	5·90853	8·1180	11·0816	13·5644	16·3659
5	4·36958	6·73499	8·0599	10·3819	12·6462
10	3·18856	5·20047	6·94103	7·88857	9·67249
			(b) $r_1/r_0 = 0{\cdot}20$, $r_2/r_0 = 0{\cdot}50$		
0·10	6·62886	11·3741	16·8716	22·6801	28·5861
0·50	6·54038	10·8498	15·4238	19·2739	23·2117
0·90	6·42742	10·0823	13·6818	17·3173	21·1563
1·50	6·1995	8·93583	12·3812	15·5326	19·0616
2	5·95446	8·28035	11·6849	14·4222	18·0224
5	4·46584	7·00521	8·68057	11·8409	13·7741
10	3·27235	5·70926	7·39306	9·00634	11·6997
			(c) $r_1/r_0 = 0{\cdot}30$, $r_2/r_0 = 0{\cdot}50$		
0·10	6·66781	11·4488	16·9574	22·7846	28·7404
0·50	6·59934	11·0552	15·9978	20·8337	25·2087
0·90	6·51686	10·5207	14·7483	19·0143	23·3935
1·50	6·36119	9·62068	13·5136	17·7137	21·5586
2	6·19904	8·98417	12·9686	16·8197	20·3776
5	4·98875	7·5298	10·8368	13·5689	17·3269
10	3·74198	6·94943	8·42828	12·0690	13·8120

In the case of a doubly-connected membrane of two materials of densities ρ_1, and ρ_2 the determinantal equation corresponding to an nth degree of antisymmetry is

$$\begin{bmatrix} J_n(\Omega) & Y_n(\Omega) & 0 & 0 \\ J_n(R_1\Omega) & Y_n(R_1\Omega) & -J_n(R_1\rho\Omega) & -Y_n(R_1\rho\Omega) \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & J_n(R_2\rho\Omega) & Y_n(R_2\rho\Omega) \end{bmatrix} = 0,$$

where

$$m_{31} = \frac{J_{n-1}(R_1\Omega) - J_{n+1}(R_1\Omega)}{\rho}, \qquad m_{32} = \frac{Y_{n-1}(R_1\Omega) - Y_{n+1}(R_1\Omega)}{\rho},$$

$$m_{33} = J_{n+1}(R_1\rho\Omega) - J_{n-1}(R_1\rho\Omega), \qquad m_{34} = Y_{n+1}(R_1\rho\Omega) - Y_{n-1}(R_1\rho\Omega),$$

and

$$R_1 = r_1/r_0, \quad R_2 = r_2/r_0, \quad \rho = \sqrt{\rho_2/\rho_1}, \quad \Omega = \sqrt{\rho_1/S}\omega r_0.$$

3. NUMERICAL RESULTS

Tables 1, 2 and 3 depict values of $\Omega_{ni} = \sqrt{\rho_1/S}\omega_{ni}r_0$, for $n = 1, 2$ and 3, respectively.

The following geometric and mechanical combinations have been considered: $r_1/r_0 = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ for $r_2/r_0 = 0{\cdot}5$ and $\rho_2/\rho_1 = 0{\cdot}10$, $0{\cdot}50$, $0{\cdot}90$, $1{\cdot}50$, 2, 5 and 10.

The first five roots have been determined for each case ($i = 1, 2 \ldots 5$). The calculation procedure has been greatly facilitated by the use of *Mathematica* [4].

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