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FREQUENCY EQUATION OF THIN SHELL VIBRATION IN THE TRANSITION RANGE

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In this paper, the frequency equation of thin shell vibration in the transition range is deduced, the frequency equation of bending vibration is given, and finally the results are verified with numerical calculation.

1. INTRODUCTION

Gol'denveizer *et al*. [1] pointed out that the axisymmetric vibration of revolution thin shells may be described by only including the normal displacement w in the following higher order equation

$$-\mu^5\left[\sum_{n=0}^{6} \mathrm{d}_n(s)\frac{\mathrm{d}^n W}{\mathrm{d}s^n}\right]+\sum_{n=0}^{2} b_n(s)\frac{\mathrm{d}^n W}{\mathrm{d}s^n}=0 \qquad (\mathrm{d}_6=1), \tag{1}$$

where $b_2(s)=\Omega^2-R_2^{-2}(s)$. When the frequency parameter Ω is in the frequency interval (the transition range) where

$$\min R_2^{-1}(s)\leqslant\Omega\leqslant\max R_2^{-1}(s) \qquad (a\leqslant s\leqslant b),$$

for any frequency there is a circle ($s=s_*$) which divides the middle surface of the shell into three parts along the longitude: $a<s<s_*$, $|s-s_*|\sim 0\ (\mu)$ and $s_*<s<b$. In the three parts $b_2(s)$ can be positive, zero and negative, respectively. The corresponding solutions of equation (1) in these three parts have different characteristics. The point $s=s_*[b_2(s_*)=0]$ is therefore called the turning point, whose position can be shifted by changing frequency f. For $f=f_{tb}=c/[2\pi R_2^*(b)]$ ($c=(E/\rho)^{1/2}$), the turning point is on the outer edge. As the frequency is increased, the turning point shifts from the outer edge of the shell to the inner edge. For $f=f_{ta}=c/[2\pi R_2^*(a)]$, the turning point is on the inner edge. Frequency f will find

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itself in one of the three frequency ranges: low frequency range ($f < f_{tb}$), transition range ($f_{tb} < f < f_{ta}$) and high frequency range ($f > f_{ta}$).

The loudspeakers are shaped as a typical cone with a thin shell. According to the theory of thin shells, in the low frequency range there is no bending resonance of cones [2], because the actual size of loudspeaker cones is less than half the wavelength [3], also there is no longitudinal resonance. The transition range is actually beyond the effective frequency range of loudspeakers. So the natural frequencies which really exert an influence on sound radiation of loudspeakers only exist in the transition range. Recently, the uniformly valid solutions of equation (1) in the transition range have been obtained, which was considered to be difficult to resolve [4].

In the following context, the four bending solutions in the transition range will be shown to be improved by redefining the first category of the generalized functions. The frequency equation of vibration of cone shaped thin shells, which is based on the uniformly valid solutions in the whole interval of s, has also been obtained. Finally, the independent frequency equation of bending vibration will be given.

2. THE BENDING SOLUTIONS IN TRANSITION RANGE

The uniformly valid solution of a thin shell in the transition range has been obtained by Zhang and Zhang [4]. Their results will be used for obtaining the bending solutions in the transition range.

According to Sander's theory of thin shells, after separating time variable $e^{i\omega t}$ the system of equations, which represents the free vibration of revolution thin shells, may be written as

$$(L + \mu^5 N)U = -(1 - \nu^2)\Omega^2 U, \tag{2}$$

where the operators are

$$L = \begin{bmatrix} L_{11} & L_{13} \\ L_{31} & L_{33} \end{bmatrix} \qquad N = \begin{bmatrix} N_{11} & N_{13} \\ N_{31} & N_{33} \end{bmatrix} \qquad U = \begin{bmatrix} u \\ W \end{bmatrix},$$

$$L_{11} = \frac{d^2}{ds^2} + \frac{B'}{B}\frac{d}{ds} + \frac{\nu}{R_1 R_2} - \left(\frac{B'}{B}\right)^2,$$

$$L_{13} = \left[-\frac{1}{R_1} + \frac{\nu}{R_2}\right]\frac{d}{ds} - \frac{B'}{B}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] - \left[\frac{1}{R_1}\right],$$

$$L_{31} = \left[\frac{1}{R_1} - \frac{\nu}{R_2}\right]\frac{d}{ds} + \frac{B'}{B}\left[\frac{\nu}{R_1} + \frac{1}{R_2}\right],$$

$$L_{33} = \left[-\frac{1}{R_1} + \frac{v}{R_2} \right] \frac{1}{\mathrm{R}_1} + \left[\frac{v}{R_1} + \frac{1}{R_2} \right] \frac{1}{R_2},$$

$$N_{13} = (1 - v^2) \frac{1}{R_1} \frac{\mathrm{d}^3}{\mathrm{d}s^3} + \dots,$$

$$N_{33} = (1 - v^2) \left\{ -\frac{\mathrm{d}^4}{\mathrm{d}s^4} - 2 \frac{B'}{B} \frac{\mathrm{d}^3}{\mathrm{d}s^3} + \left[\left(\frac{B'}{B} \right)^2 - \frac{1+v}{R_1 R_2} \right] \frac{\mathrm{d}^2}{\mathrm{d}s^2} \right.$$

$$\left. - \left[\left(\frac{B'}{B} \right)^3 - \frac{2}{R_1 R_2} \frac{B'}{B} - v \left(\frac{1}{R_1} \right)^2 \frac{B'}{B} + \frac{v}{R_2} \left(\frac{1}{R_1} \right)' \right] \frac{\mathrm{d}}{\mathrm{d}s} \right\}.$$

Here N_{11}, N_{31} and the lower order differential terms of N_{13} are neglected because they are so small that they do not contribute to our problem.

By introducing the Langer's variable z and ζ

$$\zeta = \mu^{-1} z = \mu^{-1} \Phi(s) = \mu^{-1} \left[\frac{5}{4} \int_{s_*}^{s} b_2^{1/4}(x) \, \mathrm{d}x \right]^{4/5},$$

and the new dependent variable

$$Y(\zeta) = \left[\frac{b_2(s)}{\mu \zeta} \right]^{1/8} W(s) = \sum_{n=0}^{\infty} \mu^n Y_n(\zeta), \tag{3}$$

and substituting them into equation (2), a series of equations are obtained as follows:

$$\begin{aligned} ADY_0(\zeta) &= 0, \\ ADY_1(\zeta) &= [a(z)D^5 + \beta(z)] Y_0(z), \\ ADY_2(\zeta) &= [a(z)D^5 + \beta(z)] Y_1(z) + \gamma(z) D^4 Y_0(z), \\ &\dots \end{aligned} \tag{4}$$

where A and D are differential operators

$$D = \frac{\mathrm{d}}{\mathrm{d}\zeta}, \qquad A = D^5 - \zeta D - 2,$$

and $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are slowly varying coefficients.

The equation,

$$Af(\zeta) = 0, \tag{5}$$

is referred to as the related equation corresponding to equation (2), the solutions of which are called the related function.

The solutions of the related equation (5), and their corresponding differentials and integrals can all be expressed as

$$F_K(\zeta, P) = \frac{1}{2\pi \mathrm{i}} \int_{L_K} t^{-p} \exp\left(\zeta t - \frac{t^5}{5}\right) \mathrm{d}t, \quad (K = 0, 1, \ldots, 5, P = 0, \pm 1, \pm 2. \ldots), \tag{6}$$

where L_K represents the contours in complex t-plane, as shown in Figure 1. When $p = -1$, F_K are the solutions of equation (5). When $p \neq -1$, F_K are their differentials and integrals.

The first category of the generalized related function $Z_h(\zeta, p)$ $(h = 1, 2, 3, 4)$ is defined by the combination of F_K as follows:

$$Z_1(\zeta, P) = -F_3(\zeta, P), \qquad Z_2(\zeta, P) = -\mathrm{i}[F_2(\zeta, P) - F_4(\zeta, P)],$$

$$Z_3(\zeta, P) = -\mathrm{i}[F_1(\zeta, P) - F_5(\zeta, P)], \qquad Z_4(\zeta, P) = F_1(\zeta, P) + F_5(\zeta, P),$$

for any real value of ζ, $Z_h(\zeta, P)$ $(h = 1, 2, 3, 4)$ is identically a real function.

It is straightforward to verify that $Z_h(\zeta, P)$ satisfies the following relations:

$$(A + P + 1)Z_h(\zeta, P) = 0, \qquad D^n Z_h(\zeta, P) = Z_h(\zeta, P - n), \tag{7a, b}$$

$$Z_h(\zeta, P - 5) - \zeta Z_h(\zeta, P - 1) + (P - 1)Z_h(\zeta, P) = 0, \tag{7c}$$

$$ADZ_h(\zeta, P + 1) = -(P + 1)Z_h(\zeta, P). \tag{7d}$$

From equation (4) and relations (7), $Y_0(\zeta)$, $Y_1(\zeta)$, $Y_2(\zeta) \ldots$ can be obtained in terms of $Z_h(\zeta, P)$. Inserting them back into equation (3) and expressing $Z_h(\zeta, P)$

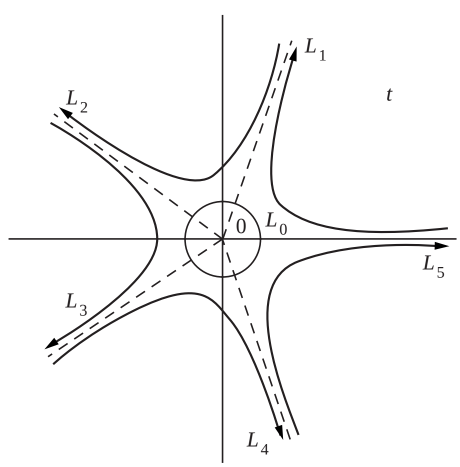

Figure 1. Contours L_K $(K = 0, 1, 2, 3, 4, 5)$ in the t-plane.

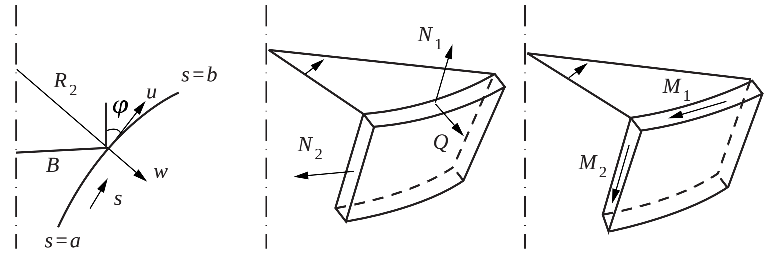

Figure 2. (a) Geometric parameters and displacements; (b) positive direction of stress resultants; (c) the moments.

$(h = 1, 2, 3, 4)$ as $Z_h(\zeta, 0)$, $Z_h(\zeta, 1), \ldots, Z_h(\zeta, 4)$ by using the recursion formula (7c), we may obtain $Y^{(h)}(\zeta)$ in the following form

$$Y^{(h)}(\zeta) = \sum_{n=0}^{4} \mu^n \pi_n(z, \mu) Z_h(\zeta, n), \qquad (h = 1, 2, 3, 4),$$

where $\pi_n(z, \mu)$ are slowly varying coefficients.

The bending solutions can be expressed as

$$u_h(\zeta, \mu) = a_1(S) Z_h(\zeta, 0) + \mu\beta_1(s) Z_h(\zeta, 1) + \ldots,$$
$$W_h(\zeta, \mu) = a_3(S) Z_h(\zeta, 0) + \ldots \qquad (h = 1, 2, 3, 4). \tag{8}$$

Finally, by substituting equation (8) into equation (2), the bending solutions can be written as (due to their uniform validity):

$$u_h(\zeta, \mu) = \mu^{5/8}\left[\frac{1}{R_1} - \frac{v}{R_2}\right]\left[\frac{B(S_*)}{B(s)}\right]^{1/2}\left[\frac{\Phi'(S_*)}{\Phi'(s)}\right]^{3/2}\frac{1}{\Phi'(s)} Z_h(\zeta, 1),$$

$$W_h(\zeta, \mu) = \mu^{-3/8}\left[\frac{B(S_*)}{B(s)}\right]^{1/2}\left[\frac{\Phi'(S_*)}{\Phi'(s)}\right]^{3/2} Z_h(\zeta, 0) \qquad (h = 1, 2, 3, 4). \tag{9}$$

3. THE FREQUENCY EQUATION

The inner edge of the loudspeaker, to which the voice coil and inner suspension are attached, is relatively stiff so that it can only move in the axial direction. In fact the influence of the boundary conditions of the inner edge is not obvious for bending resonance in the transition range [3]. The voice coil, as a simple loading mass, has no obvious influence on the first few bending resonance frequencies. The cone of the loudspeaker is moved by the axial force. The influence of outer suspension can be neglected in the transition range, and so one can assume that the outer edge is free. Thus, the eigen-boundary conditions of the loudspeaker cone, which are dimensionlessly expressed, can be given by

$$Q(b) = N_1(b) = M_1(b) = 0, \qquad \hat{W}(a) = u(a)\sin\varphi(a) + W(a)\cos\varphi(a) = 0,$$
$$\beta(a) = W'(a) = 0, \qquad (1 - v^2)\varepsilon^4 Q(a)\sin\varphi(a) - N_1\cos\varphi(a) = 0, \tag{10}$$

where $\hat{W}(a)$ is the displacement normal to the cone axis, $\varphi(a)$ the angle between the shell axis and the tangent along the longitude on the inner edge (see Figure 2).

The stress resultants and moments can be expressed as

$$N_1 = u' + v\frac{B'}{B}u - \left(\frac{1}{R_1} - \frac{v}{R_2}\right)W,$$

$$M_1 = -\left(\frac{u}{R_1}\right)' - \frac{vB'u}{BR_1} - W'' - v\frac{B'}{B}W',$$

$$Q = -W''' - \frac{B'}{B}W'' - \left[\frac{v}{R_1R_2} - \left(\frac{B'}{B}\right)^2\right]W' - \left(\frac{u}{R_1}\right)'' - \frac{B'}{B}\left(\frac{u}{R_1}\right)'$$
$$- \left[\frac{v}{R_1R_2} - \left(\frac{B'}{B}\right)^2\right]\frac{u}{R_1}. \tag{11}$$

The asymptotic expressions for bending solution, as $\zeta \to \pm\infty$, can be obtained by the method of steepest descent as follows:

$S > S_*$, $(\zeta \to +\infty)$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \varepsilon C_1(s) \begin{bmatrix} -1/2\, e^{-\theta} \\ \cos(\theta - \pi/4) \\ e^{\theta} \\ \sin(\theta - \pi/4) + \sqrt{2\pi/\varepsilon}\Phi^{5/8} \end{bmatrix},$$

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} = C_2(s) \begin{bmatrix} 1/2\, e^{-\theta} \\ \cos(\theta + \pi/4) \\ e^{\theta} \\ \sin(\theta + \pi/4) \end{bmatrix}, \tag{12}$$

$s < s_*$, $(\zeta \to -\infty)$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \varepsilon C_1(s) \begin{bmatrix} e^{\alpha}\sin(\alpha + 7/8\pi) \\ e^{\alpha}\cos(\alpha + 7/8\pi) \\ e^{-\alpha}\cos(\alpha + 5/8\pi) \\ e^{-\alpha}\sin(\alpha + 5/8\pi) \end{bmatrix},$$

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} = C_2(s) \begin{bmatrix} e^{\alpha}\sin(\alpha + \pi/8) \\ e^{\alpha}\cos(\alpha + \pi/8) \\ -e^{-\alpha}\sin(\alpha - \pi/8) \\ e^{-\alpha}\cos(\alpha - \pi/8) \end{bmatrix}, \tag{13}$$

where

$$C_1(s) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{R_1} - \frac{v}{R_2}\right)\left[\frac{B(S_*)}{B(s)}\right]^{1/2} \frac{b(s_*)^{3/10}}{|b_2(s)|^{5/8}}$$

$$C_2(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{B(S_*)}{B(s)}\right]^{1/2} \frac{b(s_*)^{3/10}}{|b_2(s)|^{3/8}}$$

$$\theta = \theta(s) = \varepsilon^{-1}\int_{s_*}^{s} b_2(x)^{1/4}\,\mathrm{d}x, \qquad \alpha = \alpha(s) = \frac{\varepsilon^{-1}}{\sqrt{2}}\int_{S}^{s_*}[-b_2(x)]^{1/4}\,\mathrm{d}x,$$

$$b(s_*) = b_2'(s)|_{s=s_*}. \tag{14}$$

$b(s_*) \neq 0$ because the point $s = s_*$ is a simple turning point.

The expression of singular membrane solution is obtained from reference [4].

$$u_5 = P_5(S) + 1n\zeta u_6^{(0)}(s),$$

$$W_5 = r_5(s) + 1n\zeta W_6^{(0)}(s) + b^{1/2}(S_*)\Phi^{5/8}(s)b_2^{-3/8}(s)\left[\frac{B(s_*)}{B(s)}\right]^{1/2}, \tag{15}$$

where $P_5(s)$, $r_5(s)$ are regular parts of the singular membrane solution, and $u_6^{(0)}(s)$, $W_6^0(s)$ are the leading terms of the regular membrane solution when $|s - s_*| \sim 0$ (μ).

Substituting equations (12), (13) and (15) into equation (11), the order of magnitude relations for the four bending solutions in equation (11) are: u, $N_1 \sim O(\varepsilon)$, $W \sim O(1)$, $\beta \sim O(\varepsilon^{-1})$, $M_1 \sim O(\varepsilon^{-2})$, $Q \sim Q(\varepsilon^{-3})$ and u, $N_1 \sim O(\varepsilon^{1/2})$ for the fourth bending solution when $s > s_*$. For the regular membrane solution, these variables are on the order of $O(1)$, or $O(|\ln \varepsilon|)$ for the singular membrane solution.

The exponential decay variables (the variables for the first bending solution when $s > s_*$, for the third and fourth bending solutions when $s < s_*$) can be neglected because $\theta(s)$, $\alpha(s)$ are fast varying coefficients.

The general solutions of equation (2) consist of the above six solutions.

Suppose that they are in the following form.

$$u = \sum_{i=1}^{6} P_i u_i, \qquad W = \sum_{i=1}^{6} P_i W_i,$$

where $P_1, \ldots, P_6$ are constants determined by boundary conditions (10). Substituting them into equations (10) and (11), one obtains the frequency equations from the assumption that the coefficient determinant of P_i is zero:

$$\begin{aligned} &\cos\varphi(a)[N_{15}(a)N_{16}(b) - N_{15}(b)N_{16}(a)] \\ &\quad \times [M_{13}(b)Q_4(b) - Q_3(b)M_{14}(b)] \\ &\quad + [Q_2(b)M_{13}(b) - Q_3(b)M_{12}(b)][\hat{W}_5(a)N_{16}(a) \\ &\quad - N_{15}(a)\hat{W}_6(a)]N_{14}(b) \\ &\quad \times \beta_1(a)/[W_1(a)\beta_2(a) - W_2(a)\beta_1(a)] = O(\varepsilon^{-4}), \end{aligned} \tag{16}$$

TABLE 1

The parameters of the loudspeaker cone which is used for calculation

Outer edge radius	83 mm	Young's modulus	2×10^9 N/m^2
Inner edge radius	17 mm	Poisson's ratio	0·3
Thickness	0·23 mm	Mass density	600 kg/m^3
Semi-apex angle	50°		

where the last subscript of each variable denotes the nth solution, the order of magnitude relation for the first item is $O(\varepsilon^{-5})$, and for the second item is $O(\varepsilon^{-4\cdot 5})$. Noting that $\beta_1(a)/[W_1(a)\beta_2(a) - W_2(a)\beta_1(a)] = -2^{1/2}\cos[\alpha(a) - \pi/8]\,e^{-\alpha(a)}/C_2(a)$, and once again neglecting the exponential decay term, one can obtain the independent bending frequency equation and the membrane frequency equation as follows:

$$K_B \int_{s_*}^{b} (1 - \Omega^{-2} R_2^{-2})^{1/4}\, ds = \theta(b) = n\pi \qquad (n = 0, 1, 2, \ldots), \tag{17a}$$

$$N_{15}(a)N_{16}(b) - N_{15}(b)N_{16}(a) = 0, \tag{17b}$$

where K_b is the transverse wave number of the infinite flat plate.

According to the theory of vibration in a thin shell, the bending vibration only occurs outside the turning pont ($s = s_*$). The integral region of equation (17a) is from s_* to the outer edge. That conforms with the theory. From equation (17a) one can see that the characteristics of the bending vibration only depend on the material properties outside the turning point. These characteristics have been applied to the measurement of the distribution of Young's modulus along the generatrix, which has been discussed elsewhere [5].

Furthermore, when $\Omega R_2 \gg 1$, the wave number of the bending wave is nearly the transverse wave number of the infinite flat plate (when $\Omega R_2 \geqslant 1\cdot 75$, the error of the above approach is less than 10%).

4. NUMERICAL CALCULATION

In order to prove the above theory, the bending resonance frequencies have been calculated using equation (17a) and the results compared with those of numerical calculation.

TABLE 2

The initial three bending resonance frequencies worked out by two methods. The f_{brn} denotes the nth bending resonance frequency

Calculation method	f_{br1} (Hz)	f_{br2} (Hz)	f_{br3} (Hz)
Finite element	2366	2668	2993
Using equation (17a)	2251	2613	2925

The finite element method has been used as the method of numerical calculation. The parameters of the loudspeaker cone are listed in Table 1. The results of the calculation are shown in Table 2. It can be seen from Table 1 that the error between the results of the two calculation methods is about 5%.

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APPENDIX: LIST OF SYMBOLS

All the quantities are dimensionless with the exception of the characteristic shell radius R^*, the Young's modulus E, the mass density ρ and all the quantities with superscripts *.

The subscripts 1 and 2 denote the first and second principal co-ordinate, respectively. B is Lame's coefficient of the second principal co-ordinate.

$s = s^*/R^*$	the first principal co-ordinate along the longitude
$R_1 = R_1^*/R^*$, $R_2 = R_2^*/R^*$, $B = B^*/R^*$	the geometrical parameters of the middle surface
$u = u^*/R^*$, $w = w^*/R^*$	the tangential and normal displacements
$N_1, N_2 = N_1^*, N_2^*/[Eh^*(1-\nu^2)]$	the stress resultants
$M_1, M_2 = M_1^*, M_2^*/[Eh^*R^*\varepsilon^4]$	the moments
$Q = Q^*/[Eh^*\,\varepsilon^4]$	the transverse—shear resultant
$h = h^*/R^*$	thickness of the thin shell
$\varepsilon^4 = \mu^5 = h^2/[12(1-\nu^2)]$	*parameter of the thickness of the thin shell*
$\Omega^2 = \rho\omega^2(R^*)^2/E$	*the frequency parameter*

The positive directions of stress resultants, moments and displacements are shown in Figure 2.