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DUAL FORMULATION OF MULTIPLE RECIPROCITY METHOD FOR THE ACOUSTIC MODE OF A CAVITY WITH A THIN PARTITION

J. T. CHEN AND F. C. WONG

Department of Harbor and River Engineering, *National Taiwan Ocean University*,
Keelung, *Taiwan*, *Republic of China*

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The dual formulation of the multiple reciprocity method (MRM) is employed to solve the acoustic mode of a cavity with a thin partition. In order to avoid the spurious eigenvalues and non-uniqueness of the solution in the case of zero thickness for the partition, the hypersingular equation for MRM is considered. To determine the invariant quantity of influence coefficients more efficiently, an objectivity concept, i.e., a frame of indifference, is used by means of co-ordinate transformation. A generalized eigenvalue problem is derived and transformed into a standard eigenvalue problem through the state–space formulation. The spurious roots in MRM are examined and filtered out by using the hypersingular formulation. Three examples, including finite thickness of the partition, zero thickness and no partition, are shown to check the validity of the proposed method. Also, the analytical solution if available, the FEM results obtained by Petyt and by ABAQUS and experimental measurements are compared with those of the proposed method, and it is found that the agreement between them is very good.

1. INTRODUCTION

Acoustic modes of cavities with complex geometry, degenerate boundaries and boundary conditions in general, can only be achieved by using numerical methods, since an exact solution is not usually available. Although Harris and Feshbach [1] applied the perturbation method to deal with such a problem, the domain and boundary in their paper were very simple. Besides the traditional methods, the finite difference method, finite element method and boundary element method (BEM) have also been widely used to solve these problems. For example, Petyt *et al*. [2, 3] applied the finite element method to solve the problem. Also, the commercial code, ABAQUS (FEM formulation) and SYSNOISE (BEM formulation), has a module for solving the acoustic problem. The conventional BEM, in which the fundamental solution of Helmholtz equation is applied, can construct the boundary integral equations. Such a fundamental solution is frequency dependent and is expressed in terms of a complex form of the Hankel

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function. Thus, serious difficulty may be encountered in the numerical calculation. For instance, in the evaluation of eigenvalues, the eigenequation is transcendental. Therefore, an alternative way to apply the multiple reciprocity method (MRM) [4] has been used to avoid using the complex fundamental solution. However, to the authors' knowledge, the main applications of MRM are in problems without degenerate boundaries. For problems with a degenerate boundary, e.g., a partition in a cavity, a thin partition will make the solution ill-posed without considering the hypersingular equation in the MRM formulation. Recently, Chen and his coworkers have applied the dual integral formulation to solve degenerate boundary problems, e.g., potential flow with a cut-off wall [5], crack in an elastic body [6, 7] and thin airfoil problems in aerodynamics [8]. Since MRM is no more than a part of the complex-valued formulation [9], the spurious roots are embedded [9]. To avoid spurious roots, two alternatives can be employed; one is the complex-valued formulation, and the other one is the dual formulation for MRM. The former method has been adopted by many researchers [9]. For the latter case, the MRM method combined with dual BEM for the one-dimensional eigenproblem was successfully applied to solve the spurious roots and non-unique mode [10, 11].

In this paper, MRM is extended to solve the acoustic problems of a cavity with a thin partition by using dual integral formulations. In this way, the role of the hypersingular formulation in MRM in filtering the spurious roots is examined. The kernel functions in the hypersingular equation of the dual integral equation are found by taking the normal derivative of the kernels in the singular equation, and a closed-form representation for the kernels using symbolic manipulation software is derived. To determine the invariant quantity of the influence coefficient more efficiently, an objectivity concept, i.e., the frame of indifference, is used under a special co-ordinate transformation. Also, a generalized eigenvalue problem is derived and transformed into a standard eigenvalue problem by means of the concept of the state–space formulation. Three examples, including finite thickness of the partition, zero thickness and no partition, are shown to check the validity of the proposed method. Finally, the solutions are compared with the exact solution, experimental data and FEM results obtained by ABAQUS and Petyt, to check the validity of the present formulation.

2. DUAL INTEGRAL FORMULATION OF MRM FOR A TWO-DIMENSIONAL ACOUSTIC CAVITY WITH A THIN PARTITION

The governing equation for an acoustic cavity is the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x_1, x_2) = 0, \qquad (x_1, x_2) \in D,$$

where ∇^2 is the Laplacian operator, D is the domain of the cavity and k is the wave number, which is frequency over the speed of sound. The boundary conditions can be either Neumann or Dirichlet type.

Based on the multiple reciprocity method (MRM) [4], the singular equation is

$$\pi u(x) = C.P.V. \int_B T(s, x) u(s) \, \mathrm{d}B(s) - R.P.V. \int_B U(s, x) t(s) \, \mathrm{d}B(s), \tag{1}$$

where $C.P.V.$ and $R.P.V.$ denote the Cauchy principal value and the Riemann principal value, $t(s) = \partial u(s)/\partial n_s$ and

$$U(s, x) = U^0(s, x) - k^2 U^1(s, x) + k^4 U^2(s, x) + \cdots, \tag{2}$$

$$T(s, x) = T^0(s, x) - k^2 T^1(s, x) + k^4 T^2(s, x) + \cdots, \tag{3}$$

in which

$$U^j(s, x) = A(j) r^{2j} \ln(r) - B(j) r^{2j}, \qquad j = 0, 1, 2, \ldots, \tag{4}$$

$$T^j(s, x) = \frac{\partial U^j(s, x)}{\partial n_s}, \qquad j = 0, 1, 2, \ldots . \tag{5}$$

$A(j)$ and $B(j)$ in equation (4) are shown in Table 1. Since a thin partition is considered in this paper, equation (1) will result in the same constraint equation, as shown in Figure 1, when the x point collocates on the two sides of the thin partition. Therefore, the hypersingular integral equation is utilized to construct another constraint equation as follows:

$$\pi t(x) = H.P.V. \int_B M(s, x) u(s) \, \mathrm{d}B(s) - C.P.V. \int_B L(s, x) t(s) \, \mathrm{d}B(s), \tag{6}$$

where $H.P.V.$ denotes the Hadamard principal value and

$$L(s, x) = L^0(s, x) - k^2 L^1(s, x) + k^4 L^2(s, x) + \cdots, \tag{7}$$

$$M(s, x) = M^0(s, x) - k^2 M^1(s, x) + k^4 M^2(s, x) + \cdots, \tag{8}$$

in which

$$L^j(s, x) = \frac{\partial U^j(s, x)}{\partial n_x}, \quad j = 0, 1, 2, \ldots, \qquad M^j(s, x) = \frac{\partial^2 U^j(s, x)}{\partial n_x \, \partial n_s}, \quad j = 0, 1, 2, \ldots . \tag{9, 10}$$

In equation (10), n_s and n_x are the normal vectors on s and x, respectively. The explicit forms of the kernel functions in equations (4), (5), (9) and (10) are

$$U^j(s, x) = r^{2j} \ln(r) A(j) - r^{2j} B(j), \tag{11}$$

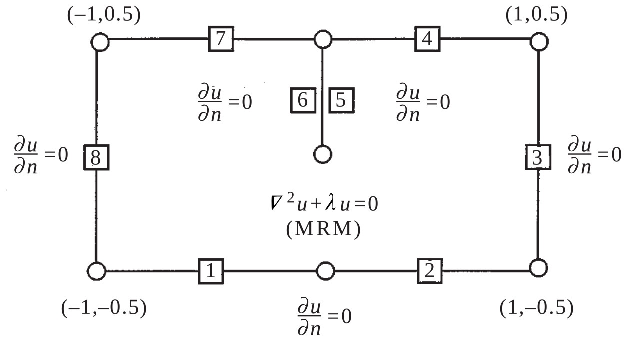
$$T^j(s, x) = -[(2j \ln(r) + 1) r^{2j-2} y_i n_i] A(j) + [2j r^{2j-2} y_i n_i] B(j), \tag{12}$$

$$L^j(s, x) = +[(2j \ln(r) + 1) r^{2j-2} y_i \bar{n}_i] A(j) - [2j r^{2j-2} y_i \bar{n}_i] B(j), \tag{13}$$

TABLE 1

Values of $A(j)$ and $B(j)$

	j										
Iteration formula	0	1	2	3	4	5	6	7	8	9	10
$A(j)$ $A_j = \frac{A_0}{4^j(j!)^2}$	1	$\frac{1}{4}$	$\frac{1}{64}$	$\frac{1}{2304}$	$\frac{1}{147\,456}$	$\frac{10^{-2}}{147\,456}$	$\frac{10^{-4}}{212\,337}$	$\frac{10^{-7}}{41\,618}$	$\frac{10^{-9}}{106\,542}$	$\frac{10^{-11}}{345\,196}$	$\frac{10^{-14}}{138\,078}$
$B(j)$ $B_{j+1} = \frac{1}{4(j+1)^2}\left(\frac{A_j}{j+1} + B_j\right)$	0	$\frac{1}{4}$	$\frac{10^4}{426\,667}$	$\frac{10^2}{125\,673}$	$\frac{10}{707\,789}$	$\frac{10^{-1}}{645\,793}$	$\frac{10^{-4}}{86\,668}$	$\frac{10^{-7}}{16\,051}$	$\frac{10^{-8}}{392\,007}$	$\frac{10^{-11}}{122\,022}$	$\frac{10^{-13}}{471\,424}$



$$[U]=\left[\begin{array}{cccccccc|c}
 & & & & 5(+) & 6(+) & & & \\
-1.6514 & .1658 & .7112 & .5946 & .0490 & 0.0490 & .2727 & -.1694 & \\
.1658 & -1.6514 & -.1694 & .2727 & .0490 & 0.0490 & .5946 & .7112 & \\
.6673 & -.1694 & -1.6514 & -.1694 & .1363 & 0.1363 & .6673 & .8294 & \\
.5946 & .2727 & -.1694 & -1.6514 & -.2184 & -0.2183 & .1658 & .7112 & \\
.1291 & .1291 & .3005 & -.5138 & -1.1861 & -1.1861 & -.5138 & .3005 & 5(+) \\
.1291 & .1291 & .3005 & -.5138 & -1.1861 & -1.1861 & -.5138 & .3005 & 6(+) \\
.2727 & .5946 & .7112 & .1658 & -.2184 & -0.2184 & -1.6514 & -.1694 & \\
-.1694 & .6673 & .8294 & .6673 & .1376 & 0.1363 & -.1694 & -1.6514 &
\end{array}\right]$$

$$[T]=\left[\begin{array}{cccccccc|c}
\xrightarrow{} n(s) & & & & 5(+) & 6(-) & & & \\
-3.1416 & .0000 & 0.4517 & 0.4973 & -.3814 & .3814 & 1.0809 & 1.2903 & \\
.0000 & -3.1416 & 1.2903 & 1.0809 & .3814 & -.3814 & .4973 & .4517 & \\
.1822 & 1.2903 & -3.1416 & 1.2903 & .5404 & -.5404 & .1822 & .0826 & \\
.4973 & 1.0809 & 1.2903 & -3.1416 & .9089 & -.9089 & .0000 & .4517 & \\
1.0944 & 1.0944 & 1.0278 & 1.4587 & -3.1416 & -3.1416 & 1.4587 & 1.0278 & 5(+) \\
1.0944 & 1.0944 & 1.0278 & 1.4587 & -3.1416 & -3.1416 & 1.4587 & 1.0278 & 6(+) \\
1.0809 & .4973 & .4517 & .0000 & -.9089 & .9089 & -3.1416 & 1.2903 & \\
1.2903 & .1822 & .0826 & .1822 & -.5404 & .5404 & 1.2903 & -3.1416 &
\end{array}\right]$$

$$\begin{array}{c} \downarrow \\ n(x) \end{array} [L]=\left[\begin{array}{cccccccc|c}
 & & & & 5(+) & 6(+) & & & \\
3.1416 & .0000 & .1301 & .4973 & .5421 & .5421 & 1.0809 & .9462 & \\
.0000 & 3.1416 & .9462 & 1.0809 & .5421 & .5421 & .4973 & .1301 & \\
.4755 & .9462 & 3.1416 & .9462 & .5404 & .5404 & .4755 & .0826 & \\
.4973 & 1.0809 & .9462 & 3.1416 & .4041 & .4041 & .0000 & .1301 & \\
-.5958 & .5958 & 1.0278 & 1.6121 & 3.1416 & -3.1416 & -1.6121 & -1.0278 & 5(+) \\
.5958 & -.5958 & -1.0278 & -1.6121 & -3.1416 & 3.1416 & 1.6121 & 1.0278 & 6(-) \\
1.0809 & .4973 & .1301 & .0000 & .4041 & .4041 & 3.1416 & .9462 & \\
.9462 & .4755 & .0826 & .4755 & .5404 & .5404 & .9462 & 3.1416 &
\end{array}\right]$$

$$\begin{array}{c} \downarrow \\ n(x) \end{array} [M]=\left[\begin{array}{cccccccc|c}
\xrightarrow{} n(s) & & & & 5(+) & 6(-) & & & \\
2.9152 & -1.5346 & -.3489 & -.3729 & .7316 & -.7316 & -1.1841 & -1.8276 & \\
-1.5346 & 2.9152 & -1.8276 & -1.1841 & -.7316 & .7316 & -.3729 & -.3489 & \\
-.4441 & -1.8276 & 2.9152 & -1.8276 & -.5921 & .5921 & -.4441 & -.8526 & \\
-.3729 & -1.1841 & -1.8276 & 2.9152 & -1.0960 & 1.0960 & -1.5346 & -.3489 & \\
1.0741 & -1.0741 & -1.0907 & -3.9583 & 7.2804 & -7.2804 & 3.9583 & 1.0907 & 5(+) \\
-1.0741 & 1.0741 & 1.0907 & 3.9583 & -7.2804 & 7.2804 & -3.9583 & -1.0907 & 6(-) \\
-1.1841 & -.3729 & -.3489 & -1.5346 & 1.0960 & -1.0960 & 2.9152 & -1.8276 & \\
-1.8276 & -.4441 & -.8526 & -.4441 & .5921 & -.5921 & -1.8276 & 2.9152 &
\end{array}\right]$$

Figure 1. Dependent equations due to a degenerate boundary using the UT and LM methods for $\lambda = 1$. ○, Geometry node; $\boxed{N}$, the Nth constant or linear element. $[U]\{t\} = [T\{u\}$, $[L]\{t\} = [M]\{u\}$.

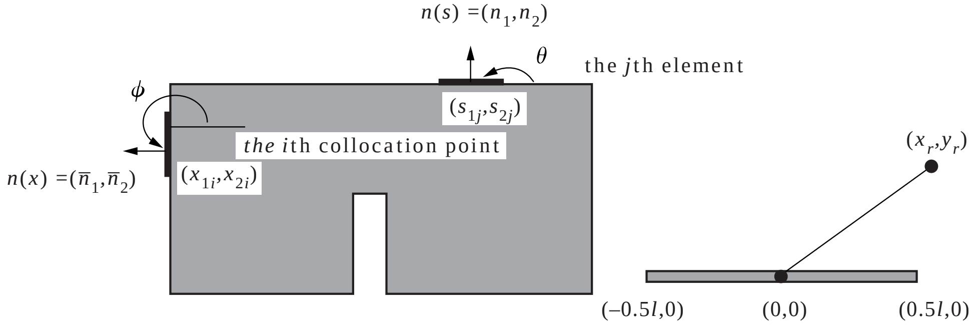


Figure 2. A convenient co-ordinate system.

$$\begin{aligned} M^j(s,x) = &-[(4j(j-1)\ln(r) + 4j - 2)r^{2j-4}y_i n_i y_k \bar{n}_k]A(j) \\ &-[(2j\ln(r)+1)r^{2j-2}n_i\bar{n}_i]A(j) \\ &+[4j(j-1)r^{2j-4}y_i n_i y_k \bar{n}_k]B(j) \\ &+[2jr^{2j-2}n_i\bar{n}_i]B(j), \end{aligned} \tag{14}$$

where r is the distance between x and s, $y_i = x_i - s_i$, n_i and $\bar{n}_i$ are the ith components for the normal vectors on s and x, respectively. The dual properties for the four kernels are shown below:

$$U^j(s,x) = U^j(x,s), \qquad T^j(s,x) = L^j(x,s), \qquad M^j(s,x) = M^j(x,s). \qquad (15\text{–}17)$$

For the special case of $j = 0$, one has

$$U^j(s,x) = \ln(r), \qquad T^j(s,x) = \frac{-y_i n_i}{r^2}, \qquad (18, 19)$$

$$L^j(s,x) = \frac{y_i \bar{n}_i}{r^2}, \qquad M^j(s,x) = \frac{2y_i n_i y_k \bar{n}_k}{r^4} - \frac{n_i \bar{n}_i}{r^2}, \qquad (20, 21)$$

which are the same as the four kernels proposed by Chen and Hong [12].

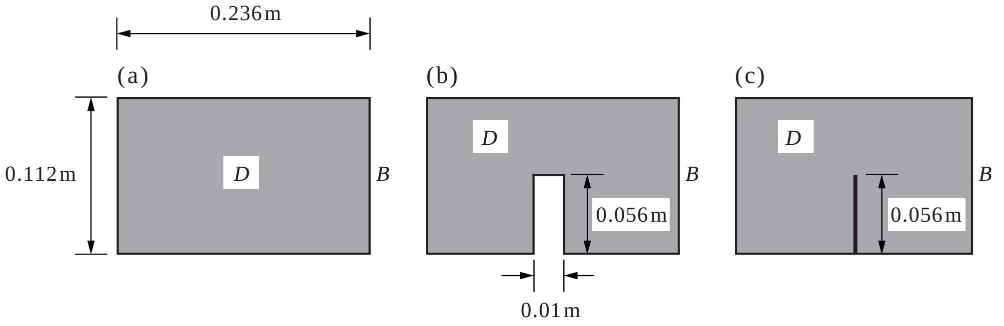


Figure 3. (a) Rectangular cavity without a partition. (b) Rectangular cavity with a partition with finite thickness. (c) Rectangular cavity with a partition of zero thickness. $(V^2 + k^2)u(x_1, x_2) = 0$, $(x_1, x_2 \in D$; $\partial u(s)/\partial n_s = 0$, s on all the boundaries B.

TABLE 2

Natural frequencies (Hz) of an acoustic cavity without a partition

Mode no.	FEM by ABAQUS(AC2D4) (32 elements)	FEM by ABAQUS(AC2D8) (32 elements)	Exact solution	Dual MRM(*UT*) (24 elements)	Dual MRM(*LM*) (24 elements)	DBEM by Chen [14] (*UT*)	DBEM by Chen [14] (*LM*)	Measurement by Petyt
1	724	729	720	721	719	720	722	720
2	1420	1458	1439	1444	1447	1442	1442	1515
3	1496	1536	1515	1519	1509	1517	1517	1677
4	1630	1700	1677	1689	1726	1680	1680	1827
5	1960	2118	1827	1997	1932	2094	2096	2264

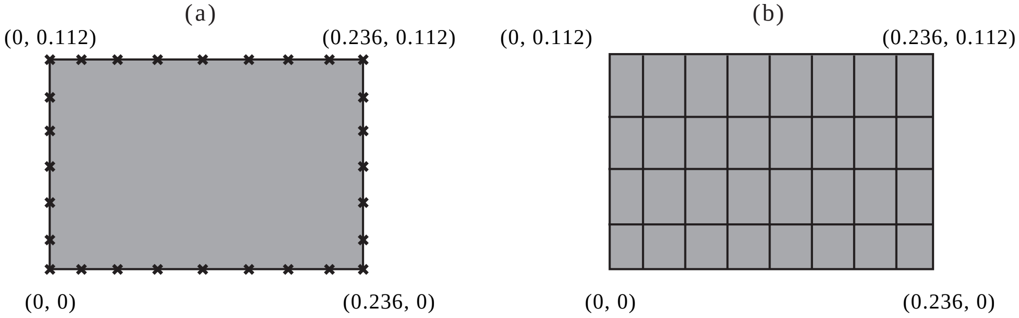


Figure 4. (a) Boundary element mesh for example 1. (b) Finite element mesh for example 1.

3. DUAL BOUNDARY ELEMENT METHOD FOR AN ACOUSTIC CAVITY WITH A THIN PARTITION

By discretizing the boundary B into boundary elements in equation (1), one has

$$\pi\{u\} = [T]\{u\} - [U]\{t\}, \tag{22}$$

where

$$[T] = [T^0] - k^2[T^1] + k^4[T^2] - k^6[T^3] + \cdots, \tag{23}$$

$$[U] = [U^0] - k^2[U^1] + k^4[U^2] - k^6[U^3] + \cdots, \tag{24}$$

in which the elements of $[U]$ and $[T]$ can be determined by

$$U^j_{pq} = \int_B U^j(s_q, x_p)\, \mathrm{d}B(s_q), \qquad T^j_{pq} = \int_B T^j(s_q, x_p)\, \mathrm{d}B(s_q). \tag{25, 26}$$

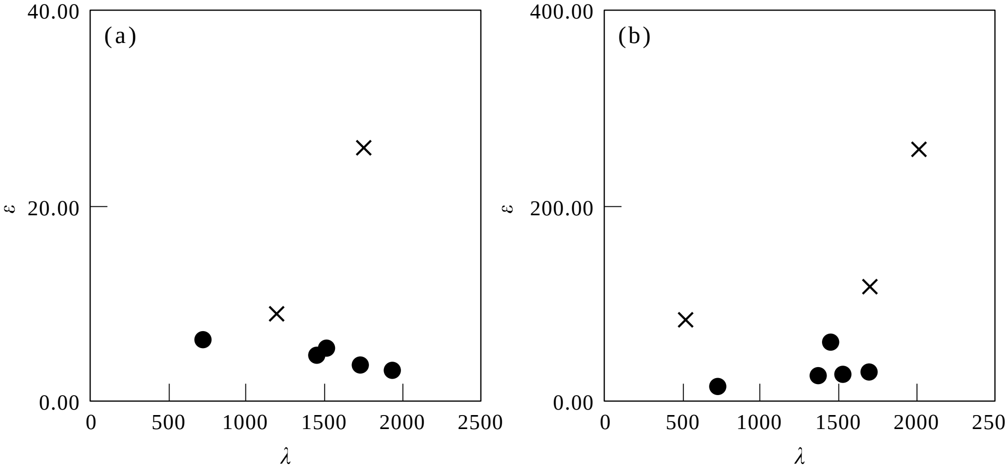


Figure 5. (a) Residual norm of $[Mu]$: ●, true eigenvalue; ×, spurious eigenvalue; $\overline{T}(\lambda)u = 0$; $\|M(\lambda)u\| = \varepsilon$. (b) Residual norm of $[\overline{T}u]$: ●, true eigenvalue; ×, spurious eigenvalue; $M(\lambda)u = 0$; $\|\overline{T}(\lambda)u\| = \varepsilon$.

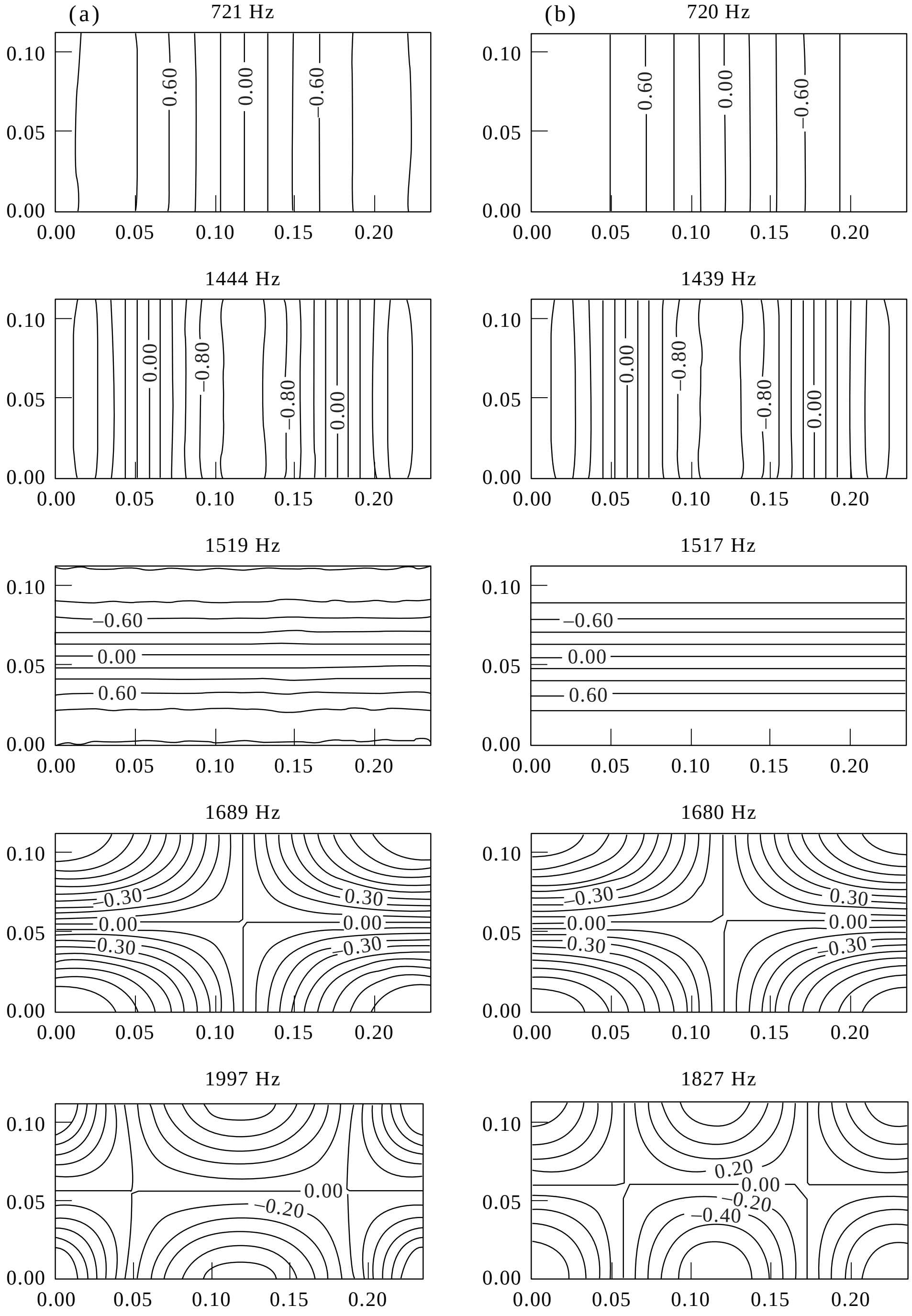


Figure 6. (a) The pressure contour of the mode shape with numerical results for example 1. (b) The pressure contour of the mode shape with the exact solution for example 1.

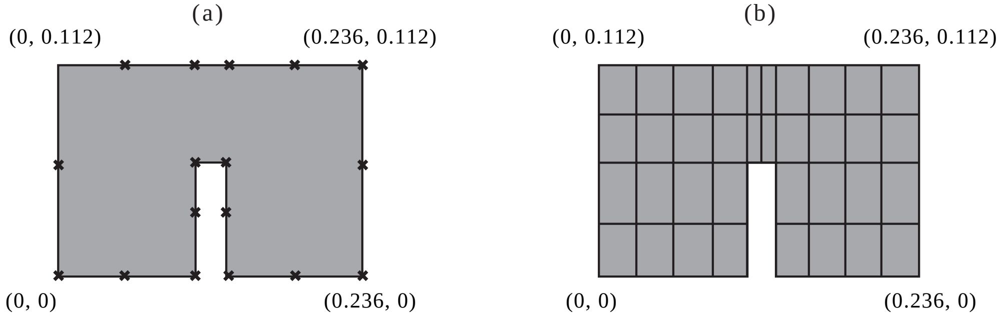


Figure 7. (a) Boundary element mesh for example 2. (b) Finite element mesh for example 2.

If one subtracts the free term in equation (26), then one has

$$\overline{T}^j_{pq} = T^j_{pq} - \pi\delta_{pq}\delta_{j0}. \tag{27}$$

Similarly, the corresponding algebraic equation for equation (6) is

$$\pi\{t\} = [M]\{u\} - [L]\{t\}, \tag{28}$$

where

$$[L] = [L^0] - k^2[L^1] + k^4[L^2] - k^6[L^3] + \cdots, \tag{29}$$

$$[M] = [M^0] - k^2[M^1] + k^4[M^2] - k^6[M^3] + \cdots, \tag{30}$$

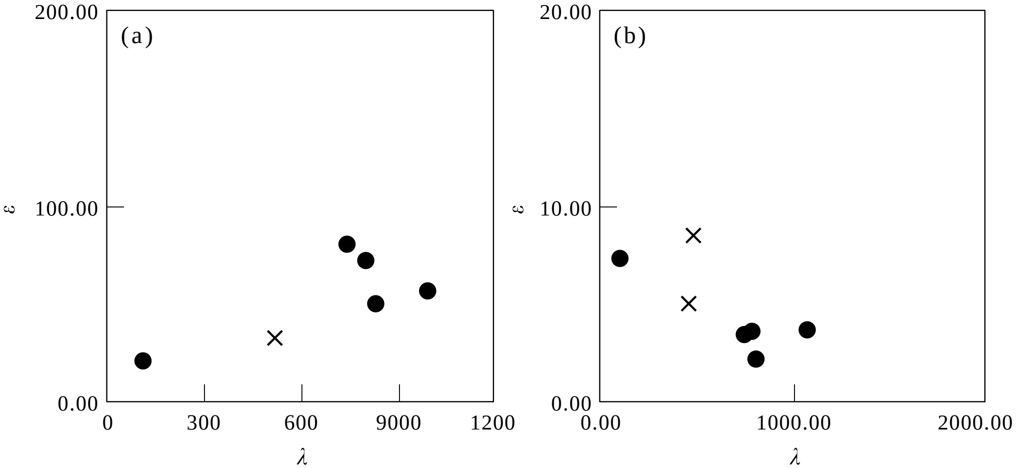


Figure 8. (a) Residual norm of $[Mu]$: ●, true eigenvalue; ×, spurious eigenvalue; $\overline{T}(\lambda)u = 0$; $\|M(\lambda)u\| = \varepsilon$. (b) Residual norm of $[\overline{T}u]$: ●, true eigenvalue; ×, spurious eigenvalue; $M(\lambda)u = 0$; $\|\overline{T}(\lambda)u\| = \varepsilon$.

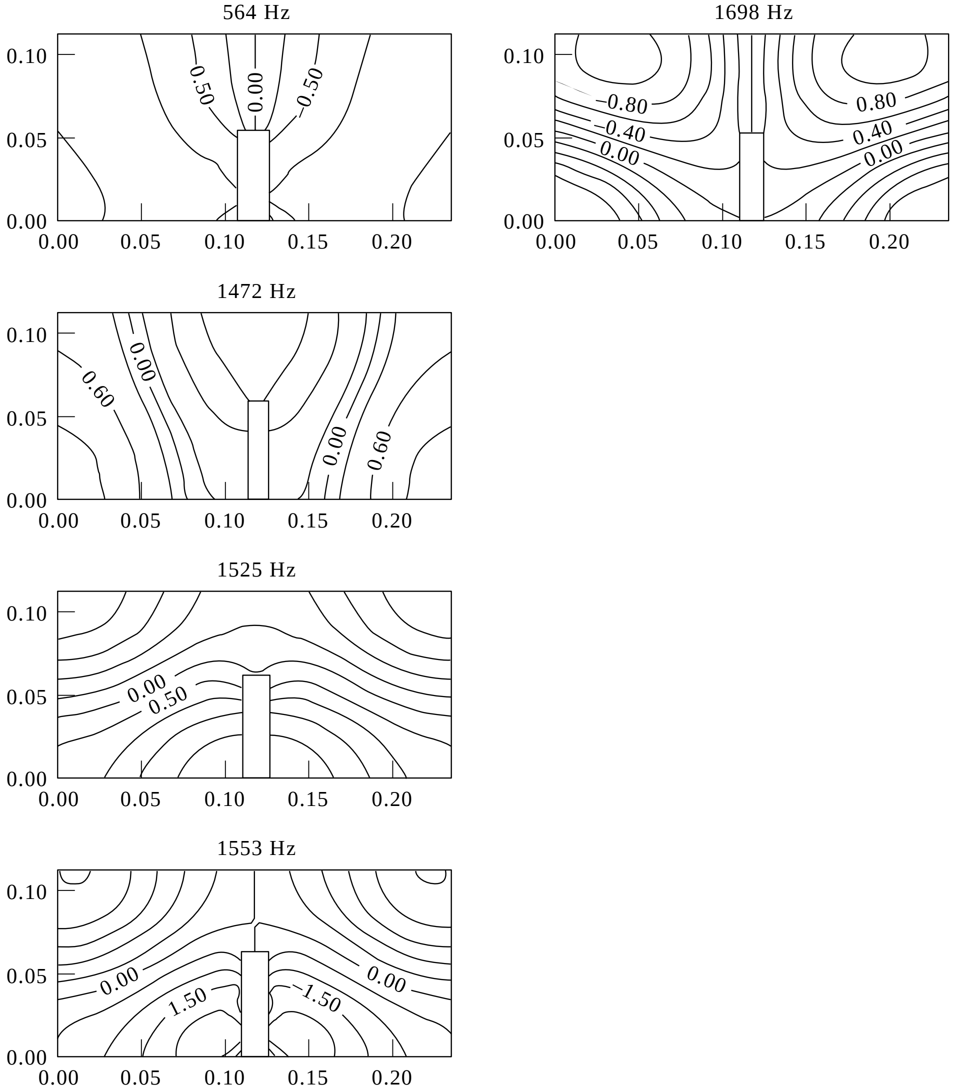


Figure 9. The pressure contour of the mode shape of the present method for example 2.

in which the elements in $[L]$ and $[M]$ can be determined by

$$L^j_{pq} = \int_B L^j(s_q, x_p)\, \mathrm{d}B(s_q), \tag{31}$$

$$M^j_{pq} = \int_B M^j(s_q, x_p)\, \mathrm{d}B(s_q). \tag{32}$$

Similarly, by adding the free term in equation (31), one has

$$\bar{L}^j_{pq} = L^j_{pq} + \pi\delta_{pq}\delta_{j0}. \tag{33}$$

TABLE 3

Natural frequencies (Hz) of an acoustic cavity with a partition of finite thickness

Mode no.	FEM by ABAQUS(AC2D4) (36 elements)	FEM by ABAQUS(AC2D8) (36 elements)	FEM by Petyt (9 elements)	Measurement by Petyt	Dual MRM(*UT*) (36 elements)	Dual MRM(*LM*) (36 elements)	DBEM by Chen [14] (*UT*)	DBEM by Chen [14] (*LM*)
1	590	579	577	570	564	558	586	587
2	1443	1480	1450	1470	1464	1477	1452	1443
3	1502	1542	1550	1534	1523	1513	1518	1516
4	1527	1564	1610	1555	1549	1535	1539	1535
5	1786	1858	1860	1840	1692	1765	1824	1818

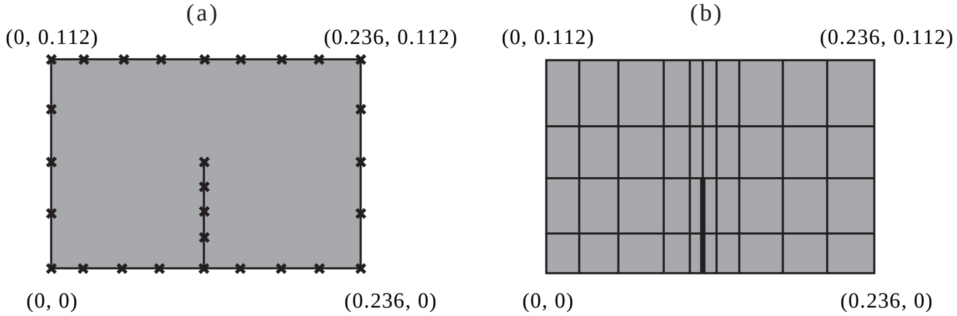


Figure 10. (a) Boundary element mesh for example 3. (b) Finite element mesh for example 3.

4. DETERMINATION OF THE INFLUENCE COEFFICIENTS BY CO-ORDINATE TRANSFORMATION USING THE OBJECTIVITY POINT OF VIEW

Since the influence coefficient is a scalar invariant under any co-ordinate transformation, the objectivity of the invariant integral for the influence coefficients should be obeyed. Therefore, one can define a convenient co-ordinate, system as shown in Figure 2, with the following components for the normal vector, $n(x)$,

$$n_1 = 0, \qquad n_2 = -1. \tag{34}$$

if (x_1, x_2) is the interior point, one has

$$\bar{n}_1 = \sin(\phi - \theta), \qquad \bar{n}_2 = -\cos(\phi - \theta), \tag{35}$$

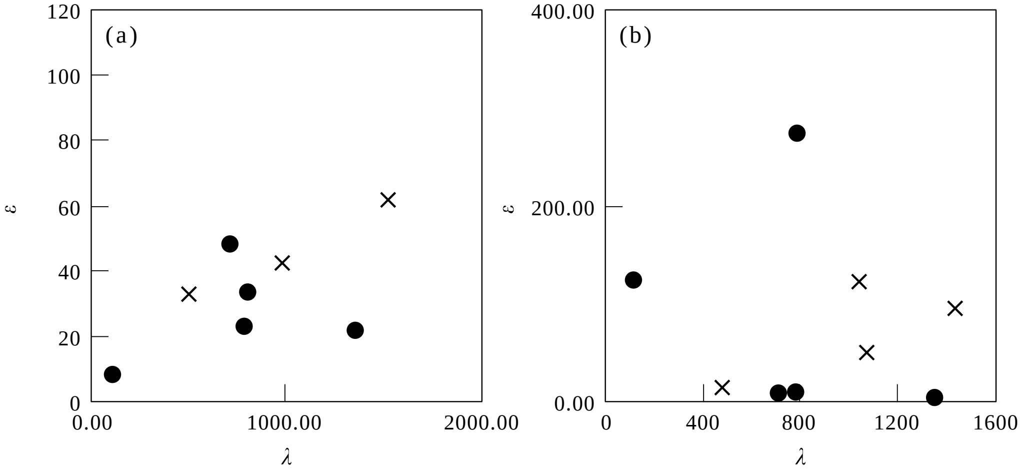


Figure 11. (a) Residual norm of $[Mu]$: ●, true eigenvalue; ×, spurious eigenvalue; $\bar{T}(\lambda)u = 0$; $\|M(\lambda)u\| = \varepsilon$. (b) Residual norm of $[\bar{T}u]$: ●, true eigenvalue; ×, spurious eigenvalue; $M(\lambda)u = 0$; $\|\bar{T}(\lambda)u\| = \varepsilon$.

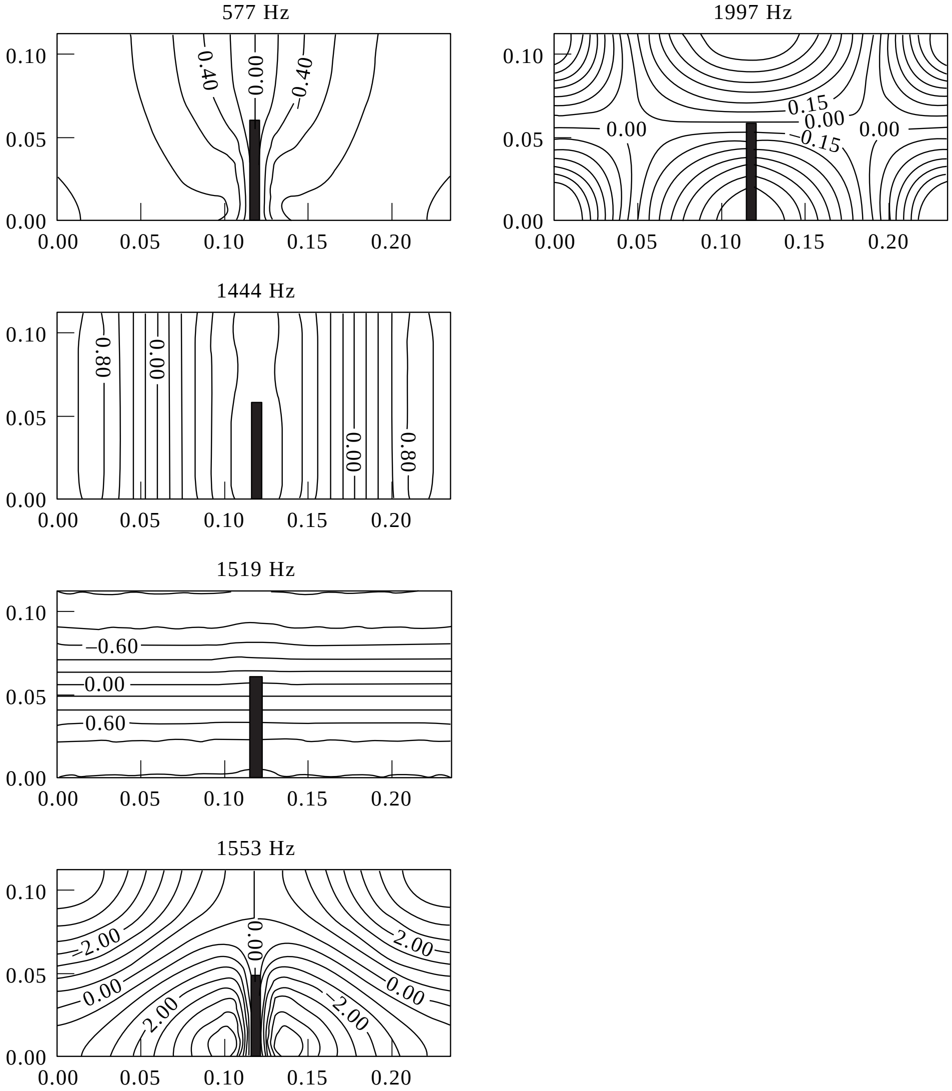


Figure 12. The pressure contour of the mode shape of the present method for example 3.

where θ and ϕ are defined in Figure 2. After translation and rotation, the co-ordinate of (x_1, x_2) changes to

$$\begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{Bmatrix} x_1 - s_1 \\ x_2 - s_2 \end{Bmatrix}. \tag{36}$$

If (x_1, x_2) is on the boundary, one has

$$\bar{n}_1 = 0, \qquad \bar{n}_2 = -1. \tag{37}$$

Table 4

Natural frequencies (*Hz*) *of an acoustic cavity with a partition of zero thickness*

Mode no.	FEM by ABAQUS(AC2D4) (40 elements)	FEM by ABAQUS(AC2D8) (40 elements)	FEM by Petyt (9 elements)	Measurement by Petyt	Dual MRM(*UT*) (32 elements)	Dual MRM(*LM*) (32 elements)	DBEM by Chen [14] (*UT*)	DBEM by Chen [14] (*LM*)
1	618	605	586	570	577	576	584	584
2	1421	1458	1478	1470	1444	1447	1439	1439
3	1496	1536	1540	1534	1519	1509	1518	1518
4	1527	1563	1570	1555	1533	1521	1537	1534
5	1780	1851	1861	1840	1997	1991	1818	1818

Therefore, the following equations can be obtained after objective orientation:

$$y_i n_i = -y_2, \qquad y_i \bar{n}_i = -y_2, \qquad n_i \bar{n}_i = 1, \qquad y_i n_i y_j \bar{n}_j = y_2^2. \qquad (38\text{–}41)$$

According to equations (38–41), it is found that only the following two types of integrals must be obtained in the case of the constant element scheme, and they are

$$\int_B r^{2j} \ln(r)\, \mathrm{d}s \quad (j = 0, 1, 2, \ldots) \qquad \text{and} \qquad \int_B r^{2j}\, \mathrm{d}s \quad (j = 0, 1, 2, \ldots).$$

By using symbolic manipulation software, the analytical formula for the kernel integration was obtained in reference [11]. The influence coefficients for the singular element can be obtained by employing L'hospital's rule for the diagonal terms of the U, $\bar{T}$, $\bar{L}$ and M matrices. In determining the influence coefficients, two formulations can be employed to change the position of zero in the denominator into the numerator for the easier implementation in the numerical computation as follows:

$$\tan^{-1}(v) + \tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2}, \qquad \text{if } v > 0,$$

$$\tan^{-1}(v) + \tan^{-1}\left(\frac{1}{v}\right) = -\frac{\pi}{2}, \qquad \text{if } v < 0.$$

It must be noted that the transformed functions should be consistent in substituting the boundary values; i.e., the following equation should be obeyed:

$$\tan^{-1}(v)\big|_a^b = -\tan^{-1}\left(\frac{1}{v}\right)\bigg|_a^b, \qquad \text{if } ab > 0.$$

However, the inequality occurs as shown below:

$$\tan^{-1}(v)\big|_a^b \neq -\tan^{-1}\left(\frac{1}{v}\right)\bigg|_a^b, \qquad \text{if } ab < 0.$$

5. TRANSFORMATION FROM A GENERALIZED EIGENVALUE PROBLEM TO A STANDARD EIGENVALUE PROBLEM

For simplicity, only the Neumann type boundary condition is considered as follows:

$$\{t\} = 0.$$

Since convergence for the series of kernels can be obtained [4], a generalized eigenvalue problem can be derived by considering the former $n+1$ terms by assuming that the residue terms can be omitted as follows:

$$\{[\overline{T}^0] - k^2[T^1] + k^4[T^2] - k^6[T^3] + \cdots + (-1)^n k^{2n}[T^n]\}\{u\} = 0. \tag{42}$$

By defining $\lambda = k^2$, one has

$$\{[\overline{T}^0] - \lambda[T^1] + \lambda^2[T^2] - \lambda^3[T^3] + \cdots + (-1)^n \lambda^n[T^n]\}\{u\} = 0. \tag{43}$$

By introducing the state variable vector as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} u \\ \lambda u \\ \lambda^2 u \\ \vdots \\ \lambda^{n-1} u \end{Bmatrix}, \tag{44}$$

the generalized eigenvalue problem in equation (43) can be transformed into a standard eigenvalue problem with a real unsymmetric matrix as follows:

$$\begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{-(T^n)^{-1}\overline{T}^0}{(-1)^n} & \dfrac{(T^n)^{-1}T^1}{(-1)^n} & \dfrac{-(T^n)^{-1}T^2}{(-1)^n} & \cdots & \dfrac{-(-1)^{n-1}(T^n)^{-1}T^{n-1}}{(-1)^n} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}$$

$$= \lambda \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix}, \tag{45}$$

where I is a unit matrix. If the collocation points, x_{p+} and x_{p-}, are located on the two sides of the degenerate boundary with the same geometry co-ordinates, the following two constraints are the same:

$$[[\overline{T}^0_{p^+q}] - \lambda[T^1_{p^+q}] + \lambda^2[T^2_{p^+q}] + \cdots]\{u_q\} = 0, \tag{46}$$

$$[[\overline{T}^0_{p^-q}] - \lambda[T^1_{p^-q}] + \lambda^2[T^2_{p^-q}] + \cdots]\{u_q\} = 0, \tag{47}$$

where $[T^j_{p^+q}]$ and $[T^j_{p^-q}]$ are the pth row vectors in $[T^j]$ for the two points on both sides of partition, respectively. Figure 1 indicates the result. Similarly, the two constraints from the *LM* equation are linearly dependent by a negative sign as follows:

$$[[M^0_{p^+q}] - \lambda[M^1_{p^+q}] + \lambda^2[M^2_{p^+q}] + \cdots]\{u_q\} = 0, \tag{48}$$

$$[[M^0_{p^-q}] - \lambda[M^1_{p^-q}] + \lambda^2[M^2_{p^-q}] + \cdots]\{u_q\} = 0, \tag{49}$$

where $[M^j_{p^+q}]$ and $[M^j_{p^-q}]$ are the pth row vectors in $[M^j]$ for the two points on both sides of partition, respectively. Figure 1 indicates the result. To provide the sufficient constraint, two equations, one from equation (46) or equation (47) and another from equation (48) or equation (49), are both required.

6. DETECTION OF SPURIOUS ROOTS USING DUAL FORMULATION FOR MRM

According to equations (22) and (28), one can obtain the eigenvalues independently for the problem without degenerate boundaries. However, spurious roots are imbedded in equation (22) or equation (28). As mentioned by Kamiya *et al*. [9], the equation derived using MRM is no more than a real part of the complex-valued formulation. The loss of the imaginary part in MRM results in the spurious roots. Yeih *et al*. [13] and Chen [14] extended the general proof for any dimensional problems and demonstrated it using a one-dimensional case. The imaginary part in the complex-valued formulation is not present in MRM, and the number of constraints for the eigenequation is insufficient. These findings can explain the reason why the spurious roots occur using the MRM method when either equation (22) or equation (28) only is employed: i.e., the mechanism of the spurious roots can be understood in this way.

Since only the real part is concerned in MRM, another approach to obtaining enough constraints for the eigenequation instead of the imaginary part of the complex-valued formulation is obtained by differentiation with respect to the conventional MRM. This method results in the hypersingular formulation for MRM. For simplicity, we deal with the Neumann problem. Therefore, equations (22) and (28) reduce to

$$[\overline{T}(\lambda)]\{u\} = 0, \qquad [M(\lambda)]\{u\} = 0. \tag{50, 51}$$

An approach to detecting the spurious roots is the criterion of satisfying both equations (50) and (51). The spurious roots from equation (50) will not satisfy equation (51). Also, the spurious roots from equation (51) will not satisfy equation (50) in controversa. Therefore, a residual norm can be defined as follows:

$$\varepsilon_T = [\overline{T}(\lambda_M)]\{u_M\}, \tag{52}$$

where $\{u_M\}$ satisfies $[M(\lambda_M)]\{u_M\} = 0$

$$\varepsilon_M = [M(\lambda_T)]\{u_T\}, \tag{53}$$

where $\{u_T\}$ satisfies $[\overline{T}(\lambda_T)]\{u_T\} = 0$, and ε_T and ε_M are the residue norms induced by equations (52) and (53), respectively, and λ_M and λ_T are the eigenvalues obtained by equations (50) and (51), respectively. By setting an appropriate value of the threshold, one can determine whether the root is true or spurious. To double check, the acoustic modes are examined by means of the distribution of nodal lines and orthogonal properties.

7. NUMERICAL EXAMPLES

7.1. EXAMPLE 1. RECTANGULAR CAVITY WITHOUT PARTITIONS

In this case, an analytical solution is available as follows:

$$\text{Eigenvalues:} \quad k_{mn} = \pi\sqrt{\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2}, \qquad (m, n = 0, 1, 2, \ldots),$$

$$\text{Eigenmode:} \quad u_{mn}(x, y) = \cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right),$$

where L_x and L_y denote the length and width of the cavity, respectively. Also, the FEM solution obtained by Petyt can be obtained using the ABAQUS program. To test the present program, DUALMRM, the results were compared with the above solutions, and experimental data [2], as shown in Table 2. Since no degenerate boundary is present, either the *UT* or *LM* method can be used to solve the problems. The present results (*UT* and *LM* methods) were compared with the exact solutions, two ABAQUS results and complex-valued dual BEM in references [14–16]. Good agreement was achieved. Figures 4(a) and (b) show the boundary element and finite element meshes, respectively. The residual norms, ε_T in equation (52) and ε_M in equation (53), of the spurious roots using the *UT* and *LM* methods are shown in Figures 5(a) and (b), respectively. It is found that the residual of the true eigenvalue is smaller than that of the spurious roots, as expected. An appropriate threshold can be chosen to distinguish which eigenvalue is true. The former five modes are shown in Figure 6(a). Two BEM results (*UT* and *LM* methods) can be found to have higher accuracy than the ABAQUS solution after comparison with the exact solution.

7.2. EXAMPLE 2. RECTANGULAR CAVITY WITH A PARTITION OF FINITE THICKNESS

In this case, a partition with a finite thickness of 10 mm is considered. *UT* combined with the *LM* method can make the BE model more well-conditioned. Figures 7(a) and (b) show the boundary element mesh and finite element mesh, respectively. The residual norms of the spurious roots obtained using the *UT* and *LM* methods are shown in Figures 8(a) and (b), respectively. Since there exists a dependent relationship between *UT* and *LM* for the thin partition, an appropriate threshold can not be determined. Therefore, the mode should be plotted and detected to determine whether or not it is a true mode. The former five modes is shown in Figure 9. The acoustic frequencies are shown in Table 3.

Since two alternatives, the *UT* or *LM* equations, can be chosen when collocating on the outer (normal) boundary, two results, obtained using the *UT* and *LM* methods, can be obtained. FEM results obtained by Petyt and ABAQUS, complex-valued dual BEM results and experimental data measured by Petyt *et al*. have also been compared with the present solutions, and agreement between them has been found.

7.3. EXAMPLE 3. RECTANGULAR CAVITY WITH A PARTITION OF ZERO THICKNESS (*UT* COMBINED WITH *LM* TECHNIQUE)

When the thickness of the partition became zero, the dual formulation for MRM was employed to solve the problem. Figures 10(a) and (b) shows the boundary element mesh and finite element mesh, respectively. The residual norms of the spurious roots using the *UT* and *LM* methods are shown in Figures 11(a) and (b), respectively. Since there exists a dependent relationship between *UT* and *LM* for the zero thickness partition, an appropriate threshold could not be determined. Therefore, the mode was plotted and detected to determine whether or not it was a true mode. The former four modes are shown in Figure 12. The acoustic frequencies are shown in Table 4. Since two alternatives, the *UT* or *LM* equation, could be chosen when collocating on the outer (normal) boundary, two results from the *UT* and *LM* methods, could be obtained. FEM results obtained by Petyt and ABAQUS, complex-valued dual BEM results and experimental data measured by Petyt *et al*. were also compared with the present solutions, and agreement was found between the numerical results and experimental data.

8. CONCLUSIONS

The dual formulation for MRM has been applied to solve the acoustic modes of a cavity with a thin partition. The frequency dependent eigenmatrix obtained using BEM and the non-uniqueness of the solution due to the zero thickness partition could be avoided simultaneously. A general purpose program, DUALMRM, has been developed to determine the acoustic frequencies and modes of an arbitrary cavity with or without a partition. Numerical results show that the present method can predict the acoustic eigenfrequencies more efficiently than FEM. Also, the numerical results match the experimental data well.

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