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A UNIFORMIZING METHOD FOR THE FREE VIBRATION ANALYSIS OF METAL–PIEZOCERAMIC COMPOSITE THIN PLATES

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A uniformizing method is presented for the free vibration analysis of metal–piezoceramic composite thin plates which are popular structures of thin plate-type ultrasonic motors. A metal–piezoceramic composite thin plate is normally composed of a metal plate with one surface adhering to a piezoelectric ceramic plate. The uniformizing method described in this paper aims to obtain an equivalent single-layer uniform thin plate which has the same free vibration characteristics as the metal–piezoceramic composite thin plate. Hence the free vibration analysis for a metal–piezoelectric composite thin plate can be performed through investigating the free vibration behaviors of the equivalent single-layer uniform thin plate by using classical thin plate theory.

In order to confirm the validity of the uniformizing method in this paper, two actual configurations of the metal–piezoceramic composite thin plate structure are constructed. By comparing the measured natural frequencies and vibration models with those yielded by the uniformizing method, one finds not only excellent agreement but also satisfying precision for engineering uses.
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1. INTRODUCTION

The structures of metal–piezoceramic composite thin plates have found ever increasing application as electro–acoustic transducers in naval and ocean engineering system and also in ultrasonic motors as piezoelectric vibrators.

Increasing use of these structures in various engineering fields necessitate the investigation of vibration analysis for metal–piezoceramic composite plates. There are three common theories for laminated plates: classical laminated plate theory [1], first order shear deformation laminated plate theory and third order shear deformation laminated plate theory [2, 3]. Some studies on the vibration analysis of laminated plates have been conducted by Liew [4, 5] and Nosier *et al*. [6], but the problem becomes more complicated when piezoelectric composite plates are involved because the piezoelectric effect must be taken into consideration. In 1973 Denkman [7], employing the Rayleigh–Ritz method, calculated the approximate natural frequencies of thin piezoceramic composite circular plates, which were not

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precise enough to be used practically. Woollette [8] obtained the first order natural frequency which has satisfactory precision for engineering use. In 1980, Adelman and Stavsky [9] and Stavsky and Loewy [10] gave the exact solution for the vibration of composite piezoelectric circular plates. In 1983, Mu [11], using Hamilton's principle, presented the motion differential equation and boundary conditions of metal–piezoceramic composite circular disks, as well as the generalized solution and frequency equation under resonant condition.

All the studies were limited to the axisymmetrical vibration analysis for some special and concrete configurations, e.g., circular or annular plates using energy and variation methods. In 1959, Pister and Dong [12] proposed in their layered plates theory that, when introducing equivalent bending stiffness D_e, layered plates can be treated as single-layer uniform plates. They also obtained the equivalent bending stiffness of the layered plate which was symmetrical about the adhered layer which divided the layered plate in halves equally.

The purpose of this paper is to develop a uniformizing method for the free vibration analysis of metal–piezoceramic composite thin plates which are not symmetrical about the adhered layer, and the piezoelectric effect will be taken into consideration. By using the uniformizing method, the vibration of metal–piezoelectric composite thin plates can be expressed as equivalent single-layer uniform thin plates having the same free vibration characteristics. Consequently, the complicated problems about the metal–piezoelectric composite plates are converted into problems of single-layer uniform thin plates.

The uniformizing method proposed in this paper is able to investigate free vibration of metal–piezoceramic composite thin plates with various shapes, e.g., circular disks or rectangular plates, various vibration modes, e.g., axisymmetric or non-axisymmetric modes and various boundary conditions, e.g., clamped, simply-supported and free. The uniformizing method of this paper is based on Kirchhoff's thin plate theory and g-piezoelectric constitutive equations. To uniformize a piezoelectric composite plate, two fundamental assumptions are made: i.e., besides the continuous strain condition at adhered points, the shearing stress components τ_{zx}, τ_{zy}, T_{zx}, T_{zy} of both the metal and the piezoceramic are also equal in the adhered layer.

2. UNIFORMIZING A METAL–PIEZOCERAMIC COMPOSITE THIN PLATE

In this section, in order to uniformize a metal–piezoceramic composite thin plate into a corresponding equivalent single-layer uniform thin plate, the following procedures are employed: first, several fundamental assumptions are proposed; next, the midplane position h_0 is calculated by using Kirchhoff's thin plate theory and the g-piezoelectric constitutive equations; after this, the differential motion equation is established for the metal–piezoelectric composite plate together with three kinds of boundary conditions; finally, through comparing the motion equation and boundary conditions of the metal–piezoelectric composite plate with those of a typical single-layer uniform thin plate, one obtains the equivalent quantities D_e, u_e, E_e, ρ_e, h_e of an equivalent single-layer uniform thin plate.

2.1. FUNDAMENTAL ASSUMPTIONS

An arbitrary element from the metal–piezoceramic composite thin plate in Cartesian co-ordinates as shown in Figure 1 is taken. The upper layer is a metal plate while the lower one is a piezoceramic plate. One assumes that the position of the mid-plane is h_0 along the z-axis. t_1, t_2 are the thicknesses of the metal layer and piezoceramic layer respectively. The metal–piezoceramic composite thin plates in the present study satisfy the following basic assumptions:

(1) The thickness of the metal–piezoceramic composite thin plate is sufficiently small compared to the other dimensions. Normally the ratio of the thickness to the smaller span length is less than 1/20 so that the composite plates are subject to the fundamental assumptions of the Kirchhoff–Love thin plate theory [13]. The components of strain in the metal layer parallel to the x and y-axis and the xy direction can be given as follows

$$\epsilon_x = -(z - h_0)\,\partial^2\omega/\partial x^2, \qquad \epsilon_y = -(z - h_0)\,\partial^2\omega/\partial y^2,$$
$$\gamma_{xy} = -2(z - h_0)\,\partial^2\omega/\partial x\,\partial y, \tag{1}$$

where ω is the transverse displacement, ϵ_x, ϵ_y, γ_{xy} are strain components of the metal thin plate parallel to the x-, y- and xy-directions respectively. In the piezoceramic layer

$$S_1 = -(z - h_0)\,\partial^2\omega/\partial x^2, \qquad S_2 = -(z - h_0)\,\partial^2\omega/\partial y^2,$$
$$S_3 = -2(z - h_0)\,\partial^2\omega/\partial x\,\partial y, \tag{2}$$

where S_k ($k = 1, 2, \ldots, 6$) denotes the strain components of the piezoelectric plate, using IEEE standard notation [14].

(2) The metal and the piezoceramic are assumed to adhere perfectly. At the connected points the strain components of both the metal layer and the piezoelectric layer are continuous.

(3) The lateral shearing stress components are also equal in the adhered layer. So one has

$$\tau_{zx}|_{z=t_1} = T_{zx}|_{z=t_1}, \qquad \tau_{zy}|_{z=t_1} = T_{zy}|_{z=t_1}. \tag{3}$$

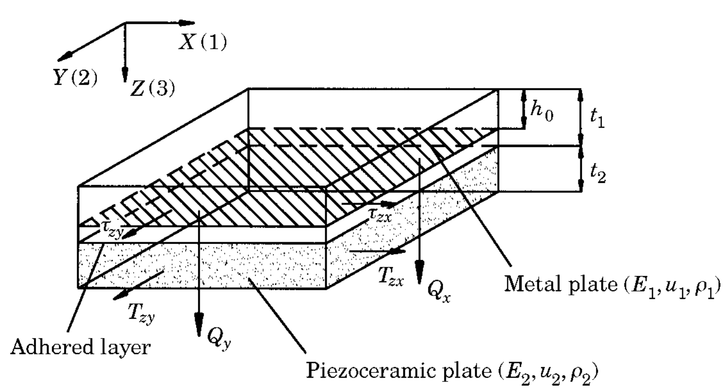

Figure 1. The metal–piezoelectric composite plate element.

where τ_{zx}, τ_{zy} are lateral shearing stress components of the metal plate. According to the IEEE standard notation, T_k $(k = 1, 2, \ldots, 6)$ represents components of stress of the piezoelectric element. In this paper one defines $T_{xx} = T_1$, $T_{yy} = T_2$, $T_{zz} = T_3$, $T_{zx} = T_5$, $T_{zy} = T_4$, $T_{xy} = T_6$.

(4) The moment of inertia is neglected. The differential equations of motion along the x- and y-axes in the metal layer can be written as

$$\partial\tau_{zx}/\partial z + \partial\tau_{yx}/\partial y + \partial\sigma_x/\partial x = 0, \qquad \partial\tau_{zy}/\partial z + \partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = 0. \quad (4, 5)$$

Correspondingly one deduces the following equations in the piezoelectric layer

$$\partial T_{zx}/\partial z + \partial T_{yx}/\partial y + \partial T_{xx}/\partial x = 0, \qquad \partial T_{zy}/\partial z + \partial T_{xy}/\partial x + \partial T_{yy}/\partial y = 0. \quad (6, 7)$$

(5) The electric field density in the piezoceramic is uniform along the thickness of the piezoelectric layer.

2.2. OBTAINING THE MIDPLANE POSITION h_0

2.2.1. τ_{zx} *in the metal layer*

According to the generalized two dimension Hooke's law in isotropic materials,

$$\epsilon_x = (1/E_1)\sigma_x - (u_1/E_1)\sigma_y, \qquad \epsilon_y = -(u_1/E_1)\sigma_x + (1/E_1)\sigma_y,$$
$$\gamma_{xy} = 2(1 + u_1)/E_1\, \tau_{xy}, \quad (8)$$

in the metal layer where E_1 is the Young's modules, u_1 is the Poisson ratio, and σ_x, σ_y, τ_{xy} are the stress components parallel to the x, y and xy directions respectively. Equation (8) can also be written as

$$\sigma_x = E_1/(1 - u_1^2)(\epsilon_x + u_1\epsilon_y), \qquad \sigma_y = E_1/(1 - u_1^2)(\epsilon_y + u_1\epsilon_x),$$
$$\tau_{xy} = E_1/[2(1 + u_1)]\gamma_{xy}. \quad (9)$$

Substituting equation (1) into equation (9), the resulting equations are

$$\sigma_x = -\frac{E_1(z - h_0)}{1 - u_1^2}\left(\frac{\partial^2\omega}{\partial x^2} + u_1\frac{\partial^2\omega}{\partial y^2}\right), \qquad \sigma_y = -\frac{E_1(z - h_0)}{1 - u_1^2}\left(\frac{\partial^2\omega}{\partial y^2} + u_1\frac{\partial^2\omega}{\partial x^2}\right),$$

$$\tau_{xy} = -\frac{E_1(z - h_0)}{1 + u_1}\frac{\partial^2\omega}{\partial x\,\partial y}. \quad (10)$$

Differentiating σ_x, τ_{xy} with respect to X, Y respectively, and introducing the results into equation (4), one obtains

$$\partial\tau_{zx}/\partial z = [E_1(z - h_0)/(1 - u_1^2)](\partial/\partial x)\nabla^2\omega, \quad (11)$$

where $\nabla^2\omega = \partial^2\omega/\partial x^2 + \partial^2\omega/\partial y^2$. Integrating equation (11) over z, one obtains

$$\tau_{zx} = [E_1/(1 - u_1^2)](z^2/2 - h_0 z)(\partial/\partial x)\nabla^2\omega + f_1. \quad (12)$$

Here f_1 is determined by the surface stress condition of the metal plate.

2.2.2. T_{zx} in the piezoelectric layer

To obtain T_{zx} in the piezoelectric layer, the classical thin plate theory needs to be combined with the g-piezoelectric equation

$$S_h = S^D_{hk} T_k + g_{jh} D_j, \qquad e_i = -g_{ik} T_k + (1/\epsilon^T_{ij}) D_j \quad (h, k = 1, 2, \ldots, 6;\ i, j = 1, 2, 3). \tag{13}$$

where D_i ($i = 1, 2, 3$) represents the electric displacement along the x-, y- and z-axes respectively. To distinguish the electric field density from the Young's modules, e_i is substituted for E_i to represent electric field density. ϵ^T_{ij} is the permittivity of piezoelectric material, g_{jh} is piezoelectric voltage constant, and S^D_{hk} is elastic constant when electric displacement is constant.

The piezoelectric plate in the present study is two-dimensional isotropic resulting in

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \end{bmatrix}. \tag{14}$$

Equation (13) can be listed as follows

$$S_1 = S^D_{11} T_1 + S^D_{12} T_2 + g_{31} D_3, \qquad S_2 = S^D_{11} T_2 + S^D_{12} T_1 + g_{31} D_3,$$
$$S_6 = 2(S^D_{11} - S^D_{12}) T_6, \qquad e_3 = -g_{31} T_1 - g_{31} T_2 + (1/\epsilon^T_{33}) D_3 \tag{15}$$

Equation (15) can be rewritten as

$$T_1 = \frac{1}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} S_1 + \frac{-S^D_{12}/S^D_{11}}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} S_2 - \frac{g_{31}}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} D_3,$$

$$T_2 = \frac{-S^D_{12}/S^D_{11}}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} S_1 + \frac{1}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} S_2 - \frac{g_{31}}{S^D_{11}[1 - (-S^D_{12}/S^D_{11})^2]} D_3,$$

$$T_6 = \frac{1}{2S^D_{11}[1 + (-S^D_{12}/S^D_{11})]} S_6, \qquad e_3 = -g_{31}(T_1 + T_2) + \beta^T_{33} D_3. \tag{16}$$

By defining

$$E_2 = 1/S^D_{11}, \qquad u_2 = -S^D_{12}/S^D_{11} \tag{17}$$

and inserting equation (2) and (17) into (16) and rewriting equation (16), one obtains

$$T_1 = -[E_2(z-h_0)/(1-u_2^2)]\,\partial^2\omega/\partial x^2 - [E_2u_2(z-h_0)/(1-u_2^2)]\,\partial^2\omega/\partial y^2$$
$$-[E_2g_{31}/(1-u_2)]D_3, \tag{18}$$

$$T_2 = -[E_2(z-h_0)u_2/(1-u_2^2)]\,\partial^2\omega/\partial x^2 - [E_2(z-h_0)/(1-u_2^2)]\,\partial^2\omega/\partial y^2$$
$$-[E_2g_{31}/(1-u_2)]D_3, \tag{19}$$

$$T_6 = -[E_2/(1+u_2)]\,\partial^2\omega/\partial x\,\partial y, \qquad e_3 = -g_{31}(T_1+T_2) + (1/\epsilon_{33}^T)D_3. \quad (20, 21)$$

Substituting equations (18), (19) into (21), and integrating equation (21) over z from t_1 to t_1+t_2, one obtains

$$\int_{t_1}^{t_1+t_2} e_3\,\mathrm{d}z = E_2g_{31}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)\Bigg/(1-u_2)\int_{t_1}^{t_1+t_2}(z-h_0)\,\mathrm{d}z$$
$$+\left(\frac{2E_2g_{31}^2}{1-u_2}+\frac{1}{\epsilon_{33}^T}\right)t_2D_3. \tag{22}$$

According to the fifth assumption

$$\int_{t_1}^{t_1+t_2} e_3\,\mathrm{d}z = V, \tag{23}$$

where V is the voltage applied to the piezoelectric ceramic along the z direction. Introducing equation (23) into (22) one obtains

$$D_3 = [-g_{31}(t_1t_2+\tfrac{1}{2}t_2^2-h_0t_2)\nabla^2\omega + (1/E_2)(1-u_2)V]/\{[2g_{31}^2$$
$$+(1/\epsilon_{33}^T)(1/E_2)(1-u_2)]t_2\}. \tag{24}$$

By defining

$$K_1 = g_{31}^2(t_1t_2+\tfrac{1}{2}t_2^2)/[2g_{31}^2+\beta_{33}^T(1/E_2)(1-u_2)]t_2 1/E_2(1-u_2),$$
$$K_2 = -g_{31}^2t_2/\{[2g_{31}^2+\beta_{33}^T(1/E_2)(1-u_2)]t_2(1/E_2)(1-u_2)\},$$
$$K_3 = 1/\{[2g_{31}^2+\beta_{33}^T(1/E_2)(1-u_2)]t_2\}, \qquad \beta_{33}^T = 1/\epsilon_{33}^T$$

and multiplying expression (24) by $-E_2g_{31}/(1-u_2)$ results in

$$-[E_2g_{31}/(1-u_2)]D_3 = (K_1+K_2h_0)\nabla^2\omega - g_{31}K_3V. \tag{25}$$

Inserting equation (25) into equations (18) and (19), one can rewrite them as

$$T_1 = -[E_2(z-h_0)/(1-u_2^2)]\,\partial^2\omega/\partial x^2 - E_2u_2(z-h_0)/[(1-u_2^2)]\,\partial^2\omega/\partial y^2$$
$$+(K_1+K_2h_0)\nabla^2\omega - g_{31}K_3V, \tag{26}$$

$$T_2 = -E_2(z-h_0)u_2/[(1-u_2^2)]\,\partial^2\omega/\partial x^2 - [E_2(z-h_0)/[(1-u_2^2)]\,\partial^2\omega/\partial y^2$$
$$+(K_1+K_2h_0)\nabla^2\omega - g_{31}K_3V. \tag{27}$$

Differentiating T_1, T_6 with respect to x, y respectively and substituting the results into equation (6), one obtains

$$\partial T_{zx}/\partial z = [E_2(z-h_0)/(1-u_2^2)](\partial/\partial x)\nabla^2\omega - (K_1 + K_2 h_0)(\partial/\partial x)\nabla^2\omega - g_{31}K_3\,\partial V/\partial x, \tag{28}$$

where $\partial V/\partial x = 0$ (excluding the boundary of the composite plate). Integrating equation (28) over z results in

$$T_{zx} = [E_2(z^2/2 - h_0 z)/(1-u_2^2)](\partial/\partial x)\nabla^2\omega - (K_1 + K_2 h_0)z(\partial/\partial x)\nabla^2\omega + f_2, \tag{29}$$

where f_2 is determined by the surface stress condition of the piezoelectric plate. Putting formula (12) and (29) together, one has

$$\tau_{zx} = [E/(1-u_1^2)](z^2/2 - h_0 z)(\partial/\partial x)\nabla^2\omega + f_1,$$

$$T_{zx} = [E_2(z^2/2 - h_0 z)/(1-u_2^2)](\partial/\partial x)\nabla^2\omega + f_2. \tag{30}$$

By defining

$$P_x = (\partial/\partial x)\nabla^2\omega, \qquad Q_1 = E_1/(1-u_1^2), \qquad Q_2 = E_2/(1-u_2^2)$$

equation (30) can be simplified as

$$\tau_{zx} = Q_1 P_x(z^2/2 - h_0 z) + f_1, \qquad T_{zx} = Q_2 P_x(z^2/2 - h_0 z) + f_2 - (K_1 + K_2 h_0)zP_x. \tag{31}$$

2.2.3. *Obtaining* h_0

By considering the third assumption (equation 3) and the free surface stress conditions of the piezoelectric composite layer, one has

$$\tau_{zx}|_{z=t_1} = T_{zx}|_{z=t_1}, \qquad \tau_{zx}|_{z=0} = 0, \qquad T_{zx}|_{z=t_1+t_2} = 0. \tag{32}$$

Introducing equation (31) into (32) results in

$$Q_1 P_x(t_1^2/2 - h_0 t_1) + f_1 = Q_2 P_x(t_1^2/2 - h_0 t_1) + f_2 - (K_1 + K_2 h_0)t_1 P,$$

$$Q_1 P_x(0/2 - 0) + f_1 = 0,$$

$$Q_2 P_x[(t_1+t_2)^2/2 - h_0(t_1+t_2)] + f_2 - (K_1 + K_2 h_0)(t_1+t_2)P_x = 0. \tag{33}$$

Solving this equations group gives the following resolutions:

$$f_1 = 0, \qquad f_2 = (K_1 + K_2 h_0)(t_1+t_2)P_x - Q_2 P_x[(t_1+t_2)^2/2 - h_0(t_1+t_2)],$$

$$h_0 = \tfrac{1}{2}(Q_1 t_1^2 + Q_2 t_2^2 + 2Q_2 t_1 t_2 - 2K_1 t_2)/(Q_1 t_1 + Q_2 t_2 + K_2 t_2). \tag{34}$$

2.3. DIFFERENTIAL EQUATION OF MOTION AND BOUNDARY CONDITIONS

2.3.1. *Differential equation of motion*

In order to derive the equilibrium equation of motion of the metal–piezoceramic composite thin plate, the shear force resultant Q_x, is first obtained and which is given by

$$Q_x = \int_0^{t_1} \tau_{zx}\,\mathrm{d}z + \int_{t_1}^{t_1+t_2} T_{zx}\,\mathrm{d}z.$$

Inserting f_1, f_2 into τ_{zx}, T_{zx} and introducing the results into Q_x, one has

$$\begin{aligned} Q_x = {} & Q_1 P_x(t_1^3/6 - (h_0/2)t_1^2) + Q_2\ P_x\{[(t_1+t_2)^3 - t_1^3]/6 - (h_0/2)[(t_1+t_2)^2 - t_1^2]\} \\ & + (K_1 + K_2 h_0)P_x(t_1+t_2)t_2 - [(K_1+K_2h_0)P_x/2][(t_1+t_2)^2 - t_1^2] \\ & - Q_2 P_x[(t_1+t_2)^2/2 - h_0(t_1+t_2)]t_2. \end{aligned} \tag{35}$$

By defining

$$\begin{aligned} D_e = {} & -\{Q_1(t_1^3/6 - (h_0/2)t_1^2) + Q_2[(t_1+t_2)^3 - t_1^3]/6 - (h_0/2)(t_2^2 + 2t_1t_2)] \\ & + [(K_1+K_2h_0)/2]t_2^2 - Q_2[(t_1+t_2)^2/2 - h_0(t_1+t_2)]t_2\} \\ = {} & -\{Q_1(t_1^3/6 - (h_0/2)t_1^2) + Q_2([h_0t_2^2 - t_1t_2^2 - \tfrac{2}{3}t_2^3]/2) + [(K_1+K_2h_0)/2]t_2^2\}. \end{aligned} \tag{36}$$

Equation (35) can be simplified as

$$Q_x = -P_x D_e. \tag{37}$$

Similarly, one can obtain

$$Q_y = -P_y D_e, \tag{38}$$

where $P_y = (\partial/\partial y)\nabla^2\omega$. In Figure 1, the element is cut from the metal–piezoceramic composite plate by two pairs of planes parallel to co-ordinate planes xz and yz. For equilibrium of this element it is necessary that the sum of the forces acting on this element and the sum of their moments about the x, y and z-axes should be equal to zero separately. Here, the motion equilibrium equation along the z-axis is set up.

$$\partial Q_x/\partial x + \partial Q_y/\partial y = \bar{m}\,\partial^2\omega/\partial t^2. \tag{39}$$

Substituting Q_x and Q_y into equation (39) results in

$$\partial P_x/\partial x + \partial P_y/\partial y = -(\bar{m}/D_e)\,\partial^2\omega/\partial t^2, \tag{40}$$

where $\bar{m}$ is the mass per unit area of the metal–piezoceramic composite plate and is given by

$$\bar{m} = \rho_1 t_1 + \rho_2 t_2, \tag{41}$$

where ρ_1, ρ_2 are the mass density of the metal and piezoceramic respectively. Inserting expressions P_x, P_y into equation (40), the equilibrium equation of motion becomes

$$\partial^4\omega/\partial x^4 + 2\,\partial^4\omega/\partial x^2\,\partial y^2 + \partial^4\omega/\partial y^4 = -(\bar{m}/D_e)\,\partial^2\omega/\partial t^2. \tag{42}$$

Defining $\nabla^4 = \nabla^2\nabla^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, equation (42) is reduced to

$$\nabla^4\omega = -(\bar{m}/D_e)\,\partial^2\omega/\partial t. \tag{43}$$

2.3.2. *Obtaining the boundary condition expressions*

In this section, the three kinds of boundary conditions of the metal–piezoelectric composite plate will be established.

(1) Clamped edge: if the edge $y = 0$ of the composite plate is clamped as shown in Figure 2, both the flexion and the rotation angle at the points of this edge is zero giving

$$\omega|_{y=0} = 0, \qquad \partial\omega/\partial y|_{y=0} = 0. \tag{44}$$

(2) Simply supported case: if the edge ($x = 0$) of the plate is simply supported, the deflection and bending moment at this edge must be zero. So, one has

$$\omega|_{x=0} = 0, \qquad M_x|_{x=0} = 0. \tag{45}$$

The bending moment of this layered plate is given by

$$M_x|_{x=0} \int_0^{t_1} \sigma_x(z - h_0)\,\mathrm{d}z + \int_{t_1}^{t_1+t_2} T_1(z - h_0)\,\mathrm{d}z. \tag{46}$$

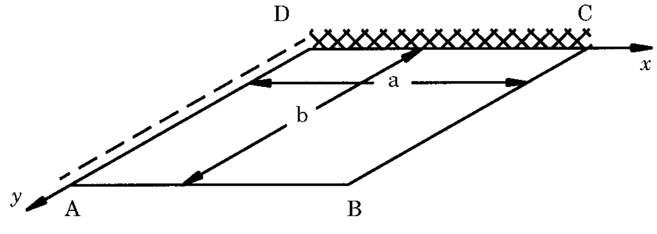

Figure 2. Three boundary conditions for the metal–piezoceramic plate.

Substituting equation (9) and (16) into equation (46), one obtains

$$M_x|_{x=0} = \int_0^{t_1} -Q_1(z-h_0)^2(\partial^2\omega/\partial x^2 + u_1\,\partial^2\omega/\partial y^2)\,\mathrm{d}z$$

$$+\int_{t_1}^{t_1+t_2}\left[-Q_2(z-h_0)\frac{\partial^2\omega}{\partial x^2} - Q_2u_2(z-h_0)\frac{\partial^2\omega}{\partial y^2} + (K_1+K_2h_0)\nabla^2\omega\right.$$

$$\left.-g_{31}K_3V\right](z-h_0)\,\mathrm{d}z = -aQ_1\left(\frac{\partial^2\omega}{\partial x^2}+u_1\frac{\partial^2\omega}{\partial y^2}\right) - bQ_2\left(\frac{\partial^2\omega}{\partial x^2}+u_2\frac{\partial^2\omega}{\partial y^2}\right)$$

$$+c(K_1+K_2h_0)\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right) - cK_3g_{31}V, \tag{47}$$

where

$$a = \tfrac{1}{3}t_1^3 + h_0^2t_1 - t_1^2h_0, \qquad b = [(t_1+t_2)^3 - t_1^3]/3 + t_2h_0^2 - h_0(t_2^2 + 2t_1t_2),$$
$$c = (t_2^2 + 2t_1t_2)/2 - h_0t_2.$$

In this paper only free vibration is investigated. So one can put $V = 0$. Substituting expression 47 into relation (45) results in

$$-[aQ_1 + bQ_2 - c(K_1+K_2h_0)]\left(\frac{\partial^2\omega}{\partial x^2} + \frac{aQ_1u_1 + bQ_2u_2 - c(K_1+K_2h_0)}{aQ_1 + bQ_2 - c(K_1+K_2h_0)}\frac{\partial^2\omega}{\partial y^2}\right) = 0. \tag{48}$$

The following equation (49) was obtained with the help of MathCAD 6.0 by Collabra Software, Inc.. The detailed process of derivation can be found in Appendix B of this paper

$$aQ_1 + bQ_2 - c(K_1+K_2h_0) = D_e. \tag{49}$$

By defining

$$u_e = [aQ_1u_1 + bQ_2u_2 - c(K_1+K_2h_0)/][aQ_1 + bQ_2 - c(K_1+K_2h_0)], \tag{50}$$

and introducing equations (49) and (50), formula (48) can be written as

$$-D_e(\partial^2\omega/\partial x + u_e\,\partial^2\omega/\partial y)|_{x=0} = 0. \tag{51}$$

(3) Free case: when the edge BC is free, according to the thin plate theory one has

$$M_x|_{x=a} = 0, \qquad (Q_x + \partial M_{xy}/\partial y)|_{x=a} = 0. \tag{52}$$

In the same way of deriving simply supported boundary condition, the relation $M_x|_{x=a} = 0$ can also be written as

$$-D_e(\partial^2\omega/\partial x + u_e\,\partial^2\omega/\partial y)|_{x=a} = 0. \tag{53}$$

M_{xy} can be written as

$$M_{xy} = \int_0^{t_1} \tau_{xy}(z - h_0)\,\mathrm{d}z + \int_{t_1}^{t_1+t_2} T_6(z - h_0)\,\mathrm{d}z$$

Inserting formulas (10) and (20) into M_{xy}, one has

$$M_{xy} = \int_0^{t_1} -\frac{E_1}{1+u_1}(z-h_0)^2\frac{\partial^2\omega}{\partial x\,\partial y}\,\mathrm{d}z + \int_{t_1}^{t_1+t_2} -\frac{E_2}{1+u_2}(z-h_0)^2\frac{\partial^2\omega}{\partial x\,\partial y}\,\mathrm{d}z$$

$$= \left\{-\frac{E_1}{1+u_1}\left(\frac{t_1^3}{3} + h_0^2 t_1 - h_0 t_1^2\right) - \frac{E_2}{1+u_2}\left[\frac{(t_1+t_2)^3 - t_1^3}{3}\right.\right.$$

$$\left.\left.+ h_0^2 t_2 - h_0(t_2^2 + 2t_1 t_2)\right]\right\}\frac{\partial^2\omega}{\partial x\,\partial y}$$

By defining

$$K_{xy} = -\frac{E_1}{1+u_1}\left(\frac{t_1^3}{3} + h_0^2 t_1 - h_0 t_1^2\right) - \frac{E_2}{1+u_2}\left[\frac{(t_1+t_2)^3 - t_1^3}{3} + h_0^2 t_2 - h_0(t_2^2 + 2t_1 t_2)\right]$$

M_{xy} can be rewritten as

$$M_{xy} = K_{xy}\,\partial^2\omega/\partial x\,\partial y. \tag{54}$$

Introducing equations (37) and (54) into equation (52), one has

$$-D_e[\partial^3\omega/\partial x^3 + (1 - K_{xy}/D_e)\,\partial^2\omega/\partial x\,\partial y] = 0. \tag{55}$$

equation (56) is obtained with the help of MathCAD 6.0 by Collabra Software, Inc.. The detailed derivation can be found in Appendix C of this paper.

$$1 - K_{xy}/D_e = 2 - u_e. \tag{56}$$

Substituting equation (56) into (55) results in

$$-D_e[\partial^3\omega/\partial x^3 + (2 - u_e)\,\partial^2\omega/\partial x\,\partial y] = 0. \tag{57}$$

2.4. COMPARING THE MOTION EQUATION AND BOUNDARY CONDITIONS BETWEEN THE COMPOSITE PLATE AND A SINGLE-LAYER PLATE

The differential equations of motion and boundary conditions of both the metal–piezoelectric composite plate and a single layer uniform thin plate are listed in Table 1. One finds that the motion equation and boundary conditions of both the metal–piezoelectric composite plate and the single-layer uniform plate have identical forms. Hence, there must be an equivalent single layer uniform thin plate which has the same free vibration characteristics as the metal–piezoelectric composite thin plate. D_e is defined as the equivalent bending stiffness of the

TABLE 1

Comparing motion equations and boundary conditions between the single and piezoelectric composite plates

	Single-layer uniform thin plates	Metal–piezoelectric composite thin plates
Differential equation of motion	$\nabla^4 w = -(\rho h/D)\,\partial^2 w/\partial t^2$	$\nabla^4 w = -(\bar{m}/D_e)\,\partial^2 w/\partial t^2$
Clamped edge ($y = 0$)	$\{\omega\|_{y=0} = 0,\ \partial\omega/\partial y\|_{y=0} = 0$	$\{\omega\|_{y=0} = 0,\ \partial\omega/\partial y\|_{y=0} = 0$
Simply supported edge ($x = 0$)	$\omega\|_{x=0} = 0,\ \left.\left(\frac{\partial^2\omega}{\partial x^2} + u\frac{\partial^2\omega}{\partial y^2}\right)\right\|_{x=0} = 0$	$\omega\|_{x=0} = 0,\ \left.\left(\frac{\partial^2\omega}{\partial x} + u_e\frac{\partial^2\omega}{\partial y}\right)\right\|_{x=0} = 0$
Free edge ($x = a$)	$(\partial^2\omega/\partial x^2 + u\,\partial^2\omega/\partial y^2)\|_{x=a} = 0$ $[\partial^3\omega/\partial x^3 + (2-u)\,\partial^3\omega/\partial x\,\partial y^2)]\|_{x=a} = 0$	$(\partial^2\omega/\partial x^2 + u_e\,\partial^2\omega/\partial y^2)\|_{x=a} = 0$ $[\partial^3\omega/\partial x^3 + (2-u_e)\,\partial^3\omega/\partial x\,\partial y^2)]\|_{x=a} = 0$
Bending stiffness	$D = \frac{Eh^3}{12(1-u^2)}$	$D_e = -\{Q_1(t_1^3/6 - (h_0/2)t_1^2)$ $+ Q_2([h_0 t_2^2 - t_1 t_2^2 - \frac{2}{3}t_2^3]2) + [K_1 + K_2 h_0]/2t_2^2\}$
Poisson ratio	u	$u_e = \frac{aQ_1u_1 + bQ_2u_2 - c(K_1 + K_2h_0)}{aQ_1 + bQ_2 - c(K_1 + K_2h_0)}$
Midplane position	$h/2$	$h_0 = \frac{1}{2}\frac{Q_1t_1^2 + Q_2t_2^2 + 2Q_2t_1t_2 - 2K_1t_2}{Q_1t_1 + Q_2t_2 - K_2t_2}$

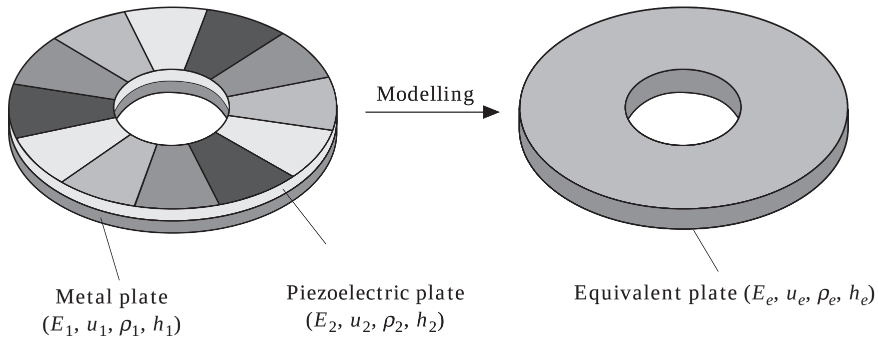

Figure 3. Metal–piezoceramic composite thin circular plate.

piezoelectric composite plate, and u_e the equivalent Poisson ratio. Other elastic parameters E_e, h_e, ρ_e can be determined by the equations

$$D_e = E_e h_e^3/[12(1 - u_e^2)], \qquad \bar{m} = \rho_e h_e, \tag{58}$$

where h_e, E_e, ρ_e represent equivalent thickness, equivalent modulus of elasticity and equivalent mass density respectively. When (58) is satisfied, arbitrary values of E_e, h_e, ρ_e will have no influence on the results of free vibration analysis of the equivalent thin plate. Normally, one defines

$$E_e = (E_1 + E_2)/2. \tag{59}$$

Introducing equation (59) for E_e into equation (58), one obtains

$$h_e = [24D_e(1 - u_e^2)/(E_1 + E_2)]^{1/3}, \qquad \rho_e = (\rho_1 t_1 + \rho_2 t_2)/h_e. \tag{60}$$

Now all the elastic quantities of the equivalent single-layer thin plate have been calculated.

3. RESULTS AND EXPERIMENTS

In this section, two actual configurations of metal–piezoceramic composite thin plates are constructed in order to confirm the validity of the uniformizing method. Firstly, one expresses these two composite plates as a corresponding equivalent single-layer uniform thin plates using the uniformizing method. Secondly FEM is employed to compute the natural frequencies of the vibration models of the equivalent thin plates. Lastly, the actual resonant frequencies are measured for the

TABLE 2

Parameters of the experimental materials

Materials	ρ (Kg/m^3)	E (GPA)	u	S_{11}^D ($\times 10^{-12}$ m^2/N)	S_{12}^D ($\times 10^{-12}$ m^2/N)	g_{31} (Vm/N)	β_{33}^T ($\times 10^8$ m/F)
Metal	$2·7 \times 10^3$	70	0·33	—	—	—	—
Piezoceramic	$7·9 \times 10^3$	—	—	12·5	−4·125	0·01139	1·1309

TABLE 3

Natural frequencies of the circular plate

Mode	Theoretical analysis (KHz)	Experimental measured (KHz)	Error (%)
(2,1)	9·61	—	—
(3,1)	15·46	16·7	7·43
(4,1)	22·40	24·5	8·57

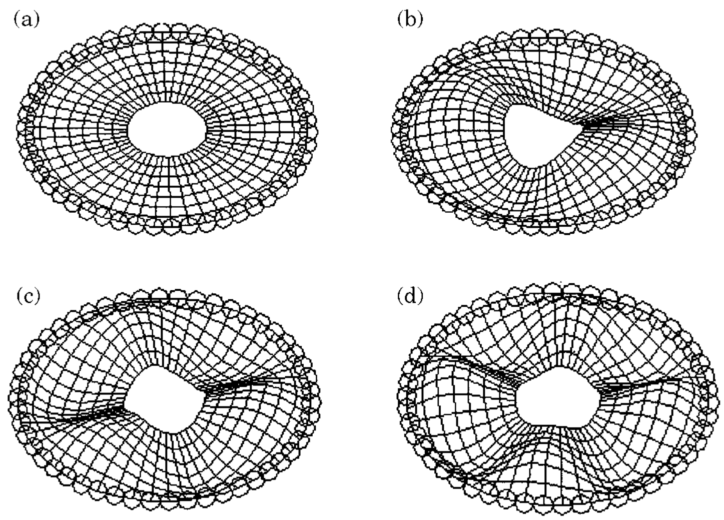

Figure 4. Vibration modes of the equivalent single layer circular plate: (a) FEM mesh; (b) Mode (2,1); (c) Mode (3,1); (d) Mode (4,1).

vibration models and compared with the experimental results obtained by using the uniformizing method.

3.1. CIRCULAR METAL–PIEZOELECTRIC COMPOSITE PLATE

Figure 3 shows a metal–piezoceramic composite circular plate with an inner radius of 8·5 mm and an outer radius of 30 mm. Thickness of both the metal layer and the piezoelectric layer is 1 mm. These two thin layers are attached by ethoxyline. The edge is simply supported and the piezoceramic is polarized along the thickness direction. The electric polar of the piezoceramic surface is divided into 12 areas or

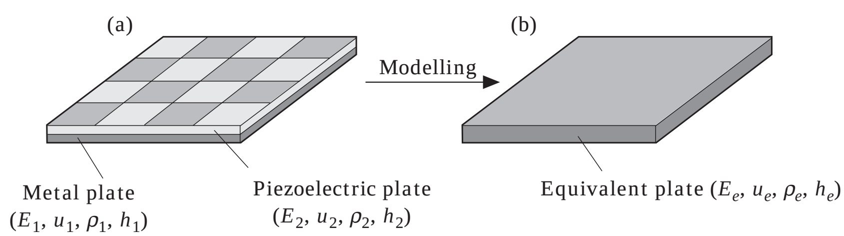

Figure 5. Rectangular piezoelectric laminated plate.

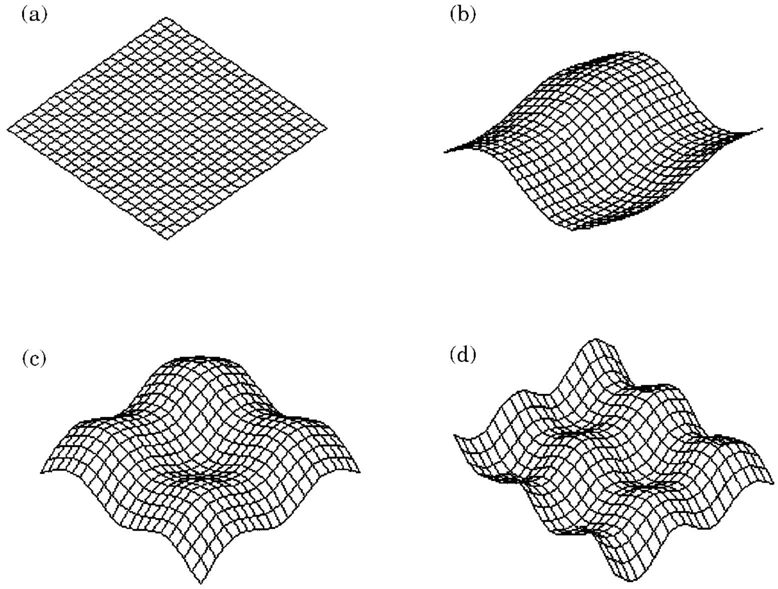

Figure 6. Vibration modes of the equivalent rectangular layer plate: (a) FEM mesh; (b) Mode (2,1); (c) Mode (3,3); (d) Mode (4,4).

16 areas equally in order to produce various vibration modes through applying voltage to different areas. Other parameters are listed in Table 2.

According to the uniformizing method, we model this piezoelectric composite thin plate into an equivalent single-layer uniform thin plate as shown in Figure 3.

According to Table 2 and equations (59) and (60), related parameters are substituted into the expressions of D_e, u_e, E_e, h_e, ρ_e:

$$u_e = 0{\cdot}288, \qquad E_e = 75\,\mathrm{GPa}, \qquad h_e = 1{\cdot}979\,\mathrm{mm}, \qquad \rho_e = 5{\cdot}354 \times 10^4\,\mathrm{Kg/m^3}. \tag{61}$$

FEM is used to calculate the free vibration frequencies and vibration modes of the equivalent single layer thin plate. The equivalent single layer circular is meshed into 350 elements as shown in Figure 4(a), and the circles on the mesh represent the simply supported boundary condition. Calculated vibration modes are given in Figure 4(b), (c) and (d). Natural frequencies of measurement and theoretical calculation are listed in Table 3.

TABLE 4

Natural frequencies of the rectangular plate

Mode	Theoretical analysis (KHz)	Experimental measured (KHz)	Error (%)
(2,1)	4·35	4·7	9·37
(2,2)	15·24	16·4	7·07
(3,3)	25·57	28·0	8·67
(4,4)	44·26	—	—

3.2. RECTANGULAR METAL–PIEZOELECTRIC COMPOSITE THIN PLATE

A rectangular piezoelectric composite plate is constructed as shown in Figure 5(a). The piezoceramic surface is divided into 16 areas to produce various vibration modes and the four edges are free. Both the metal and the piezoceramic are of 1 mm thickness with a length of 40 mm and a width of 40 mm. Other parameters are the same as those of the circular composite plate in Table 2. Figure 5(b) is the equivalent single layer uniform thin plate obtained using the uniformizing method. Computed equivalent quantities are equal to those of the circular experimental configuration in expression (61). Vibration modes by FEM are shown in Figure 7. The calculated and the measured frequencies are listed in Table 4.

A comparison between measured frequencies and those obtained by the uniformizing method shows good agreement and errors are less than 10%. Some other experimental structures of the metal–piezoceramic composite thin plate (not described in this paper) show the same agreement and precision. It should be mentioned that the error is also partly attributed to boundary conditions which are not absolutely free or simply supported edges. Another source of errors is the computation errors of FEM, especially for the high order modes such as modes (3,1), (4,1), (3,3), (4,4).

4. DISCUSSION AND CONCLUSIONS

A new approach is presented for the free vibration analysis of metal–piezoceramic composite thin plates. It is well demonstrated that the metal–piezoelectric composite thin plates can be modelled as corresponding equivalent single-layer thin plates which have the same free vibration characteristics. A large number of experiments have proved that the uniformizing method can obtain free vibration frequencies and vibration models with enough accuracy for use in engineering fields (in spite of the added assumption of continuous lateral shearing stress components at adhered points).

The advantages of exploiting this new approach become more pronounced as the problems become considerably more complicated, e.g., seeking antisymmetric vibration natural frequencies for metal–piezoceramic circular plates or investigating free vibration of a metal–piezoelectric laminated rectangular plate with a free boundary condition. With the uniformizing method, one can obtain free vibration characteristics of the metal–piezoelectric composite plates by only analyzing the corresponding equivalent single-layer thin plates.

It should be mentioned that the uniformizing method has been applied successfully in the design of ultrasonic motors. The two experimental configurations described in section 3 come from the stators of a NBDS01 ultrasonic motor [15] and RPS00 USM respectively. When designing an ultrasonic motor before using the uniformizing method, the piezoelectric effect was neglected and the mid-plane position was unknown.

There have often been large differences of vibration levels between the actual motor and the designed model. Design precision is greatly enhanced by employing the uniformizing method as stated in this paper as well as being able to obtain the mid-plane position. Hence the authors are able to design the mid-plane to coincide with the adhered layer to reduce the width of the glue layer. The authors currently have work underway related to the error analysis of the uniformizing method.

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APPENDIX A: LIST OF SYMBOLS

ω	transverse displacement of thin metal plates
$\epsilon_x, \epsilon_y, \gamma_{xy}$	strain components of a thin metal plate parallel to the x-, y- and xy directions respectively
$\sigma_x, \sigma_y, \tau_{xy}$	stress components of a thin metal plate parallel to the x-, y- and xy directions respectively
S_1, S_2, S_6	strain components of the piezoceramic thin plate parallel to the x-, y- and xy direction respectively
T_1, T_2, T_6	stress components of the piezoceramic thin plate parallel to the x-, y- and xy direction respectively
E_1	modulus of elasticity of the metal layer
ρ_1, ρ_2	mass density of the metal and the piezoceramic plate
u_1	Poisson ratio of the metal plate
t_1, t_2	thickness of metal layer and piezoceramic layer
h_0	midplane position of the metal–piezoceramic composite thin plate

h_e, E_e, ρ_e	equivalent thickness, equivalent modulus of elasticity and equivalent mass density
D_e	equivalent bending stiffness
u_e	equivalent Poisson ratio
$\bar{m}$	mass per unit area of the metal–piezoelectric composite thin plate
S^D_{hk}	elastic constants of the piezoceramic when electric displacement is constant ($h, k = 1, 2, \ldots, 6$)
g_{ik}	piezoelectric voltage constants ($k = 1, 2, \ldots, 6$ $i = 1, 2, 3$)
ϵ^T_{ij}	permittivity of the piezoceramic material ($i, j = 1, 2, 3$)
D_3	electric displacement along the z-axis ($j = 1, 2, 3$)
e_3	electric field density along the z-axis ($j = 1, 2, 3$)

APPENDIX B

Proof of the relation (equation 49)

$$aQ_1 + bQ_2 - c(K_1 + K_2h_0) = D_e \tag{B.1}$$

In Appendices B and C, subscripts are replaced by the following symbols:

$$Q1 - Q_1,\quad Q2 - Q_2,\quad K1 - k_1,\quad K2 - k_2,\quad De - D_e,\quad h0 - h_0,\quad E1 - E_1,$$
$$E2 - E_2,\quad Ue - u_e,\quad U1 - u_1,\quad U2 - u_2,\quad t1 - t_1,\quad t2 - t_2,\quad Kxy - k_{xy}.$$

Before proceeding with the proof following equations have already been obtained:

$$Q1 = \frac{E1}{1-(U1)^2},\qquad Q2 = \frac{E2}{1-(U2)^2},$$

$$h0 = \frac{Q1(t1)^2 + Q2(t2)^2 + 2t1t2Q2 - 2K1t2}{(Q1t1 + Q2t2 + K2t2)2},$$

$$a = (t1)^3/3 + (h0)^2t1 - (t1)^2h0,$$

$$b = [(t1+t2)^3 - (t1)^3]/3 + (h0)^2t2 - h0[(t2)^2 + 2t1t2],$$

$$c = [(t2)^2 + 2t1t2]/2 - h0t2,$$

$$Kxy = -E1/(1+U1)a - E2/(1+U2)b,$$

$$K1 = (G31)^2[t1t2 + (t2)^2/2]/[2(G31)^2 + \beta 33S11(1-U2)](1-U2)t2S11,$$

$$K2 = -(G31)^2(t2)/[2(G31)^2 + \beta 33S11(1-U2)](1-U2)t2S11,$$

$$K1 = -K2(t1 + t2/2).$$

By defining $K = K1 + K2h0$ then

$$K1 = -K2(t1 + t2/2) + K2\cdot h0.$$

By rewriting $h0$ and De one has

$$h0 = \tfrac{1}{2}(Q1t1^2 + Q2t2^2 + 2t1t2Q2 + 2K2t2t1 + K2t2^2)/(Q1t1 + Q2t2 + K2t2)$$

$$De = \frac{-E1}{(1-U1^2)}\left(\tfrac{1}{6}t1^3 - \tfrac{1}{2}h0t1^2\right) - \frac{E2}{(2-2\cdot U2^2)}\left(h0t2^2 - t1t2^2 - \tfrac{2}{3}t2^3\right)$$
$$- \left[\tfrac{1}{2}K2\left(t1 + \tfrac{1}{2}t2\right) + \tfrac{1}{2}K2h0\right]t2^2.$$

To prove equation (B1) the expression De is firstly simplified by substituting $h0$, $Q1$, $Q2$ into De

$$
\begin{aligned}
De = \frac{1}{12} & \frac{E1^2t1^4 - E2t2^4K2 - 4E1t1^3K2t2 + 8E1t1^3U2^2K2t2 + 12E1t1^2U2^2K2t2^2 - E2^2t2^4 - 6E1t1^2K2t2^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{2E2t2^4U1^2K2 - 4E1t1^3E2t2 + 4E1t1^3U2^2E2t2 - 6E2t2^2E1t1^2 + 3K2t2^3U1^2E1t1 + 6E2t2^2U1^2E1t1^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)} \\
& + \frac{-4E2t2^3U1^2E1t1 + 2E2t2^4U1^2K2 - 4E1t1^3E2t2 + 4E1t1^3U2^2E2t2 - 6E2t2^2E1t1^2 + 3K2t2^3U1^2E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{+6E2t2^2U1^2E1t1^2 + 4E2t2^3U1^2E1t1 + 2E1^2t1^4U2^2 + 6K2t2^3U2^2E1t1 - 3K2t2^3E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{-12K2t2^2t1^2U1^2U2^2E1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{-6K2t2^3U1^2U2^2E1t1 + 2E2^2t2^4U1^2 + 6E1t1^2U2^2E2t2^2 - E1^2t1^4U2^4 + 6K2t2^2t1^2U1^2E1 - 4E2t2^3E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{+E2t2^4K2U2^2 - 2E2t2^4K2U1^2U2^2 + 4E1t1^3K2t2U1^2 - 8E1t1^3K2t2U1^2U2^2 - 4E1t1^3U2^4K2t2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{+4E1t1^3U2^4K2t2U1^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{-6E1t1^2U2^4K2t2^2 + 6E1t1^2U2^4K2t2^2U1^2 + (E2t2^4U1^4K2 + E2t2^4U1^4K2U2^2 - 3K2t2^3U2^4E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{-E2^2t2^4U1^4) - 3K2t2^3U1^2U2^4E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{-6E1t1^2U2^2E2t2^2U1^2 - (4E2t2^3U1^2E1t1U2^2 + 4E2t2^3E1t1U2^2 + 4E1t1^3E2t2U1^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{-4E1t1^3U2^2E2t2U1^2)}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]}
\end{aligned}
$$

By defining the left part of equation as (B1) *Left*, ($Left = aQ1 + bQ2 - kc$) and substituting for expressions a, b, c, k, $Q1$, $Q2$:

$$
\begin{aligned}
Left = \frac{1}{12} & \frac{-E1^2t1^4 - E2t2^4K2 - 4E1t1^3K2t2 + 8E1t1^3U2^2K2t2 + 12E1t1^2U2^2K2t2 - E2^2t2^4 - 6E1t1^2K2t2^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{2E2t2^4U1^2K2 - (4E1t1^3E2t2 + 4E1t1^3U2^2E2t2 - 6E2t2^2E1t1^2 + 3K2t2^3U1^2E1t1 + 6E2t2^2U1^2E1t1^2)}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{4E2t2^3U1^2E1t1 + 2E1^2t1^4U2^2 + 6K2t2^3U2^2E1t1 - 3K2t2^3E1t1 - 12K2t2^2t1^2U1^2U2^2E1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{6K2t2^3U1^2U2^2E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{2E2^2t2^4U1^2 + 6E1t1^2U2^2E2t2^2 - E1^2t1^4U2^4 + 6K2t2^2t1^2U1^2E1 - 4E2t2^3E1t1 + E2t2^4K2U2^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{2E2t2^4K2U1^2U2^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{(4E1t1^3K2t2U1^2 - 8E1t1^3K2t2U1^2U2^2 - 4E1t1^3U2^4K2t2 + 4E1t1^3U2^4K2t2U1^2) - 6E1t1^2U2^4K2t2^2}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{6E1t1^2U2^4K2t2^2U1^2 + (E2t2^4U1^4K2 + E2t2^4U1^4K2U2^2 - 3K2t2^3U2^4E1t1 - E2^2t2^4U1^4)}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& + \frac{3K2t2^3U1^2U2^4E1t1}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]} \\
& - \frac{6E1t1^2U2^2E2t2^2U1^2 - (4E2t2^3U1^2E1t1U2^2 + 4E2t2^3E1t1U2^2 + 4E1t1^3E2t2U1^2 - 4E1t1^3U2^2E2t2U1^2)}{(Q1t1 + Q2t2 + K2t2)[(-1 + U1^2)(-1 + U2^2)]}
\end{aligned}
$$

Comparing the expression *De* and *Left*, the numerators and denominators are the same, one has

$$aQ_1 + bQ_2 - c(K_1 + K_2h_0) = D_e$$

APPENDIX C

The relation to be proved is

$$1 - K_{xy}/D_e = 2 - u_e \tag{C1}$$

The left part is defined as

$$Left = 1 - Kxy/De \tag{C2}$$

and the right part as

$$Right = 2 - Ue \tag{C3}$$

In accordance with the proof in Appendix B and the relationships already obtained, one has

$$De = aQ1 + bQ2 - kc, \qquad K_{xy} = [E1/(1 + U1)]a - [E2/(1 + U2)]b,$$
$$U_e = (aQ1U1 + bQ2U2 - kc)/De \tag{C4}$$

Introducing $Q1$, $Q2$ and equation (C4) into equations (C2) and (C3), one obtains

$$Left =$$
$$-\frac{kcU1^2U2^2 - kcU2^2 - U1E1aU2^2 + 2E1aU2^2 + U2E2b - U2E2bU1^2 - kcU1^2}{(1 + U2)(1 + U1)(U2 - 1)(U1 - 1)De}$$
$$+\frac{-2E1a + 2E2 + 2E2bU1^2 + kc + U1E1a - 2E2b}{(1 + U2)(1 + U1)(U2 - 1)(U1 - 1)De}$$

$$Right =$$
$$-\frac{kcU1^2U2^2 + kcU2^2 + U1E1aU2^2 - 2E1aU2^2 - U2E2b + U2E2bU1^2 + kcU1^2}{(1 + U2)(1 + U1)(U2 - 1)(U1 - 1)De}$$
$$+\frac{+2E1a - 2E2bU1^2 - kc - U1E1a - 2E2b}{(1 + U2)(1 + U1)(U2 - 1)(U1 - 1)De}$$

By comparing the expressions *right* and *left*, one sees that

$$left = right. \qquad \square$$