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NON-LINEAR EFFECTS OF VIBRATION OF A CONTINUOUS TRANSVERSE CRACKED SLENDER SHAFT

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This paper deals with flexural vibrations of a continuous slender shaft with a crack located at a distance l_n from the left end of the shaft. The mathematical model of this problem is formulated by means of the large finite element method (LFEM). The crack effect is modelled by a switching crack [1] (Jun *et al*. 1992 *Journal of Sound and Vibration* **155**, 273–290). The increase in crack depth causes decrease in bending stiffness, whereas the non-linearity is related to opening and closing of the crack faces in the process of flexural vibrations (the so called crack "breathing"). These are generated by the rotating shaft unbalance and by deflection due to shaft own weight. As the zero approximation of the solution a linearized model is used, in which a permanently opened crack is assumed. Based on this simplified model a condition is given discriminating whether the crack remains permanently open/closed during the shaft rotation or it "breathes". For the first approximation of the solution of the non-linear mathematical model the averaging method, based on the small parameter theory, is used. The theoretical results are illustrated by calculation of the amplitudes and phases of the first, second and third harmonics of the forced shaft flexural vibrations.

1. INTRODUCTION

As can be seen from review studies, for example from references [2, 3], various aspects of vibration of mechanical systems with a crack have been thoroughly studied in the last 25 years by a number of authors. From a practical point of view, the investigations were mainly focussed on vibrations of beams and shafts with transverse crack.

Kikidis and Papadopoulos [4] treated a clamped–clamped beam of circular cross-section with a permanently open crack. The crack was described by a 2×2 compliance matrix. The transverse vibrations were calculated analytically by using both the Euler–Bernoulli beam approach and the Timoshenko beam approach. They compared the influence of crack depth on beam natural frequencies of both models in relation to the beam slenderness ratio.

Shen and Chu investigated [5] vibrations of an Euler–Bernoulli beam with a periodically opening and closing (i.e., "breathing") transverse crack. When the crack was closed, the same equations as for the beam without a crack were valid.

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In case of an opened transverse crack they stipulated a theoretical course of stress and strain in the crack vicinity. Using the extended Hu–Washizu variation principle they derived a partial differential equation and boundary conditions for this particular case. Discretization and transformation into a set of ordinary differential equations was provided by expansion using eigenfunctions and Galerkin's principle.

Detailed investigation of processes in a breathing crack was undertaken by Abraham and Brandon [6]. As they also investigated the shear forces' influence on the crack behaviour, they used for analysis of transverse vibrations the Timoshenko beam model. They sought analytical solutions, separately for the open crack and closed crack cases.

Non-linear effects caused by flexural vibrations of slender beams with a breathing crack were investigated by Actis and Dimarogonas [7]. The discretization was facilitated by a FEM method, where the crack was modelled by a special finite element. This element accounted for full bending stiffness in the case of the closed crack and a reduced stiffness in the case of the open crack. The solution in the form of expansion into the set of eigenfunctions of the beam without a crack was sought and the crack influence was described by small perturbation terms. The set of non-linear equations obtained in this way was then solved numerically.

A similar problem was also investigated by Ballo [8]. He used a similar approach to discretization, the difference being in the way of solving the set of discrete non-linear ordinary differential equations obtained. An approximate analytical method, employing the small parameter approach, was used, which is also suitable for analysis of stability problems.

The problem becomes more complicated if instead of a fixed beam a rotating shaft with a transverse crack is assumed. Flexural vibrations of a Laval rotor with a permanently opened crack and with a so-called switching crack in the place of the massive disc were treated by Papadopoulos and Dimarogonas [9]. The open crack was modelled by decreased flexural stiffness in two mutually perpendicular directions in the crack position, whereas the closed crack had the same flexural stiffness as the intact shaft. The authors mainly investigated parametric excitation due to different flexural stiffness in the crack position. Non-linear effects were treated in less detail.

Also the work of Jun *et al*. [1] dealt with flexural vibrations of a Laval rotor. They also introduced a coupling stiffness between the two mutually perpendicular directions of different transversal flexural stiffnesses. In this way they attempted to describe a continuously opening and closing crack. After a thorough analysis they concluded that in most cases the switched crack approach is a sufficiently accurate approximation of the more general breathing crack case. The contribution of this paper is the consideration of influence of the position of a rotor unbalance, as well as the influence of gravitational acceleration (weight) in the case of horizontally situated rotors. They also derived the conditions when the crack is permanently open, conditions when it is permanently closed and when the crack is breathing.

Of fundamental theoretical importance is the work of Wauer [10]. Here the equations and boundary conditions of a rather general mechanical model of a cracked rotating shaft are derived. Both the permanently open crack and a breathing crack are subjected to thorough analysis. The aim of the derivations is to prepare the fundamentals for possible use of direct variation methods. As an illustrative example the analysis of torsional vibrations of a rotating shaft of circular cross-section with a circumferential crack is given.

Valuable knowledge on vibrations of rotors, as well as verification of theoretical investigations, brought about experimental research in this field. In reference [11] the behaviour of a rotating shaft was studied, whereas reference [12] dealt with an experimental investigation of the initialisation of cracks in large scale lattice structures. Muszynska introduced [13] an effective monitoring system for the detection of transverse cracks in shafts of rotating machines. The experimental results were based on a large scale thorough theoretical investigation of rotors in which the rotor unbalance and deflection due to own mass were also accounted for.

In the literature only limited attention has been paid to the investigation of the non-linear effects due to flexural vibrations of slender bodies with breathing cracks [7, 9]. They were mostly limited to the numerical investigation of the problem. However, the experimental studies [12, 13] have shown the significance of the non-linear effects to the crack detection at a very early stage.

Therefore, in this paper non-linear effects occurring in flexural vibration of slender straight massive rotating shafts supported by two short bearings with a breathing crack, modelled in a simplified form by a switching crack [1, 4, 5, 7, 9] will be investigated. The influence of damping, gyroscopic effects and influence of the hydrodynamic effects in journal bearings will be neglected. The mathematical description and discretization of the model will be performed by a modified FEM method, in which a more precise way of description of the rotating shaft boundary conditions will be taken into account. After transformation of the system of equations from the rotating co-ordinate system into the stationary coordinate system, a discrete non-linear matrix equation with varying coefficients is obtained. For a low depth crack it is possible to assume that the non-linear and time dependent terms are small enough to allow the problem to be solved by an approximative analytical method, by using the small parameter approach [14–16]. The advantage of this method is, among others, the possibility to investigate the questions of rotating system dynamic stability.

By the approach introduced, the courses of amplitudes of various harmonic components of forced flexural vibrations of the shaft, caused by the rotor unbalance and bending deflection due to its own mass were calculated. The conditions under which the crack is either fully open or fully closed or periodically breathes while the shaft is rotating were also determined.

2. PROBLEM ANALYSIS

The shaft is schematically depicted in Figure 1. The shaft will be divided into two fields: one from the left bearing to the transverse crack and the second one from the crack to the right bearing. The crack itself is located at distance l_n from the left end of the shaft.

It is assumed that the crack opens and closes within one shaft revolution; i.e., the so called "breathing crack" case is considered [1, 6, 9]. However, it is assumed that with sufficient accuracy the crack could be modelled as the so called "switching crack" [1]. The Cartesian co-ordinate system ξ, η, fixed with the rotating shaft will be oriented so that the smallest bending stiffness of the open crack k_ξ will be assigned to the ξ-axis and the other bending stiffness in the perpendicular direction k_η will be assigned to the η-axis. The values of both stiffnesses are determined by the procedure described in reference [17]. Hence, the bending moments carried by the open crack are:

$$M_\xi = -k_\xi(\xi_p' - \xi_n'), \qquad \xi_p(l_n, t) \equiv \xi_n(l_n, t) > 0, \tag{1a}$$

$$M_\xi = -k_\eta(\eta_p' - \eta_n'). \tag{1b}$$

In the case of a closed crack it will be assumed, according to reference [1], that it carries the same bending moments in respect to two perpendicular axes ξ, η as the intact shaft. To simplify the mathematical derivations further with a certain approximation, it is assumed that the condition of the intact shaft could be described by a bending stiffness in the crack position, which is substantially larger than the bending stiffness of the open crack. Therefore, for the case of a closed crack the following bending stiffness in the crack location is assumed:

$$k_\xi + k_{\xi N}. \tag{1c}$$

In these expressions the (prime) denotes differentiation with respect to the variable z and ξ_n', ξ_p', η_n', ξ_p', denote the respective displacement derivatives at the crack location from the side of the respective shaft field.

In Figure 1 m (kgm^{-1}) denotes the mass density of the shaft, and EJ (kgm^2 s^{-2}) denotes the bending stiffness. The meaning of other variables is clearly seen from Figure 1, which depicts the situation in the ξ–z plane. In the η–z plane the situation is much the same.

Non-linear effects and the effects related to different bending stiffness in the two perpendicular directions for this case of flexural vibration of a slender shaft will be analyzed by means of a method of small parameter theory [15, 16]. This approach, as well as similar approaches in the zero order approximation of the non-linear problem, starts with the linear system. Hence, in the first instance the flexural vibrations will be analyzed for the case of a permanently opened crack.

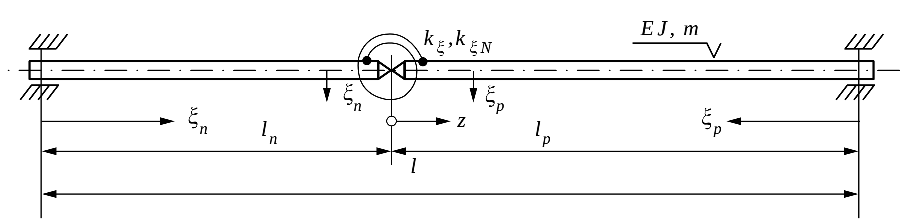

Figure 1. Cracked uniform slender shaft.

The equation of motion of the mechanical system defined in this way will be formed in the co-ordinate system rotating synchronously with the shaft around the z-axis by application of the large finite elements method (LFEM [18]). The shaft deformation in two perpendicular directions in both shaft fields will be, according to this method, approximated by the following expressions:

$$\xi_n(\zeta_n, t) = p_{n1}(\zeta_n) \cdot \xi(1, t) + p_{n2}(\zeta_n) \cdot \xi_n'(1, t) + q_n(\zeta_n) \cdot X_n(t), \tag{2a}$$

$$\xi_p(\zeta_p, t) = p_{p1}(\zeta_p) \cdot \xi(1, t) + p_{p2}(\zeta_p) \cdot \xi_p'(1, t) + q_p(\zeta_p) \cdot X_p(t), \tag{2b}$$

$$\eta_n(\zeta_n, t) = p_{n1}(\zeta_n) \cdot \eta(1, t) + p_{n2}(\zeta_n) \cdot \eta_n'(1, t) + q_n(\zeta_n) \cdot Y_n(t), \tag{2c}$$

$$\eta_p(\zeta_p, t) = p_{p1}(\zeta_p) \cdot \eta(1, t) + p_{p2}(\zeta_p) \cdot \eta_p'(1, t) + q_p(\zeta_p) \cdot Y_p(t), \tag{2d}$$

$$\zeta_n = (z/l_n) + 1, \qquad \zeta_p = 1 - z/l_p. \tag{3a, b}$$

In these expressions the primes denote spatial derivatives with respect to the new spatial co-ordinates ζ_n, ζ_p for the left and the right shaft fields, respectively. The functions $p_{ni}(\zeta_n)$, $p_{pi}(\zeta_p)$, for $i = 1, 2$ are Hermitian polynomials, which satisfy the essential boundary conditions on the ends of both fields, and $q_n(\zeta_n)$, $q_p(\zeta_p)$ are additional, improving co-ordinate functions. These functions are defined by the following boundary problems:

$$q_n^{IV} + \lambda^4 q_n = 0, \qquad q_n(0) \equiv q_n''(0) \equiv q_n(1) \equiv q_n'(1) = 0, \tag{4a}$$

$$q_p^{IV} + \lambda^4 q_p = 0, \qquad q_p(0) \equiv q_p''(0) \equiv q_p(1) \equiv q_p'(1) = 0. \tag{4b}$$

In principal more additional, improving co-ordinate functions could be used, but for the problem treated one additional co-ordinate function is sufficient.

The unknown time functions in equations (2a–2d) for both perpendicular directions can be arranged into two column vectors:

$$\mathbf{w} = \mathbf{w}_\xi + \mathrm{i}\mathbf{w}_\eta, \tag{5}$$

According to the LFEM method the matrix equation for the calculation of the unknown complex column vector $\mathbf{w}$ has the form

$$\mathbf{M}\ddot{\mathbf{w}} - 2\mathrm{i}\omega\mathbf{M}\dot{\mathbf{w}} + \left[\frac{EJ}{ml^4}\mathbf{K} + \frac{\mathbf{C}(k_\xi + k_\eta)}{2ml^3} - \omega^2\mathbf{M}\right]\mathbf{w} + \frac{\mathbf{C}(k_\xi - k_\eta)}{2ml^3}\bar{\mathbf{w}} = \mathbf{F}, \tag{6}$$

where $\mathbf{F} = \mathbf{F}_\xi + \mathrm{i} \cdot \mathbf{F}_\eta$ is the column matrix of excitation time functions which are generated by the rotor weight and unbalance and $\bar{\mathbf{w}}$ is the complex conjugate of the vector $\mathbf{w}$.

The rectangular symmetric fifth order matrices $\mathbf{M}$, $\mathbf{K}$, $\mathbf{C}$ in equation (6) are related to the scalar products of the co-ordinate functions.

The only observable shaft flexural vibrations are those which manifest themselves in the stationary co-ordinate system. Therefore, equation (6) has to be transformed from the rotating co-ordinate system ξ, η, z into the stationary co-ordinate system x, y, z. The unknown time functions will be arranged into vectors $\mathbf{v}_x$, $\mathbf{v}_y$ in the stationary co-ordinate system, whose structure is similar to the structure of vectors $\mathbf{w}_\xi$, $\mathbf{w}_\eta$ defined in the rotating co-ordinate system. The two vectors can be united into one complex vector $\mathbf{v}$. The transformation is furnished by the equation

$$\mathbf{w} = \mathbf{v}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{7}$$

After this transformation the matrix equation for calculation of the unknown complex vector $\mathbf{v}$ will have the form

$$\mathbf{M}\ddot{\mathbf{v}} + \left[\frac{EJ}{ml^4}\mathbf{K} + \frac{(k_\xi + k_\eta)}{2ml^3}\mathbf{C}\right]\mathbf{v} + \frac{k_\xi - k_\eta}{2ml^3}\mathbf{C}\cdot\mathrm{e}^{-2\mathrm{i}\omega t}\,\bar{\mathbf{v}} = \mathbf{F}\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{8}$$

3. THE NON-LINEAR CASE

In the case of a permanently opening and closing transversal crack in the rotor, the system becomes non-linear. This non-linearity is manifested in the rotor mathematical model by inclusion of a further term $\mathbf{M}_N$ on the left side of equation (6). After transformation into the stationary co-ordinate system a new term of the form $-\mathbf{M}_N\cdot\mathrm{e}^{-\mathrm{i}\omega t}$ appears on the right side of equation (8). If one supposes that the emerging crack is only slightly developed, this term could also be considered to be small. In the same way the last term on the left side of equation (8) could also be considered to be small, because it is a difference of two stiffnesses of approximately the same value. Both terms can be formally re-arranged into one column matrix $\varepsilon\boldsymbol{\Psi}$, where ε is a small parameter. This column matrix has the form

$$\varepsilon\boldsymbol{\Psi} = \frac{k_\eta - k_\xi}{2ml^3}\mathbf{C}\,\mathrm{e}^{-2\mathrm{i}\omega t} - \varepsilon\mathbf{M}_N\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{9}$$

Then the complete equation, describing the non-linear case in the stationary co-ordinate system will have the form

$$\mathbf{M}\ddot{\mathbf{v}} + \left[\frac{EJ}{ml^4}\mathbf{K} + \frac{k_\xi + k_\eta}{2ml^3}\mathbf{C}\right]\mathbf{v} = \mathbf{F}\,\mathrm{e}^{-\mathrm{i}\omega t} + \varepsilon\boldsymbol{\Psi}. \tag{10}$$

For further discussion it is important to establish the conditions when the shaft will rotate with the permanently open crack or the crack will open and close during one revolution of the shaft. These conditions could be established by detailed analysis of equation (1a), especially of the condition $(\xi_p - \xi_n) \leqslant 0$ in the position of the crack. If the discussion is limited to the frequency range in which the flexural deformation could be approximated mainly by the first eigen-function it could be shown that the crack will be closed if the following condition is fulfilled:

$$\xi(1, t) \equiv \frac{\omega^2}{|\omega_{01}^2 - \omega^2|}\mathbf{v}_{01}^{\mathrm{T}}\mathbf{F}_\xi - \frac{\mathbf{v}_{01}^{\mathrm{T}}\cdot\mathbf{F}_g}{\omega_{01}^2}\sin(\omega t) \leqslant 0. \tag{11}$$

In this equation, $\mathbf{v}_{01}$ is the first eigenvector and ω_{01} is the first circular eigenfrequency of the linearized system and ω is the shaft circular rotational frequency. The column vector $\mathbf{F}_\xi$ depends on dynamical rotor unbalance in the z–ξ plane and the column vector $\mathbf{F}_g$ depends on the rotor weight.

From equation (11) the influence of respective factors on crack opening and closing can be clearly seen. If the shaft is not very well balanced dynamically, and rotates with a circular frequency ω which is rather near to the first natural circular frequency ω_{01}, the first time-independent term is dominant and the crack is permanently opened or closed. The crack could change its shape during one revolution, that is the crack "breathes", only if the second term in equation (11) dominates. This situation could occur if the operational circular rotational frequency of the shaft ω differs sufficiently from the first natural circular frequency, or if the shaft is very well balanced dynamically, i.e., the excitation force $\mathbf{F}_\xi$ is rather small.

In the following discussion of the non-linear factors' influence and manifestation it is assumed that the crack opens and closes regularly during one revolution of the shaft. Let the time interval, during which the crack is closed, be described by the relation

$$t_{cl} = \Delta_1/\omega. \tag{12}$$

The time interval t_{cl} is given by the condition that the left side of equation (11) is zero.

Further solution of the non-linear equation (10) is based on the use of the averaging method of reference [16]. To be able to use this method it is necessary to transform equation (10) into the standard form in which all the time derivatives of the sought unknown functions are small and proportional to a small parameter. It is considered, in line with reference [14], that in the linearized form (for $\varepsilon = 0$) no internal resonance exists. Hence, in the vicinity of the first natural circular frequency ω_{01} the mono-frequency solution can be used, which is approximated by the first natural form of the shaft flexural vibrations $\mathbf{v}_{01}$. The solution of the non-linear equation (10) will be sought in the form

$$\mathbf{v} = \mathbf{v}_{01}[u(t) + s(t)]. \tag{13}$$

By multiplication of equation (10) from the left side by the vector $\mathbf{v}_{01}^{\mathrm{T}}$, substitution from equation (13) and use of orthogonality relations, equation (10) transforms into

$$\ddot{u} + \ddot{s} + \omega_{01}^2(u + s) = \mathbf{v}_{01}^{\mathrm{T}}\mathbf{F}\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathbf{v}_{01}^{\mathrm{T}}\varepsilon\boldsymbol{\Psi}. \tag{14}$$

The complex time variable $u(t)$ is the steady-state solution of

$$\ddot{u} + \omega_{01}^2 u = \mathbf{v}_{01}^{\mathrm{T}}\mathbf{F}\,\mathrm{e}^{-\mathrm{i}\omega t} \equiv \omega^2\mathbf{v}_{01}^{\mathrm{T}}(\mathbf{F}_{\xi 0} + \mathrm{i}\mathbf{F}_{\eta 0})\,\mathrm{e}^{-\mathrm{i}\omega t} + i\mathbf{v}_{01}^{\mathrm{T}}\mathbf{F}_g. \tag{15}$$

In equation (15) the first term on the right side describes the effect of unbalance, whereas the second term is the rotor deflection due to its own weight. As a consequence of equation (15), equation (14) transforms into

$$\ddot{s} + \omega_{01}^2 s = \varepsilon\mathbf{v}_{01}^{\mathrm{T}}\boldsymbol{\Psi} \equiv \varepsilon\mathbf{v}_{01}^{\mathrm{T}}(\boldsymbol{\Psi}_R + \mathrm{i}\cdot\boldsymbol{\Psi}_I). \tag{16}$$

A small distortion $\varepsilon\delta$ between the first natural circular frequency ω_{01} and an integer multiple p or integer ratio q of the shaft operational circular rotational frequency ω, is introduced into equation (16) by the following formula [14, 16]:

$$\omega_{01} = (p/q)\omega + \varepsilon\delta. \tag{17}$$

Hence, one obtains the following equation, whose accuracy is of the order of the first power of ε:

$$\ddot{s} + \left(\frac{p}{q}\omega\right)^2 s = -\varepsilon 2\delta \frac{p}{q}\omega s + \varepsilon \mathbf{v}_{01}^{\mathrm{T}} \mathbf{\Psi}. \tag{18}$$

Equation (18) is transformed by the complex transform $\mathbf{s} = \mathbf{s}_R + \mathrm{i} \cdot \mathbf{s}_I$ into four first order equations:

$$s_R = a_R(t) \cos \psi_R, \qquad \dot{s}_R = -\frac{p}{q}\omega a_R(t) \sin \psi_R, \tag{19a, b}$$

$$s_I = -a_I(t) \sin \psi_I, \qquad \dot{s}_I = -\frac{p}{q}\omega a_I(t) \cos \psi_I, \tag{19c, d}$$

$$\psi_R = \frac{p}{q}\omega t + \vartheta_R(t), \qquad \psi_I = \frac{p}{q}\omega t + \vartheta_I(t), \tag{20a, b}$$

After re-writing these equations the standard form is obtained:

$$\dot{a}_R = \frac{\varepsilon}{(p/q)\omega}\left\{2\delta \frac{p}{q}\omega a_R \cos\psi_R \sin\psi_R - \mathbf{v}_{01}^{\mathrm{T}} \mathbf{\Psi}_R \sin\psi_R\right\}, \tag{21a}$$

$$\dot{\vartheta}_R = \frac{\varepsilon}{(p/q)\omega a_R}\left\{2\delta \frac{p}{q}\omega a_R \cos^2\psi_R - \mathbf{v}_{01}^{\mathrm{T}} \mathbf{\Psi}_R \cos\psi_R\right\}, \tag{21b}$$

$$\dot{a}_I = \frac{\varepsilon}{(p/q)\omega}\left\{-2\delta \frac{p}{q}\omega a_I \cos\psi_I \sin\psi_I - \mathbf{v}_{01}^{\mathrm{T}} \mathbf{\Psi}_I \cos\psi_I\right\}, \tag{21c}$$

$$\dot{\vartheta}_I = \frac{\varepsilon}{(p/q)\omega a_I}\left\{-2\delta \frac{p}{q}\omega a_I \sin^2\psi_I - \mathbf{v}_{01}^{\mathrm{T}} \mathbf{\Psi}_I \sin\psi_I\right\}. \tag{21d}$$

By inspection of these equations it follows that the variation of these four sought unknown functions $a_R, \vartheta_R, a_I, \vartheta_I$, in time is rather small. Hence, the dominant influence on their right sides has the steady term augmented by small, time-dependent terms. Therefore, in the first approximation, the right side of the standard form equations (21) can be replaced by these constant terms, which do not explicitly depend on time. This can be accomplished by time-averaging of the right sides according to the explicitly present time. As a steady state solution is sought the unknown functions have to be constants and their respective time derivatives have to be zeros. Hence, a system of four non-linear algebraic equations can be formed in this way for the approximate calculation of two vibration amplitudes a_R, a_I and two corresponding phase angles ϑ_R, ϑ_I. To which component these amplitudes and phases belong to depends on the choice of integers p and q.

4. NUMERICAL EXAMPLES

In this paper two cases are considered in some detail: (i) $p = 2$ and $q = 1$, i.e., the second harmonic component to the rotational circular frequency ω; (ii) $p = 3$ and $q = 1$, i.e., the third harmonic component to the rotational circular frequency ω.

Both cases are described by different, rather complicated, sets of non-linear algebraic equations. For the sake of simplicity only the set for the description of the second harmonic components are given, as follows:

$$\frac{D_1}{2\omega}\{[G_1 - G_2]\cos\vartheta_R + G_3 a_R \sin 2\,\vartheta_R - G_8 a_I \sin(\vartheta_R + \vartheta_I)\} - \frac{D_2}{2\omega} G_{12}\cos\vartheta_R = 0, \tag{22a}$$

$$\delta + \frac{D_1}{2\omega a_R}\{[G_2 - G_1]\sin\vartheta_R + G_4 a_R - G_5 a_R \cos 2\,\vartheta_R - G_8 a_I \cos(\vartheta_R + \vartheta_I)\} - \frac{D_2}{2\omega a_R} G_{12}\sin\vartheta_R = 0, \tag{22b}$$

$$-\frac{D_1}{2\omega}\{[G_7 - G_6]\cos\vartheta_I + G_8 a_R \sin(\vartheta_R + \vartheta_I) + G_9 a_I \sin 2\,\vartheta_I\} - \frac{D_2}{2\omega} G_{12}\cos\vartheta_I = 0, \tag{22c}$$

$$-\delta - \frac{D_1}{2\omega a_I}\{[G_7 - G_6]\sin\vartheta_I - G_8 a_R \cos(\vartheta_R + \vartheta_I) + G_{10} a_I + G_{11} a_I \cos 2\,\vartheta_I\} - \frac{D_2}{2\omega a_I} G_{12}\sin\vartheta_I = 0. \tag{22d}$$

In these equations the components G_i ($i = 1, 2, \ldots, 12$) of the real vector $\mathbf{G}$ are complicated functions of the time interval during which the crack is closed (the variable Δ in expression (12)), system tuning and rotor unbalance. The constant D_1 is related to non-linear effects, whereas the constant D_2 is related to the effect of different rotor stiffness in respect to the mutually perpendicular axis ξ, η when the crack is open. They are given by

$$D_1 = \frac{\beta_N k_\xi}{ml^3}\mathbf{v}_{01}^{\mathrm{T}}\mathbf{C}\mathbf{v}_{01}, \qquad D_2 = \frac{k_\xi - k_\eta}{2ml^3}\mathbf{v}_{01}^{\mathrm{T}}\mathbf{C}\mathbf{v}_{01}, \qquad \beta_N = 1 + \frac{k_{\xi N}}{k_\xi}. \tag{23a–c}$$

For illustration the course of all four sought variables as a function of the relative crack depth a/R was numerically calculated. The calculations were made for a rotating shaft with the following properties: $m = 61{\cdot}3\ \mathrm{kg\,m^{-1}}$, $E = 2{\cdot}06 \times 10^{11}\ \mathrm{N\,m^{-2}}$, the relative rotor radius $R/l = 0{\cdot}025$, the relative crack position in respect to the left end of the shaft $l_n/l = 0{\cdot}4$, the non-linearity coefficient $\beta_N = 20$.

The steady state values of both amplitudes a_R, a_I of the second harmonic component as a function of the relative crack depth a/R, calculated according to formulas (22), are depicted in Figure 2. The respective phase angles ϑ_R, ϑ_I did not change in respect to the relative crack depth and remained at the values $\vartheta_R = -\pi/2$ and $\vartheta_I = \pi/2$. For comparison the first harmonic components at the rotational

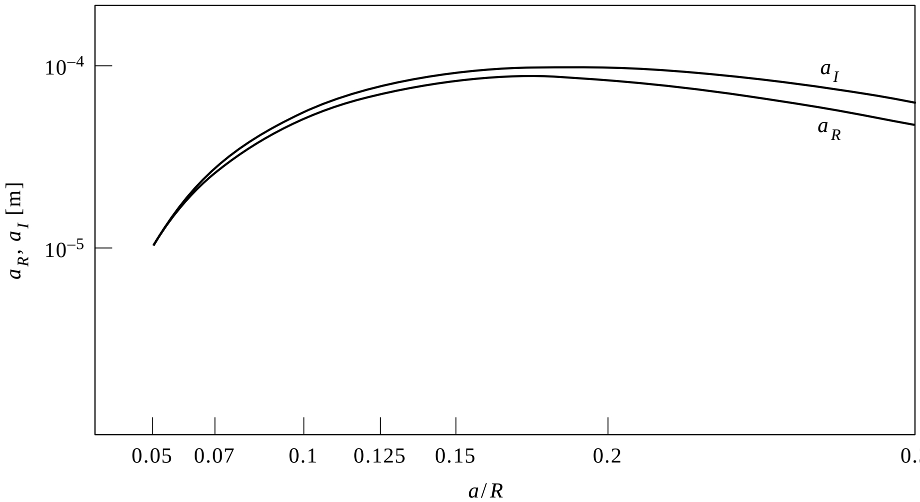

Figure 2. Steady state values of amplitudes a_R, a_I of the second harmonic component generated by the influence of non-linearity and the different rotor stiffness at the position of the crack as a function of the crack depth ratio a/R.

frequency (the zero order approximation) had in both perpendicular directions the same value, expressed numerically as $86{\cdot}36 \times 10^{-6}$ m.

According to equation (23a) the coefficient β_N is part of the D_1 constant. If this constant is set to zero the influence of non-linearity is eliminated and only the influence of different rotor stiffness on both sides of the permanently open crack is considered. In that case both amplitudes of the second harmonic component are the same. Their dependence on the relative crack depth is depicted in Figure 3. Both phase angles in this case are the same and have a rather small value.

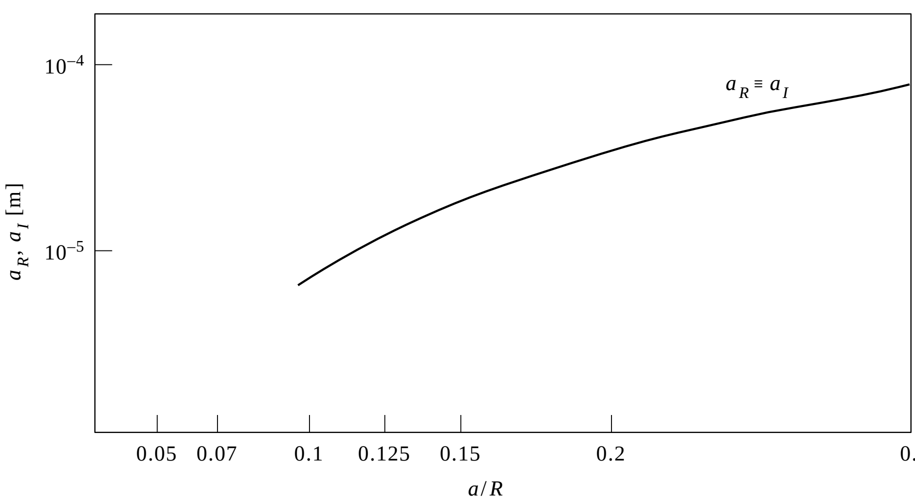

Figure 3. Steady state values of amplitudes a_R, a_I of the second harmonic component generated by the influence of the different rotor stiffness at the position of the crack as a function of the crack depth ratio a/R.

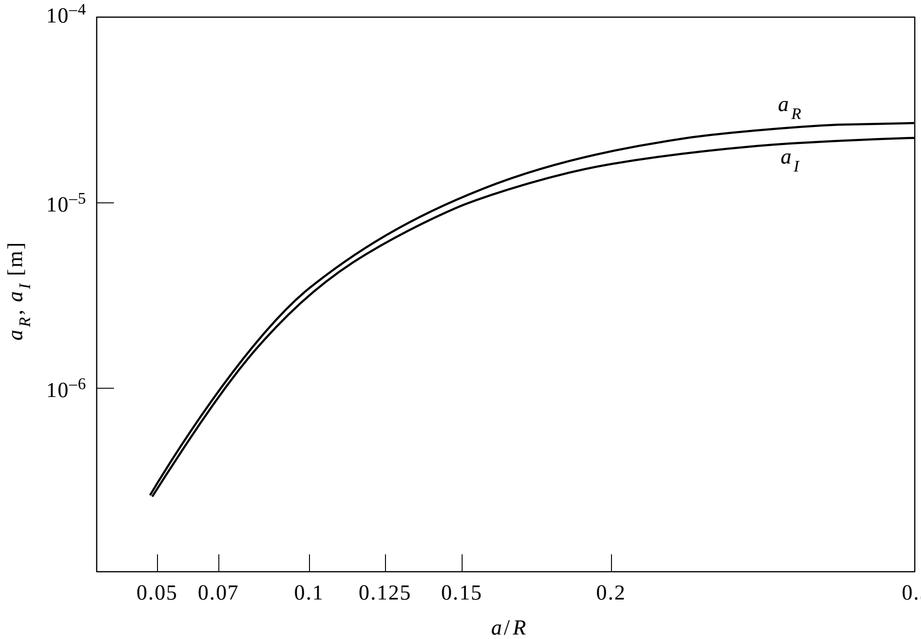

Figure 4. Steady state values of amplitudes a_R, a_I of the third harmonic component generated by the influence of non-linearity and the different rotor stiffness at the position of the crack as function of the crack depth ratio a/R.

The course of both amplitudes of the third harmonic component in dependence on the relative crack depth is depicted in Figure 4. Both phase angles for this case were again the same and equal to zero.

The comparison of the courses of the second and third harmonic components reveals that their dependence on the relative crack depth is rather strong, whereas the eigenfrequencies of the linearized system are much less dependent on the crack depth. So, if the system is suitably tuned for an emerging crack with a depth of 5% of the shaft radius, an increase in the amplitude of both the second and third harmonic components could be already observed, whereas the value of ω_{01} changes by only 0·16%: i.e., only negligible.

5. CONCLUSION

In this paper a mathematical model of flexural vibrations of a slender massive rotating shaft with a crack was derived by the application of the large finite element method under some simplifying conditions. The non-linear model with time dependent coefficients was formulated in such a way as to be solvable by the small parameter method. The corresponding matrix equation was transformed into standard form, which could be solved by the averaging method of reference [16]. Based on this approach a set of non-linear algebraic equations for the second and third harmonic components of the shaft rotational frequency was elaborated. From these equations the amplitudes and phase angles of these components could be calculated.

5. CONCLUSION

In this paper a mathematical model of flexural vibrations of a slender massive rotating shaft with a crack was derived by the application of the large finite element method under some simplifying conditions. The non-linear model with time dependent coefficients was formulated in such a way as to be solvable by the small parameter method. The corresponding matrix equation was transformed into standard form, which could be solved by the averaging method of reference [16]. Based on this approach a set of non-linear algebraic equations for the second and third harmonic components of the shaft rotational frequency was elaborated. From these equations the amplitudes and phase angles of these components could be calculated.

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