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ANALYSIS OF SUBHARMONIC RESONANCE OF MODERATELY THICK ANTISYMMETRIC ANGLE-PLY LAMINATED PLATES BY USING METHOD OF MULTIPLE SCALES

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The subharmonic resonance of simply supported, rectangular laminated plates is investigated by the method of multiple scales. The governing equations for the plate, which is based on the first order shear deformation and the von Kármán-type geometric non-linear theories, are derived by Hamilton's principle. The application of Galerkin's procedure to the governing equations yields the Duffing-type equation in terms of the transverse displacement. In order to use the method of multiple scales properly, we introduce new detuning parameter for the analysis of the subharmonic resonance. The influence of the lamination sequence, thickness ratio, number of layers and in-plane boundary condition is examined on the subharmonic resonance. The analytical results are compared with those obtained by the Runge–Kutta method, and the validity of the present analysis is clearly shown.

1. INTRODUCTION

Composite materials such as fiber reinforced plastics (FRP) have been widely used as structural members due to their excellent mechanical properties (e.g., high stiffness-to-weight ratio, high strength-to-weight ratio). When composite laminated plates are subjected to transverse harmonic loads and vibrate with large amplitudes, non-linear responses (superharmonic, subharmonic and internal resonances, etc.), which cannot be predicted by the linear theory, frequently occur. Since the laminated plates are used as one of the main structural members, it has become important to understand their non-linear dynamic properties as well as their linear ones. On the other hand, as is well known, the transverse shear rigidities of laminated plates are weaker than that of metal plates. In analyzing moderately thick laminated plates, it is necessary to take the effects of the shear deformation into account.

Non-linear vibrations of laminated plates have been extensively studied by many researchers. Chia [1, 2] and Sathyamoorthy [3] have conducted a comprehensive review of the literature dealing with non-linear problems in plates. Sivakumaran and Chia [4] studied the effects of transverse shear, rotatory inertia and transverse normal stress on non-linear free vibrations of laminated plates. Singh *et al*. [5]

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analyzed the non-linear vibrations of moderately thick unsymmetric laminated composite plates by the finite element method. The effects of geometric imperfections on non-linear free vibrations of laminated plates and cylindrical thick panels were examined by Hui [6] and Fu and Chia [7], respectively. Eslami and Kandil [8, 9] investigated primary, subharmonic and superharmonic resonances of simply supported orthotropic rectangular plates by the method of multiple scales (MMS) [10]. Moreover, some researchers have studied the responses of laminated plates with internal resonances. Hadian *et al*. [11] investigated the two-mode response of antisymmetric cross-ply laminated plates by using the averaged Lagrangian. The present authors [12] studied the two-mode response of antisymmetric angle-ply laminated plates, and they introduced new definitions of detuning parameters in order to use the MMS properly. However, there have been few studies dealing with responses of laminated plates subjected to non-resonant excitations.

This paper presents a remedy for solving the subharmonic resonance of moderately thick laminated plates by the MMS. Taking into account the first order shear deformation theory (FSDT) and the von Kármán-type geometric non-linear theory, the governing equations for antisymmetric angle-ply laminated plates are derived by Hamilton's principle. Then, applying Galerkin's procedure, the governing equations are reduced to the Duffing-type equation in terms of the transverse displacement. Finally, steady-state solutions for the subharmonic resonance are obtained by using the MMS, where we suggest a new definition of a detuning parameter to apply it properly. Accuracy of analytical results is confirmed in comparison with those of numerical integration of the equation of motion.

2. BASIC EQUATIONS

Figure 1 shows a rectangular laminated plate, which consists of N layers of an orthotropic sheet, with lengths being a, b and thickness h. The co-ordinate system (x, y, z) is taken in the midplane of the plate, as shown in the figure. The distance from the midplane to the upper plane of the kth layer is h_k. The principal directions

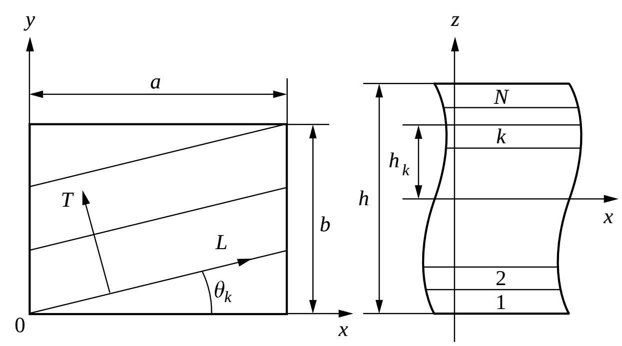

Figure 1. Geometry of a laminated plate and co-ordinate systems.

of elasticity are denoted by L and T, and θ_k is the angle between L and x axes in the kth layer of the plate.

According to the first order shear deformation theory, it is assumed that the in-plane displacements of the plate are linear functions of co-ordinate z, and that the transverse displacement is constant through the thickness of the plate. Thus, the displacements u, v and w in the x, y and z directions, respectively, can be expressed as

$$u = u_0 + z\psi_x, \qquad v = v_0 + z\psi_y, \qquad w = w_0, \tag{1}$$

where u_0, v_0 and w_0 are the displacements at the midplane, and ψ_x and ψ_y are the rotations of midplane about the y and x axes, respectively.

The strain–displacement relations, based on the von Kármán-type geometric non-linear theory, are written as

$$\left.\begin{aligned} &\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \qquad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \qquad \varepsilon_z = 0, \\ &\varepsilon_{xy} = \varepsilon_{xy}^0 + z\kappa_{xy}, \qquad \varepsilon_{xz} = \psi_x + \frac{\partial w_0}{\partial x}, \qquad \varepsilon_{yz} = \psi_y + \frac{\partial w_0}{\partial y}, \end{aligned}\right\}, \tag{2}$$

in which

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2, \qquad \varepsilon_y^0 = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2, \qquad \varepsilon_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}, \tag{3}$$

$$\kappa_x = \frac{\partial \psi_x}{\partial x}, \qquad \kappa_y = \frac{\partial \psi_y}{\partial y}, \qquad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}. \tag{4}$$

In the following analysis, we consider antisymmetric angle-ply laminated plates. The constitutive relations can be expressed as follows [7]:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & 0 & 0 & 0 & B_{16}^* \\ A_{12}^* & A_{22}^* & 0 & 0 & 0 & B_{26}^* \\ 0 & 0 & A_{66}^* & B_{61}^* & B_{62}^* & 0 \\ 0 & 0 & -B_{61}^* & D_{11}^* & D_{12}^* & 0 \\ 0 & 0 & -B_{62}^* & D_{12}^* & D_{22}^* & 0 \\ -B_{16}^* & -B_{26}^* & 0 & 0 & 0 & D_{66}^* \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{5}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}, \tag{6}$$

where N, M and Q are the stress, moment and shear stress resultants respectively. Constants A^*_{ij}, B^*_{ij}, D^*_{ij} and S_{ij} are derived from

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & 0 & 0 & C_{26} \\ 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & 0 & 0 & C_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix}, \tag{7}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} C_{ij}^{(k)} (1, z, z^2)\, \mathrm{d}z, \qquad i, j = 1, 2, 6, \tag{8}$$

$$S_{ij} = k^2 A_{ij} = k^2 \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} C_{ij}^{(k)}\, \mathrm{d}z, \qquad i, j = 4, 5, \tag{9}$$

$$\mathbf{A}^* = \mathbf{A}^{-1}, \qquad \mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}, \qquad \mathbf{D}^* = \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}, \tag{10}$$

in which $C_{ij}^{(k)}$ are the stiffness matrix elements expressing the stress–strain relation in the kth layer, and k^2 is the shear correction factor.

Bert and Chen [13] investigated the effects of rotatory and in-plane inertias on the vibration of antisymmetric angle-ply laminated plates, and they reported that both inertias had little effect on the fundamental frequency. Since the present paper examines the subharmonic resonance of the fundamental mode, the effects of rotatory and in-plane inertias are neglected. The kinetic energy of the plate can be written as

$$T = \frac{\rho}{2} \int_0^b \int_0^a \dot{w}_0^2\, \mathrm{d}x\, \mathrm{d}y, \tag{11}$$

where ρ is mass per unit area of the plate and a dot denotes differentiation in time. The strain energy of the plate is given by

$$U = \frac{1}{2} \int_0^b \int_0^a (N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy} \varepsilon_{xy}^0 + Q_x \varepsilon_{xz} + Q_y \varepsilon_{yz} + M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy})\, \mathrm{d}x\, \mathrm{d}y. \tag{12}$$

The work done by an external pressure $q(x, y, t)$ acting in the z direction is

$$W = \int_0^b \int_0^a q(x, y, t) w_0\, \mathrm{d}x\, \mathrm{d}y. \tag{13}$$

By substituting equations (11)–(13) into Hamilton's principle

$$\int_{t_0}^{t_1} \delta(T - U + W)\,\mathrm{d}t = 0, \tag{14}$$

and taking the variation in consideration of equations (2)–(4), the governing equations are derived as follows:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \tag{15}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w_0}{\partial y} + N_{xy}\frac{\partial w_0}{\partial x}\right) = \rho \ddot{w}_0 - q, \tag{16}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0. \tag{17}$$

A force function ϕ satisfying equations (15) automatically is defined as

$$N_x = \frac{\partial^2 \phi}{\partial y^2}, \qquad N_y = \frac{\partial^2 \phi}{\partial x^2}, \qquad N_{xy} = -\frac{\partial^2 \phi}{\partial x\,\partial y}. \tag{18}$$

By using relations of equations (5), (6), (15) and (18), equations (16) and (17) can be rewritten as

$$\rho \ddot{w}_0 - q - \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2} + 2\frac{\partial^2 \phi}{\partial x\,\partial y}\frac{\partial^2 w_0}{\partial x\,\partial y} - \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}$$

$$- S_{44}\left(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \psi_y}{\partial y}\right) - S_{55}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \psi_x}{\partial x}\right) = 0, \tag{19}$$

$$(B^*_{61} - B^*_{26})\frac{\partial^3 \phi}{\partial x^2\,\partial y} - B^*_{16}\frac{\partial^3 \phi}{\partial y^3} + D^*_{11}\frac{\partial^2 \psi_x}{\partial x^2} + D^*_{66}\frac{\partial^2 \psi_x}{\partial y^2}$$

$$+ (D^*_{12} + D^*_{66})\frac{\partial^2 \psi_y}{\partial x\,\partial y} - S_{55}\left(\frac{\partial w_0}{\partial x} + \psi_x\right) = 0, \tag{20}$$

$$- B^*_{26}\frac{\partial^3 \phi}{\partial x^3} + (B^*_{62} - B^*_{16})\frac{\partial^3 \phi}{\partial x\,\partial y^2} + (D^*_{12} + D^*_{66})\frac{\partial^2 \psi_x}{\partial x\,\partial y}$$

$$+ D^*_{66}\frac{\partial^2 \psi_y}{\partial x^2} + D^*_{22}\frac{\partial^2 \psi_y}{\partial y^2} - S_{44}\left(\frac{\partial w_0}{\partial y} + \psi_y\right) = 0. \tag{21}$$

The compatibility equation is obtained by eliminating u_0 and v_0 in equation (3) and using equations (4), (5) and (18):

$$A_{22}^* \frac{\partial^4 \phi}{\partial x^4} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + A_{11}^* \frac{\partial^4 \phi}{\partial y^4}$$
$$= (B_{61}^* - B_{26}^*) \frac{\partial^3 \psi_x}{\partial x^2\, \partial y} - B_{16}^* \frac{\partial^3 \psi_x}{\partial y^3} - B_{26}^* \frac{\partial^3 \psi_y}{\partial x^3}$$
$$+ (B_{62}^* - B_{16}^*) \frac{\partial^3 \psi_x}{\partial x\, \partial y^2} + \left(\frac{\partial^2 w_0}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} . \tag{22}$$

Equations (19)–(22) expressed in w_0, ψ_x, ψ_y and ϕ are the governing equations for antisymmetric angle-ply laminated plates based on the first order shear deformation and the von Kármán-type geometric non-linear theories.

We assume that the plate is simply supported along its four edges, and both movable and immovable edges are adopted for the in-plane boundary conditions. Each boundary condition can be given by, respectively,

$$\left.\begin{aligned} w_0 = M_x = \frac{\partial^2 \phi}{\partial x\, \partial y} = \int_0^b \frac{\partial^2 \phi}{\partial y^2}\, \mathrm{d}y = 0 \quad \text{at } x = 0, a \\ w_0 = M_y = \frac{\partial^2 \phi}{\partial x\, \partial y} = \int_0^a \frac{\partial^2 \phi}{\partial x^2}\, \mathrm{d}x = 0 \quad \text{at } y = 0, b \end{aligned}\right\}, \tag{23}$$

$$\left.\begin{aligned} w_0 = M_x = \frac{\partial^2 \phi}{\partial x\, \partial y} = 0 \quad \text{at } x = 0, a, \qquad \int_0^b \int_0^a \frac{\partial u_0}{\partial x}\, \mathrm{d}x\, \mathrm{d}y = 0 \\ w_0 = M_y = \frac{\partial^2 \phi}{\partial x\, \partial y} = 0 \quad \text{at } y = 0, b, \qquad \int_0^b \int_0^a \frac{\partial v_0}{\partial y}\, \mathrm{d}x\, \mathrm{d}y = 0 \end{aligned}\right\}. \tag{24}$$

Further we assume that the plate is subjected to the following force

$$q(x, y, t) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \Omega' t, \tag{25}$$

in which q_0 and Ω' are the amplitude and angular frequency of the force respectively.

In the following analysis, we study the subharmonic resonance when the excitation frequency is nearly three times the natural frequency of the fundamental mode. Therefore displacement functions are expressed using the eigenfunctions of the linear vibration as

$$w_0 = \frac{h^2}{a} W \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \qquad \psi_x = \frac{h}{a} X \cos \frac{\pi x}{a} \sin \frac{\pi y}{b},$$
$$\psi_y = \frac{h}{a} Y \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}, \tag{26}$$

where W is a non-dimensional displacement, and X and Y are non-dimensional rotation angles. The stress function satisfying the boundary conditions (23) and (24) is assumed to be of the form

$$\phi = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} B_{pq} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{b} + C_1 x^2 + C_2 y^2, \tag{27}$$

in which B_{pq}, C_1 and C_2 are unknown coefficients. If equations (26) and (27) are substituted into the compatibility condition (22), then B_{pq} can be determined by comparing the coefficients of trigonometric functions in both sides of equation (22). Using the following relations derived from equations (3) and (5)

$$\left.\begin{aligned} \frac{\partial u_0}{\partial x} &= A_{11}^* \frac{\partial^2 \phi}{\partial y^2} + A_{12}^* \frac{\partial^2 \phi}{\partial x^2} + B_{16}^* \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) - \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \\ \frac{\partial v_0}{\partial y} &= A_{12}^* \frac{\partial^2 \phi}{\partial y^2} + A_{22}^* \frac{\partial^2 \phi}{\partial x^2} + B_{26}^* \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) - \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 \end{aligned}\right\}, \tag{28}$$

C_1 and C_2 are solved by performing the integrations in equation (24). In the case of movable edge condition (23), C_1 and C_2 are equal to zero. Details of B_{pq}, C_1 and C_2 are given in the Appendix.

Substituting equations (25)–(27) into equations (19)–(21) and performing Galerkin's procedure [i.e., multiplying equations (19), (20) and (21) by $\sin(\pi x/a) \sin(\pi y/b)$, $\cos(\pi x/a)\sin(\pi y/b)$ and $\sin(\pi x/a)\cos(\pi y/b)$, respectively, and then integrating over the area of the plate], three sets of ordinary differential equations are obtained for the time-dependent variables W, X and Y:

$$\left.\begin{aligned} &\frac{d^2 W}{d\tau^2} + \frac{L_{WX}}{H} X + \frac{L_{WY}}{H} Y + L_{WW} W + H^2 G W^3 = 2F \cos \Omega\tau, \\ &L_{XX} X + L_{XY} Y + H L_{XW} W = 0, \qquad L_{YX} X + L_{YY} Y + H L_{YW} W = 0 \end{aligned}\right\}. \tag{29}$$

In the above equations, L_{ij} are the non-dimensional coefficients of the linear terms, G is the non-dimensional coefficient of the non-linear term, and F and Ω are the non-dimensional amplitude and frequency of the load, respectively, and details of the coefficients are given in the Appendix. Other non-dimensional parameters are defined as

$$\tau = \sqrt{\frac{E_T h^3}{\rho a^4}}\, t, \qquad H = \frac{h}{a}. \tag{30}$$

Eliminating X and Y from equation (29) and adding on the effect of viscous damping, the Duffing-type equation in terms of the transverse displacement W is obtained as

$$\frac{d^2 W}{d\tau^2} + 2\xi\omega \frac{dW}{d\tau} + \omega^2 W + H^2 G W^3 = 2F \cos \Omega\tau, \tag{31}$$

where

$$\omega^2 = L_{WW} + \frac{L_{XW}(L_{YY}L_{WX} - L_{YX}L_{WY}) + L_{YW}(L_{XX}L_{WY} - L_{XY}L_{WX})}{L_{XY}L_{YX} - L_{XX}L_{YY}}, \tag{32}$$

and ξ is the damping ratio [8]. The non-dimensional linear natural frequency ω is related to the linear natural frequency $\bar{\omega}$ by $\omega = \bar{\omega}a^2(\rho/E_T h^3)^{1/2}$.

3. METHOD OF MULTIPLE SCALES

In this section, we derive steady-state solutions for the subharmonic resonance ($\Omega \approx 3\omega$) by using the MMS. In order to apply the MMS to equation (31), we introduce a non-dimensional small parameter ε, which is defined as

$$\varepsilon = H^2 = (h/a)^2. \tag{33}$$

Using the above parameter, equation (31) is rewritten as

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\tau^2} + 2\varepsilon\mu \frac{\mathrm{d}W}{\mathrm{d}\tau} + \omega^2 W + \varepsilon G W^3 = 2F\cos\Omega\tau, \tag{34}$$

where $\mu = \xi\omega/\varepsilon$.

Kondou and Yagasaki [14] pointed out that the way to set small parameters in the MMS has a great influence on the analytical results in the case of a primary resonance of the Duffing-type equation. In order to use the MMS properly, they suggested that a detuning parameter σ expressing the relationship between linear natural frequency ω and driving frequency Ω had better be changed from $\Omega = \omega + \varepsilon\sigma$ to

$$\Omega^2 = \omega^2 + \varepsilon\sigma. \tag{35}$$

Moreover, the present authors [12] showed that the detuning parameter (35) was effective for two-mode responses ($\Omega \approx \omega_1$ or ω_2) of laminated plates with an internal resonance ($3\omega_1 \approx \omega_2$). Therefore, a detuning parameter for the subharmonic resonance ($\Omega \approx 3\omega$) is defined in the quadratic forms of Ω and ω such as equation (35):

$$\Omega^2 = 9\omega^2 + \varepsilon\sigma. \tag{36}$$

According to the MMS, $W(\tau)$ is expanded in the form

$$W(\tau; \varepsilon) = x_0(T_0, T_1, \ldots) + \varepsilon x_1(T_0, T_1, \ldots) + \cdots, \tag{37}$$

in which $T_0 = \tau$, $T_1 = \varepsilon\tau, \ldots$ are different time scales. Substituting equations (36) and (37) into equation (34) and equating the coefficients of ε^0 and ε^1 on both sides, we obtain

$$\varepsilon^0 : D_0^2 x_0 + \frac{\Omega^2}{9} x_0 = 2F\cos\Omega T_0, \tag{38}$$

$$\varepsilon^1 : D_0^2 x_1 + \frac{\Omega^2}{9} x_1 = -2D_0 D_1 x_0 - 2\mu D_0 x_0 - G x_0^3 + \frac{\sigma}{9} x_0, \tag{39}$$

TABLE 1

Comparison of non-dimensional linear natural frequencies ω of square laminated plates ($\theta = \pm 45^\circ$) ($E_L/E_T = 40$, $G_{LT}/E_T = G_{LZ}/E_T = 0{\cdot}6$, $G_{TZ}/E_T = 0{\cdot}5$, $\nu_{LT} = 0{\cdot}5$)

	$N = 2$		$N = 8$	
a/h	Present	Reference [15]	Present	Reference [15]
10	13·162	13·044	19·346	19·289
20	14·223	14·179	23·295	23·259
25	14·368	14·338	23·951	23·924
50	14·569	14·561	24·917	24·909
100	14·620	14·618	25·178	25·176

where $D_i = \partial/\partial T_i$. The first-order approximate solution [i.e., the solution of equation (38)] is given by

$$x_0 = P(T_1)\exp(\mathrm{i}\Omega T_0/3) + \Lambda \exp(\mathrm{i}\Omega T_0) + cc, \tag{40}$$

in which $\Lambda = -9F/(8\Omega^2)$, P is an unknown function and cc denotes the complex conjugate of the preceding terms. Substituting equation (40) into equation (39) and eliminating the secular terms, the solvability condition is obtained:

$$\frac{2}{3}\mathrm{i}\Omega(P' + \mu P) + 3G(P^2\bar{P} + \Lambda\bar{P}^2 + 2\Lambda^2 P) - \frac{\sigma}{9}P = 0, \tag{41}$$

where $\bar{P}$ is a complex conjugate of P and a prime denotes the derivative with respect to T_1.

To solve equation (41), P and $\bar{P}$ are expressed in the polar form

$$P = \frac{1}{2}p\exp(\mathrm{i}\beta), \qquad \bar{P} = \frac{1}{2}p\exp(-\mathrm{i}\beta), \tag{42}$$

in which p and β are the amplitude and phase of the subharmonic response respectively. Substituting equation (42) into equation (41) and separating the real and imaginary parts, we obtain the following equations:

$$\left.\begin{aligned} p' &= -\mu p + \frac{9}{4\Omega}G\Lambda p^2 \sin 3\beta \\ p\beta' &= \frac{9}{8\Omega}Gp(p^2 + 2\Lambda p\cos 3\beta + 8\Lambda^2) - \frac{\sigma}{6\Omega}p \end{aligned}\right\}. \tag{43}$$

The substitution of conditions $p' = \beta' = 0$ for the steady-state solutions into equation (43) leads to algebraic equations in terms of unknown variables p and β. The steady-state solutions are obtained from the algebraic equations by using the Newton–Raphson method. In order to examine the stability of the steady-state solutions, we derive characteristic equations in terms of small disturbances on the steady-state solutions and calculate the eigenvalues of the coefficient matrix of the

TABLE 2

Material properties

	Graphite-epoxy (GR)	Glass-epoxy (GL)
E_L/E_T	40	3
G_{LT}/E_T	0·5	0·5
v_{LT}	0·25	0·25
G_{LZ}/E_T	0·5	0·5
G_{TZ}/E_T	0·33	0·33

equations. If stable steady-state solutions are obtained, the transverse deflection of the plate is given by

$$\frac{w_0}{h} = \left\{ W_s \cos\left(\frac{\Omega}{3}\tau + \beta\right) + W_n \cos \Omega\tau \right\} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{44}$$

where $W_s = Hp$ and $W_n = H\Lambda$. In the above equation, W_s is the amplitude due to the subharmonic resonance (i.e., subharmonic response) and W_n is the amplitude with the same frequency as the driving frequency (i.e., nonresonant response).

4. NUMERICAL RESULTS AND DISCUSSION

The non-dimensional linear natural frequencies of antisymmetric angle-ply laminated plates ($\theta = \pm 45°$, $N = 2$ and 8) obtained by the present analysis are compared with those of reference [15] for various thickness ratios in Table 1. The shear correction factor k^2 equals 5/6 in the present analysis. Though the present theory neglects the effect of in-plane and rotatory inertias, reference [15] took the

TABLE 3

Non-dimensional linear natural frequencies ω of plates used in figures

θ	N	a/h	Material	ω	Figure
$\pm 45°$	4	10	GR	17·484	4
$\pm 45°$	4	20	GR	21·454	4, 5, 6, 7
$\pm 45°$	4	30	GR	22·535	4
$\pm 45°$	4	50	GR	23·155	2, 3, 4
$\pm 45°$	4	CPT	GR	23·528	4
$0°$	1	20	GR	17·216	5
$0°$	1	20	GL	7·246	5
$\pm 30°$	4	20	GR	20·370	5
$\pm 30°$	4	20	GL	7·702	5
$\pm 45°$	2	20	GR	14·095	6
$\pm 45°$	8	20	GR	22·747	6
$\pm 45°$	∞	20	GR	23·148	6

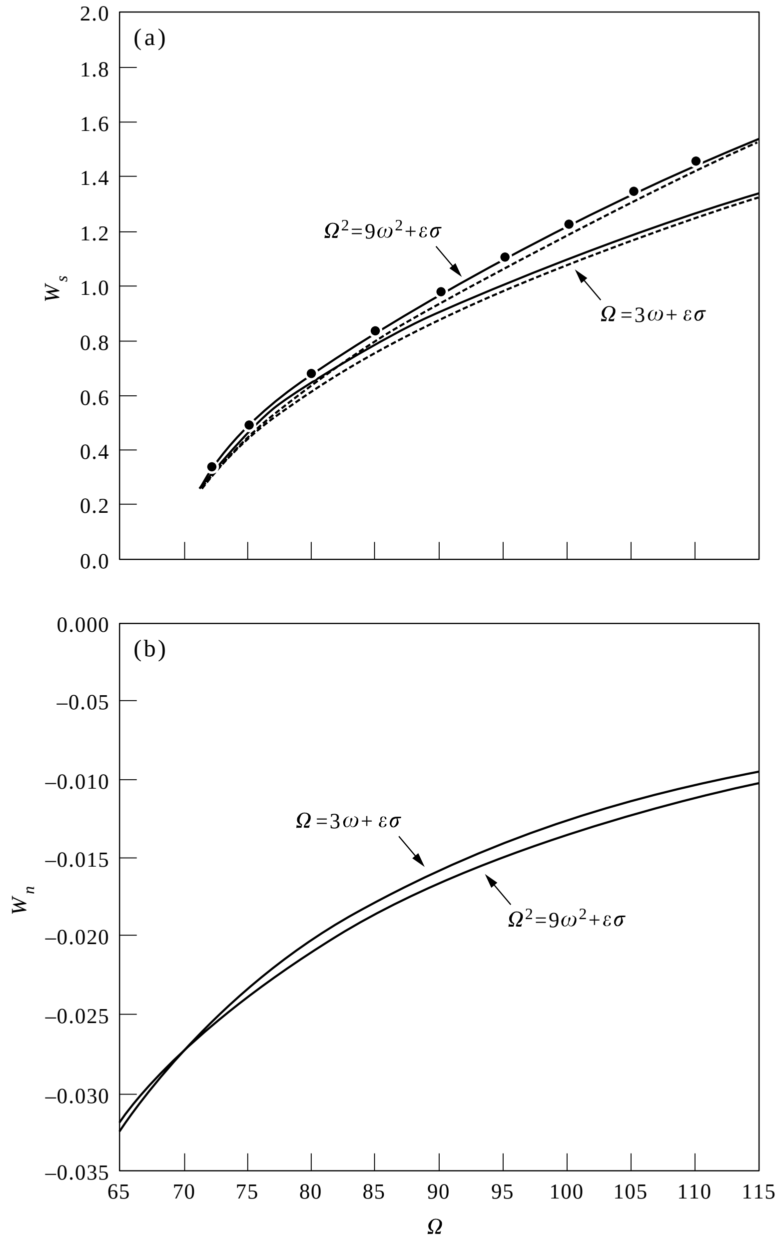

Figure 2. Comparison of two detuning parameters on frequency–response curves for the laminated plate ($\theta = \pm 45°$, $N = 4$, $a/h = 50$, GR, $F = 6000$); (a) subharmonic response and (b) non-resonant response. ●, Numerical integration; ——, stable; - - - - , unstable.

effect of both inertias into account. Therefore, the values of the present analysis are slightly larger than those of reference [15]. However, the maximum error is less than 1%, and the present analysis has sufficient accuracy to examine the subharmonic resonance.

Square composite plates (antisymmetric angle-ply laminated plates and orthotropic plates) are treated in the following numerical examples. Each layer is assumed to be made of graphite-epoxy or glass-epoxy, whose material properties [7] are shown in Table 2. The shear correction factor k^2 and the damping ratio ξ are taken as $k^2 = 5/6$ and $\xi = 0{\cdot}005$, respectively. The non-dimensional linear natural frequencies ω of the plates used in each figure are listed in Table 3. The amplitude F of the excitation is shown in each figure.

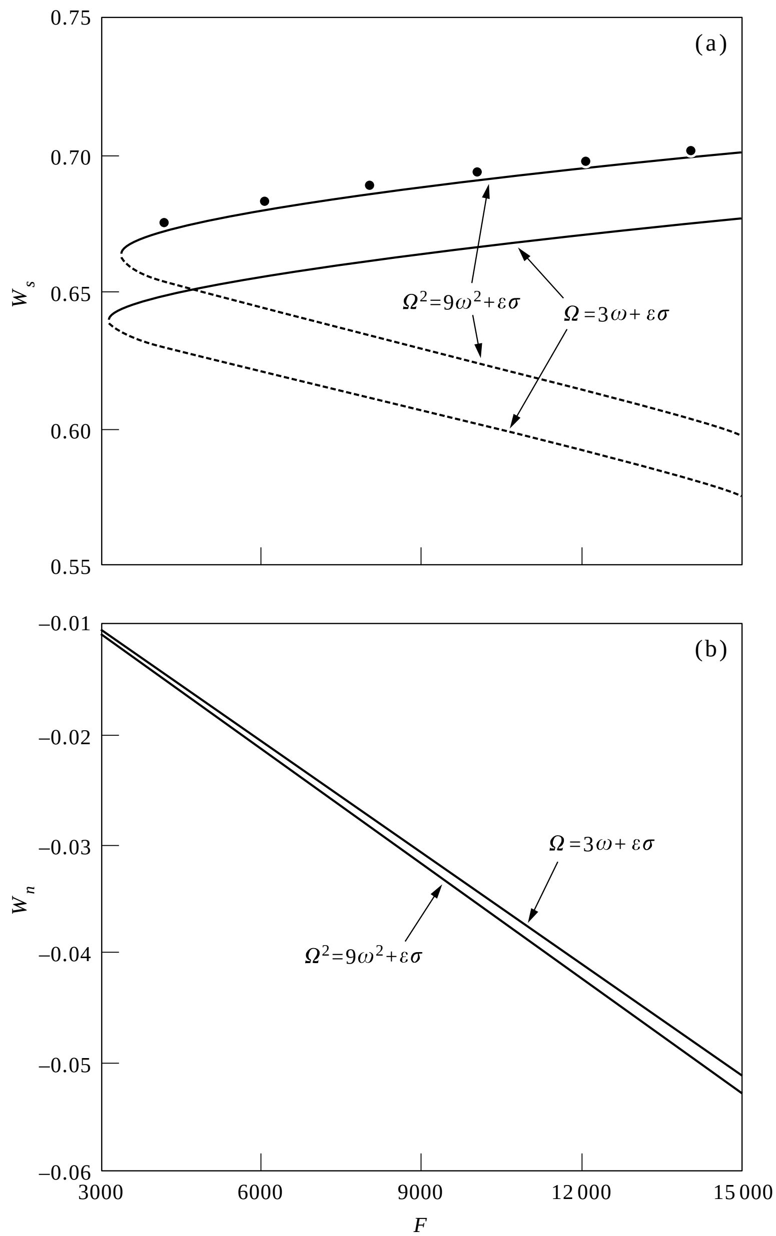

Figure 3. Comparison of two detuning parameters on force–response curves for the laminated plate ($\theta = \pm 45^\circ$, $N = 4$, $a/h = 50$, GR, $\Omega = 80$); (a) subharmonic response and (b) non-resonant response. Key as in Figure 2.

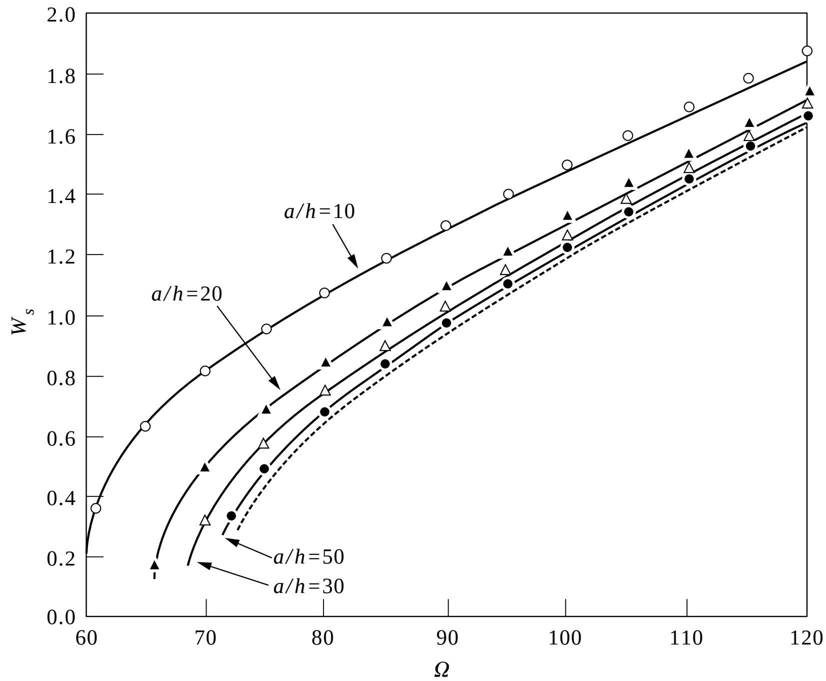

Figure 4. Effect of thickness ratio on frequency–response curves for the laminated plate ($\theta = \pm 45°$, $N = 4$, GR, $F = 6000$). ○, ▲, △, ●, Numerical integration; ——, FSDT; - - - - , CPT.

Figure 2 presents the comparison of the frequency–response curves obtained by the present method and a conventional detuning parameter (i.e., $\Omega = 3\omega + \epsilon\sigma$) [10]. An antisymmetric angle-ply laminated plate ($\theta = \pm 45°$, $N = 4$, $a/h = 50$, GR) with in-plane immovable boundary along all edges is treated here. Figure 2(a) and (b) depict subharmonic and non-resonant responses respectively. In Figure 2(a), solid and broken lines indicate stable and unstable responses respectively. It can be seen in Figure 2 that the amplitudes of the non-resonant response are much smaller than those of the subharmonic response. The non-resonant responses obtained by two different detuning parameters almost agree with each other. However, with an increase in Ω, the amplitude of the subharmonic response obtained by the present method deviates from that by the conventional MMS. There is a difference between them in the high frequency region. In order to check the validity of the analytical results, the equation of motion (34) is integrated numerically using the fourth-order Runge–Kutta method. The numerical results are denoted in Figure 2(a) by ●. As seen in Figure 2(a), they are in good agreement with the results obtained using the present method. Therefore, it can be said that the present method is more suitable for the analysis of the subharmonic resonance than the conventional MMS.

Similarly, the comparison of the force–response curves obtained in each method is presented in Figure 3. The excitation frequency is fixed at $\Omega = 80$. It can be seen in Figure 3(a) that subharmonic resonances obtained by each method occur at almost the same force, but the amplitudes of responses obtained by the conventional MMS are smaller than those by the present method. As the results by the present method show good agreement with those by the Runge–Kutta method, the validity of the present analysis is also confirmed on the force–response curves.

Figure 4 shows the effect of the thickness ratio on frequency–response curves for antisymmetric angle-ply laminated plates ($\theta = \pm 45^\circ$, $N = 4$, GR) with in-plane immovable boundary along all edges. Only the stable subharmonic responses are plotted in the figure. Solid and broken lines indicate the results obtained from FSDT and CPT respectively. The symbols ○, ▲, △ and ● denote the results for $a/h = 10$, 20, 30 and 50 respectively, obtained by the numerical integration of the equation of motion. With a decrease in a/h, the non-dimensional linear natural frequency decreases as shown in Table 3, and the results of the first order shear deformation theory deviate from that of the classical plate theory in Figure 4. It is found that the effect of the shear deformation is significant for the subharmonic response as well as the linear natural frequency.

Stable subharmonic responses of plates with different fiber angles and material properties are presented in Figure 5, where the thickness ratio of all plates is $a/h = 20$ and the boundary is in-plane immovable. Solid and broken lines indicate plates made of graphite-epoxy and glass-epoxy, respectively. The results obtained by the Runge–Kutta method for the 0°, $\pm 30^\circ$ and $\pm 45^\circ$ plates are denoted by the symbols △, ● and ○ respectively. As seen in Figure 5, the amplitude for 0° is lower than those for $\pm 30^\circ$ and $\pm 45^\circ$. The occurring frequencies of the subharmonic resonance for glass-epoxy are higher than those for graphite-epoxy.

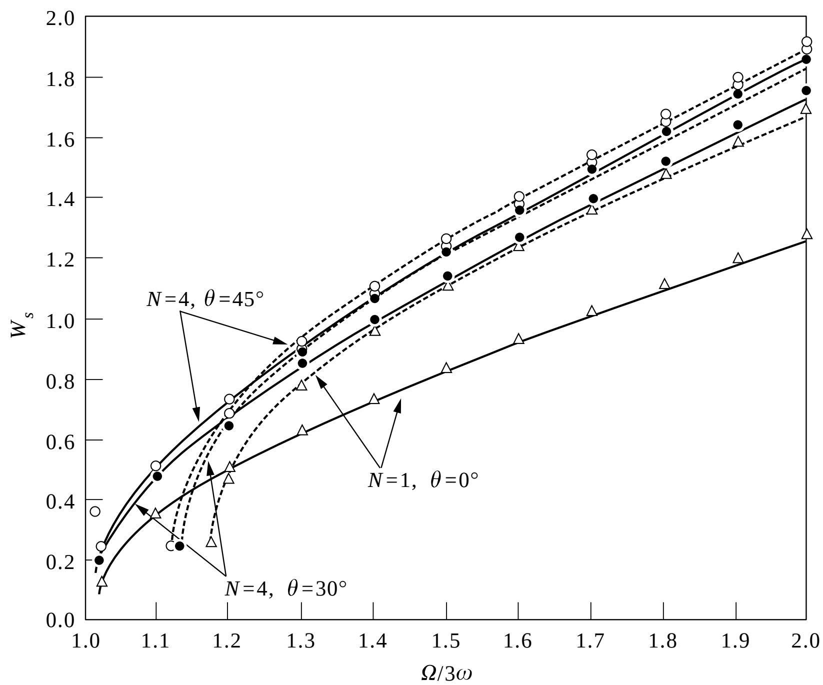

Figure 5. Effects of fiber angle and material property on frequency–response curves for the plate ($a/h = 20$, $F = 2500$). △, ●, ○, Numerical integration; ——, graphite-epoxy; - - - - , glass-epoxy.

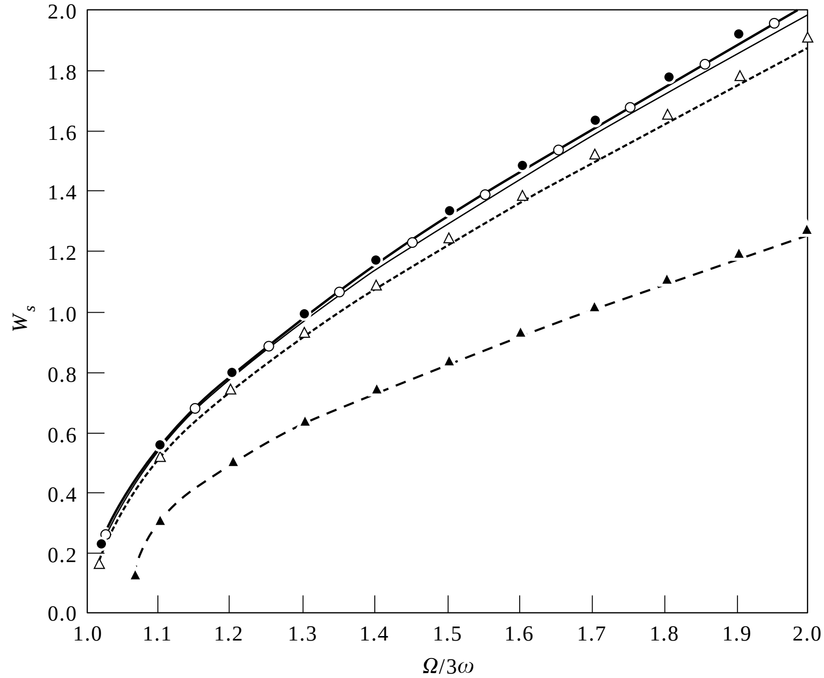

Figure 6. Effect of number of layers on frequency–response curves for the laminated plate ($\theta = \pm 45°$, $a/h = 20$, GR, $F = 3500$). ●, ○, ▲, △, Numerical integration; ——, $N = \infty$; ——, $N = 8$; ·····, $N = 4$; - - - -, $N = 2$.

There is a tendency that the increase in $\Omega/3\omega$ of the amplitudes is more rapid for glass-epoxy than for graphite-epoxy.

Figure 6 shows stable subharmonic responses of laminated plates ($\theta = \pm 45°$) having different number of layers for the same thickness ratio ($a/h = 20$). The symbols ▲, △, ○ and ● denote the results for $N = 2$, 4, 8 and ∞ respectively. In the case of $N = \infty$, the bending-stretching coupling stiffness B^*_{ij} is equal to zero. The amplitudes of the subharmonic response get bigger as the number of layers increases. Therefore, it can be said that the increase in B^*_{ij} reduces the amplitude.

Figure 7 presents the effect of in-plane boundary conditions on frequency–response and force–response curves for a laminated plate ($\theta = \pm 45°$, $N = 4$, $a/h = 20$, GR). Solid and broken lines indicate the stable and unstable responses respectively. The results obtained by numerical integration for movable and immovable edges are denoted by the symbols ● and ○ respectively. The amplitudes for the movable edge are much larger than those for the immovable edge. It is found that the in-plane boundary condition has a great influence on the subharmonic response. In Figure 7(b), the subharmonic resonances for the immovable and movable edges are generated at $F \approx 1200$ and 5600 respectively. It can be said that the immovable edge condition induces the subharmonic resonance more easily than the movable edge condition.

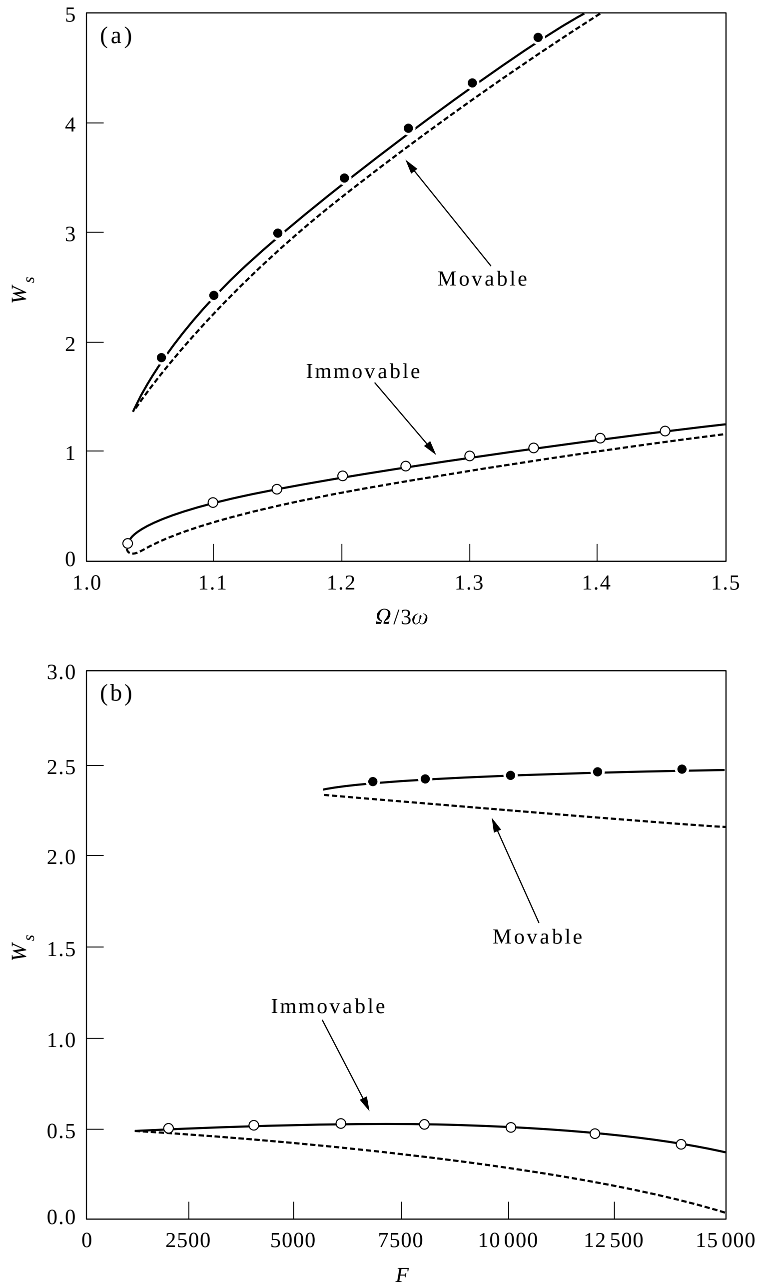

Figure 7. Effect of in-plane boundary condition on frequency–response and force–response curves for the laminated plate ($\theta = \pm 45°$, $N = 4$, $a/h = 20$, GR); (a) frequency–response curves ($F = 8000$) and (b) force–response curves ($\Omega/3\omega = 1{\cdot}1$). Key as Figure 2.

5. CONCLUSIONS

Governing equations for simply supported, rectangular antisymmetric angle-ply laminated plates, based on the first order shear deformation and the von Kármán-type geometric non-linear theories, were derived by Hamilton's principle.

Applying Galerkin's procedure and eliminating the variables except for the transverse displacement, the governing equations were reduced to the Duffing-type equation. Subharmonic resonances of the laminated plates were investigated by the MMS, where we suggested new definition of a detuning parameter to apply it properly. The analytical results were compared with those obtained by the Runge–Kutta method, and the validity of the present analysis was clearly shown. The influence of the lamination sequence, thickness ratio, number of layers and in-plane boundary condition on the subharmonic resonance was made clear in some figures.

One subject for future study is to investigate subharmonic responses for laminated plates with internal resonances by the present method.

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APPENDIX

The coefficients in equation (27) are as follows:

$$B_{11} = \frac{bh\{(a^2B^*_{16} + b^2B^*_{26} - b^2B^*_{61})aX + (a^2B^*_{16} + b^2B^*_{26} - a^2B^*_{62})bY\}}{\pi\{a^4A^*_{11} + a^2b^2(2A^*_{12} + A^*_{66}) + b^4A^*_{22}\}},$$

$$B_{02} = \frac{b^2h^4W^2}{32a^4A^*_{11}}, \qquad B_{02} = \frac{h^4W^2}{32b^2A^*_{22}},$$

$$C_1 = \frac{(b^2A^*_{12} - a^2A^*_{11})h^4\pi^2W^2}{16a^4b^2(A^{*2}_{12} - A^*_{11}A^*_{22})}, \qquad C_2 = \frac{(a^2A^*_{12} - b^2A^*_{22})h^4\pi^2W^2}{16a^4b^2(A^{*2}_{12} - A^*_{11}A^*_{22})},$$

and all other B_{pq} vanish. The coefficients in equation (29) are as follows:

$$F = \frac{4a^3rq_0}{E_Th^5}\int_0^b\int_0^a \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y, \qquad \Omega = \Omega'\sqrt{\frac{\rho a^4}{E_Th^3}},$$

$$L_{XX} = \frac{-\pi^2ra_{11}a_{12}a_{22}a_{66}(r^2b_{16} + b_{26} - b_{61})^2}{4\{a_{11}a_{12}a_{66} + r^2a_{11}a_{22}(a_{12} + 2a_{66}) + r^4a_{12}a_{22}a_{66}\}} - \frac{\pi^2(d_{11} + r^2d_{66}) + s_{55}}{4r},$$

$$L_{YY} = \frac{-\pi^2a_{11}a_{12}a_{22}a_{66}(r^2b_{62} - r^2b_{16} - b_{26})^2}{4r\{a_{11}a_{12}a_{66} + r^2a_{11}a_{22}(a_{12} + 2a_{66}) + r^4a_{12}a_{22}a_{66}\}} - \frac{\pi^2(r^2d_{22} + d_{66}) + s_{44}}{4r},$$

$$L_{XY} = L_{YX} = \frac{\pi^2a_{11}a_{12}a_{22}a_{66}(b_{61} - r^2b_{16} - b_{26})(r^2b_{16} + b_{26} - r^2b_{62})}{4\{a_{11}a_{12}a_{66} + r^2a_{11}a_{22}(a_{12} + 2a_{66}) + r^4a_{12}a_{22}a_{66}\}} - \frac{\pi^2(d_{12} + d_{66})}{4},$$

$$L_{XW} = -\frac{\pi s_{55}}{4r}, \qquad L_{YW} = -\frac{\pi s_{44}}{4},$$

$$L_{WX} = \pi s_{55}, \qquad L_{WY} = \pi r s_{44}, \qquad L_{WW} = \pi^2(r^2s_{44} + s_{55}),$$

$$G = \frac{\pi^4\{a_{11}(a_{11}a_{22} - 3a_{12}^2 + 4r^2a_{12}a_{22}) + r^4a_{22}(a_{11}a_{22} - 3a_{12}^2)\}}{16(a_{11}a_{12} - a_{12}^2)} \quad \text{(for immovable)},$$

$$G = \frac{\pi^4(a_{11} + r^4a_{22})}{16} \quad \text{(for movable)},$$

where

$$(A^*_{11}, A^*_{12}, A^*_{22}, A^*_{66}) = \frac{1}{E_Th}\left(\frac{1}{a_{11}}, \frac{1}{a_{12}}, \frac{1}{a_{22}}, \frac{1}{a_{66}}\right),$$

$$(B^*_{16}, B^*_{26}, B^*_{61}, B^*_{62}) = h(b_{16}, b_{26}, b_{61}, b_{62}),$$

$$(D^*_{11}, D^*_{12}, D^*_{22}, D^*_{66}) = E_Th^3(d_{11}, d_{12}, d_{22}, d_{66}),$$

$$(S_{44}, S_{55}) = \frac{E_Th^3}{a^2}(s_{44}, s_{55}), \qquad r = \frac{a}{b}.$$