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# STABILITY OF PIECEWISE LINEAR OSCILLATORS WITH VISCOUS AND DRY FRICTION DAMPING

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The stability analysis for periodic motions of a class of harmonically excited single degree of freedom oscillators with piecewise linear characteristics is presented. The common characteristic of these oscillators is that they possess viscous and constant damping properties, which depend on their velocity direction. The presence of constant damping terms in the equation of motion introduces acceleration discontinuities and makes possible the appearance of finite time intervals within the periodic solution where the oscillator is stuck at the same position. Harmonic and subharmonic motions with an arbitrary number of solution pieces are examined. The analysis takes advantage of the fact that the exact solution form for any solution piece included between two consecutive zero velocity values is known. It is based on the derivation of a matrix relation which determines how an arbitrary but small perturbation at the beginning of a periodic solution propagates to the end of a response period. Then, results obtained by bifurcation analysis of the periodic solutions are also presented. At the end, some of the analytical predictions are confirmed by considering an example mechanical model.



## 1. INTRODUCTION

The class of mechanical systems examined in the present study is modelled by single degree of freedom oscillators with parameter and motion discontinuities. Such discontinuities appear frequently in practice as a result of clearances, motion limiting constraints and impacts (e.g., [1–6]) or due to the presence of dry friction [6–10] and viscous damping properties depending on the velocity direction [11, 12]. In the former case, a discontinuity in the system stiffness and/or damping coefficients occurs when the system reaches certain critical displacement values. On the other hand, in the latter case a discontinuity may appear when the velocity of the system becomes zero. Due to the motion constraints, the equations of motion of both of these categories of mechanical oscillators appear in a strongly nonlinear form. As a result, their dynamic behaviour can be captured only after application of special analytical, numerical and experimental techniques. Moreover, this behaviour can be quite interesting, as was demonstrated by several previous relevant studies (e.g., [1–10]). More specifically, the response of these

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systems under periodic external excitation is expected to be periodic. However, for some parameter combinations, existing periodic solutions may lose stability and give their place to quasiperiodic or chaotic response. This leads to difficulties in predicting and understanding the system dynamics, which in turn causes complications at their design stage.

The main objective of this study is to present an appropriate stability analysis for periodic solutions of harmonically excited piecewise linear systems with dry friction and damping coefficients depending on the velocity direction. The presentation follows closely a previous work [4], developed for similar systems but with displacement constraints. Apart from the differences caused in the formulation by the velocity constraints, another important difference is that these systems may exhibit sticking motions, which are not observed in systems with displacement constraints. The method exploits the fact that the exact solution between two consecutive instances of zero velocity is known. The specifics of the mechanical model and the solution form within each time interval are presented in the following section. In the third section, the stability is presented for periodic motions with an arbitrary number of solution pieces, which may involve several sticking intervals. This is done by developing a systematic methodology, which determines the evolution of a small deviation from a periodic solution over a single response period. Based on this analysis, some quite general bifurcation analysis is also performed and presented in the fourth section. In the fifth section, an example mechanical oscillator is considered and some numerical results are obtained, which are in accord with the analytical predictions. The final section summarizes the highlights of the work.

## 2. MECHANICAL MODEL—SOLUTION FORM

The equation of motion of the class of dynamical systems examined has the form

$$m\ddot{x} + g(\dot{x}, x) = f \sin(\Omega t + \varphi). \tag{1}$$

The characteristic of these systems is that their damping and stiffness properties may change any time their velocity crosses the zero value. Namely, if $t_{i-1}$ and $t_i$ denote two consecutive times where the velocity becomes zero (these time instances will be referred to as 'crossing times' in the following), then within that time interval

$$g(\dot{x}, x) = c_i\dot{x} + k_i x + h_i. \tag{2}$$

Clearly, this class of oscillators is quite general and includes many other important classes of oscillators, like those with classical Coulomb friction, as special cases.

In order to facilitate the analysis, the original equation of motion (1) is put in a convenient normalized form, by introducing the notation

$$\theta = \Omega t, \quad \hat{\theta}_i = \Omega t_i, \quad \theta_i = \theta - \hat{\theta}_{i-1}, \quad \hat{\varphi}_i = \varphi + \hat{\theta}_{i-1}, \quad y_i(\theta_i) = x(t - t_{i-1})/x_c,$$

where $x_c$ represents a characteristic length of the system. Then, equation (1) is replaced by

$$\ddot{y}_i + 2\delta_i \dot{y}_i + \omega_i^2 y_i = p \sin(\theta_i + \hat{\varphi}_i) + \omega_i^2 f_i, \tag{3}$$

with

$$\bar{\omega}_i = \sqrt{\frac{k_i}{m}}, \qquad \omega_i = \frac{\bar{\omega}_i}{\Omega}, \qquad \zeta_i = \frac{c_i}{2\sqrt{k_i m}}, \qquad \delta_i = \zeta_i \omega_i,$$

$$p = \frac{f}{m\Omega^2 x_c}, \qquad f_i = \frac{-h_i}{k_i x_c}.$$

The analysis presented here is based on the fact that the exact solution form of equation (3) within the time interval $0 \leqslant \theta_i \leqslant \theta_{ic}$, with

$$\theta_{ic} \equiv \hat{\theta}_i - \hat{\theta}_{i-1},$$

is known. Namely, if the motion is non-sticking (and underdamped), then

$$y_i(\theta_i) = e^{-\delta_i \theta_i}[A_i \sin(\eta_i \theta_i) + B_i \cos(\eta_i \theta_i)] + P_i \sin(\theta_i + \alpha_i) + f_i, \tag{4}$$

with

$$\eta_i = \sqrt{\omega_i^2 - \delta_i^2}, \qquad P_i = \frac{p}{\sqrt{(\omega_i^2 - 1)^2 + 4\delta_i^2}}, \qquad \alpha_i = \hat{\varphi}_i - \varphi_i$$

and phase angle determined by the expressions

$$\cos\varphi_i = \frac{\omega_i^2 - 1}{\sqrt{(\omega_i^2 - 1)^2 + 4\delta_i^2}}, \qquad \sin\varphi_i = \frac{2\delta_i}{\sqrt{(\omega_i^2 - 1)^2 + 4\delta_i^2}}.$$

On the other hand, when the oscillator is stuck within the time interval $0 \leqslant \theta_i \leqslant \theta_{ic}$, equation (3) determines the corresponding friction force in the form

$$f_i = y_i - \hat{P}_i \sin(\theta_i + \hat{\varphi}_i), \tag{5}$$

with

$$\hat{P}_i = p/\omega_i^2.$$

The dynamical systems examined are expected to exhibit periodic steady state response, among other possible response types. For general $n$-periodic motions consisting of $k$ pieces—with several non-sticking and/or sticking intervals—(see Figure 1 for an example), the unknowns of the problem are the constants $A_i$, $B_i$ and the crossing times $\hat{\theta}_i$ of each interval, together with the phase $\varphi$. As usual, this phase is introduced in the forcing function so that the motion starts at a point of zero velocity. In any particular application, the unknowns can in principle be determined by imposing an appropriate set of periodicity and matching conditions (see the example in section 5). These conditions lead to an equal number of transcendental equations, whose solution determines the unknown constants and crossing times characterizing the periodic solution considered (e.g., [1–3, 7–9]).

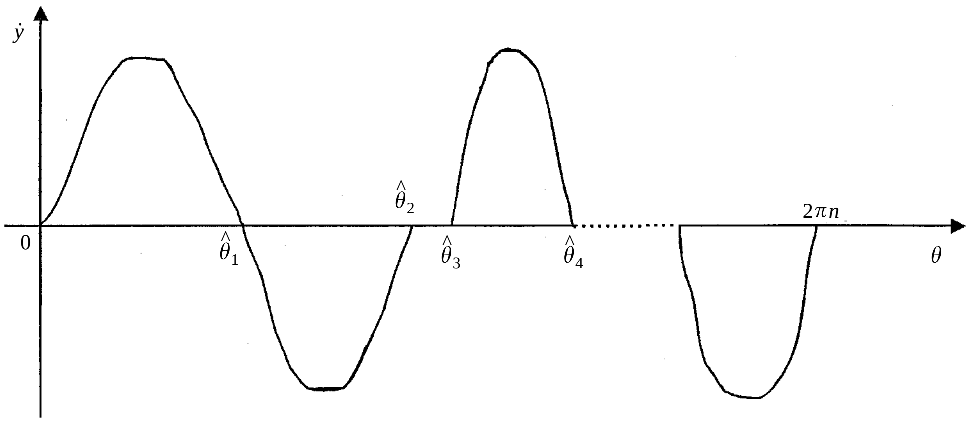


Figure 1. Typical periodic motion of the system examined.

In the following two sections it is assumed that a periodic solution of the system examined, with an arbitrary number of solution pieces, has been located. The emphasis is first placed on developing an appropriate analytical methodology for determining the stability properties of this solution. Then, some bifurcation results are also presented, helping in the efforts to investigate the way these properties change and the types of the resulting solutions, as the system parameters are varied.

## 3. STABILITY OF PERIODIC MOTIONS

Knowledge of the stability properties of the solutions of a dynamical system is of fundamental importance, since only stable solutions are observable in practice. Due to the presence of discontinuities in the equations of motion, the stability characteristics of a located periodic motion of the systems examined cannot be revealed by employing the classical linearization methods [13]. An alternative way to perform the stability analysis is to apply an equivalent methodology, which is more suitable for piecewise linear dynamical systems. According to this method, arbitrary but small perturbations are first introduced into initial conditions leading to a periodic motion. Then, employing the exact solution form within the various pieces of the motion it is determined whether these perturbations grow or diminish with time [1, 4].

In the present case, a periodic motion consists of time intervals where the system has non-zero velocity and may also involve several finite time intervals where the oscillator is stuck at the same position. For this reason, before the complete stability analysis of a periodic motion is performed, a typical non-sticking and then a sticking motion interval (as shown in Figures 2 and 3, respectively) are analyzed separately.

First, for a **non-sticking** part of the motion assume that it starts with initial conditions

$$y_i(0) = \hat{y}_{i-1}, \qquad \dot{y}_i(0) = 0. \tag{6}$$

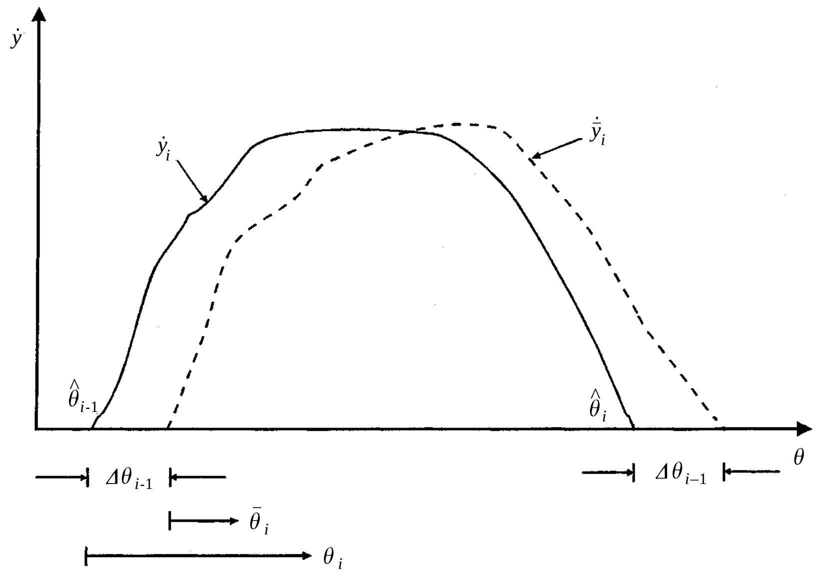


Figure 2. Exact and perturbed non-sticking interval of a periodic motion.

Then, the corresponding unknown coefficients in equation (4) are determined in the form

$$A_i = \frac{1}{\eta_i}\left[\delta_i(\hat{y}_{i-1} - f_i - P_i \sin\alpha_i) - P_i \cos\alpha_i\right], \qquad B_i = \hat{y}_{i-1} - f_i - P_i \sin\alpha_i. \quad \text{(7a, b)}$$

Next, define the new time variable $\overline{\theta}_i = \theta_i - \Delta\theta_{i-1}$ and consider the new solution $\bar{y}_i(\overline{\theta}_i)$, resulting from the neighbouring initial conditions

$$\bar{y}_i(0) = \hat{y}_{i-1} + \Delta\hat{y}_{i-1}, \qquad \dot{\bar{y}}_i(0) = 0 \tag{8}$$

(see Figure 2). This solution has the form

$$\bar{y}_i(\overline{\theta}_i) = e^{-\delta_i\overline{\theta}_i}[\overline{A}_i \sin(\eta_i\overline{\theta}_i) + \overline{B}_i \cos(\eta_i\overline{\theta}_i)] + P_i \sin(\overline{\theta}_i + \Delta\theta_{i-1} + \alpha_i) + f_i. \tag{9}$$

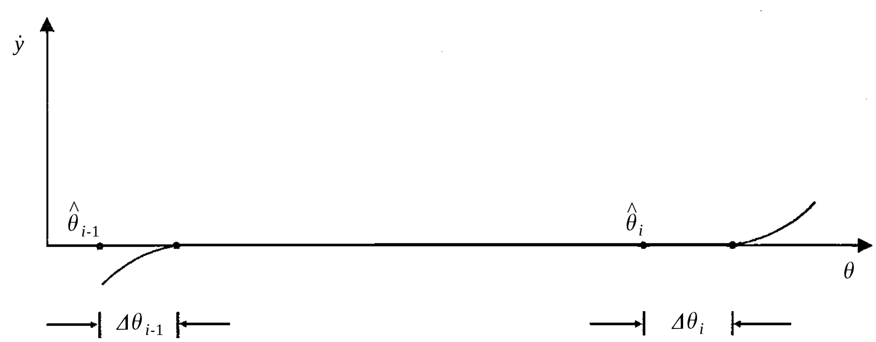


Figure 3. Exact and perturbed sticking interval of a periodic motion.

If the original perturbations $\Delta\theta_{i-1}$ and $\Delta\hat{y}_{i-1}$ are small, it is expected that the following relations hold:

$$\overline{A}_i = A_i + \frac{\partial A_i}{\partial \hat{y}_{i-1}} \Delta\hat{y}_{i-1} + \frac{\partial A_i}{\partial \hat{\theta}_{i-1}} \Delta\theta_{i-1}, \qquad \overline{B}_i = B_i + \frac{\partial B_i}{\partial \hat{y}_{i-1}} \Delta\hat{y}_{i-1} + \frac{\partial B_i}{\partial \hat{\theta}_{i-1}} \Delta\theta_{i-1}.$$

Then, by employing equation (7a), it can easily be shown that

$$\frac{\partial A_i}{\partial \hat{y}_{i-1}} = \frac{\delta_i}{\eta_i} \equiv U_i, \qquad \frac{\partial A_i}{\partial \hat{\theta}_{i-1}} = \frac{P_i}{\eta_i} (\sin \alpha_i - \delta_i \cos \alpha_i) \equiv V_i \tag{10}$$

Likewise, direct differentiation of equation (7b) yields

$$\partial B_i / \partial \hat{y}_{i-1} = 1, \qquad \partial B_i / \partial \hat{\theta}_{i-1} = -P_i \cos \alpha_i \equiv W_i. \tag{11}$$

Next, it is assumed that the velocity of the perturbed solution $\bar{y}_i(\overline{\theta}_i)$ will become zero at a new time $\overline{\theta}_{ic} = \theta_{ic} + \Delta T_{i-1}$, with $\Delta T_{i-1} \equiv \Delta\theta_i - \Delta\theta_{i-1}$. This means that the perturbed solution satisfies the conditions

$$\bar{y}_i(\overline{\theta}_{ic}) = \hat{y}_i + \Delta\hat{y}_i, \qquad \dot{\bar{y}}_i(\overline{\theta}_{ic}) = 0. \tag{12a, b}$$

Using the definitions $s_i = \sin(\eta_i\theta_{ic})$, $\sigma_i = \cos(\eta_i\theta_{ic})$, $e_i = \mathrm{e}^{-\delta_i\theta_{ic}}$ and expanding equation (12a) in Taylor series, by omitting second and higher order terms, results in the relation

$$\begin{aligned} e_i(1 - \delta_i\Delta T_{i-1})[(A_i + U_i\Delta\hat{y}_{i-1} + V_i\Delta\theta_{i-1})(s_i + \eta_i\sigma_i\Delta T_{i-1}) \\ + (B_i + \Delta\hat{y}_{i-1} + W_i\Delta\theta_{i-1})(\sigma_i - \eta_i s_i \Delta T_{i-1})] \\ + P_i[\sin(\theta_{ic} + \alpha_i) + \cos(\theta_{ic} + \alpha_i)\Delta\theta_i] + f_i = \hat{y}_i + \Delta\hat{y}_i. \end{aligned}$$

The zero order terms in the last expression cancel out, since they satisfy the condition $y_i(\theta_{ic}) = \hat{y}_i$, while the first order terms can be put in the form

$$r_{i1}\Delta T_{i-1} + r_{i2}\Delta\hat{y}_{i-1} + r_{i3}\Delta\theta_{i-1} = \Delta\hat{y}_i \tag{13}$$

with

$$r_{i1} = e_i[\eta_i(\sigma_i A_i - s_i B_i) - \delta_i(s_i A_i + \sigma_i B_i)] + P_i \cos(\theta_{ic} + \alpha_i) = \dot{y}_i(\theta_{ic}) = 0,$$

$$r_{i2} = e_i(\sigma_i + s_i U_i), \qquad r_{i3} = e_i(\sigma_i W_i + s_i V_i) + P_i \cos(\theta_{ic} + \alpha_i).$$

Upon proceeding in a similar fashion, condition (12b) yields eventually the relation

$$r_{i4}\Delta T_{i-1} + r_{i5}\Delta\hat{y}_{i-1} + r_{i6}\Delta\theta_{i-1} = 0, \tag{14}$$

with

$$r_{i4} = \delta_i(d_i B_i - c_i A_i) - \eta_i(d_i A_i + c_i B_i) - P_i \sin(\theta_{ic} + \alpha_i) = \ddot{y}_i(\theta_{ic}) \equiv \hat{a}_i,$$

$$r_{i5} = c_i U_i - d_i, \qquad r_{i6} = c_i V_i - d_i W_i - P_i \sin(\theta_{ic} + \alpha_i),$$

$$c_i = e_i(\eta_i\sigma_i - \delta_i s_i), \qquad d_i = e_i(\eta_i s_i + \delta_i\sigma_i).$$

Therefore, defining the vector

$$\underline{\epsilon}_i = \begin{pmatrix} \Delta\theta_i \\ \Delta\hat{y}_i \end{pmatrix}$$

and combining relations (13) and (14) leads to the expression

$$\epsilon_i = R_i \epsilon_{i-1}. \tag{15}$$

This expression determines the error of the solution at the end of the considered motion interval, provided that the error at the beginning of the same interval is known, with

$$R_i = \begin{bmatrix} 1 - r_{i6}/\hat{a}_i & -r_{i5}/\hat{a}_i \\ r_{i3} & r_{i2} \end{bmatrix}. \tag{16}$$

The establishment of the relation between the error at the beginning and the error at the end of an interval of a **sticking** motion is similar. Namely, in this case it is true that

$$\hat{y}_{i-1} = \hat{y}_i = \hat{P}_i \sin\psi_i + f_i, \tag{17}$$

with $\psi_i = \hat{\theta}_i + \varphi$. After the introduction of small time and displacement perturbations (see Figure 3), this solution is found to satisfy the conditions

$$\Delta\hat{y}_i = \Delta\hat{y}_{i-1}, \qquad \hat{y}_i + \Delta\hat{y}_i = \hat{P}_i \sin(\psi_i + \Delta\psi_{i-1}) + f_i. \tag{18a, b}$$

Expanding equation (18b) in Taylor series and collecting the first order terms yields $\Delta\hat{y}_i = \hat{P}_i \cos\psi_i \Delta\theta_i$, which upon employing equation (17) becomes

$$\Delta\hat{y}_i = \pm\sqrt{\hat{P}_i^2 - (\hat{y}_i - f_i)^2}\,\Delta\theta_i, \tag{19}$$

with the sign determined by the quadrant of phase $\psi_i$. Therefore, combining equation (18a) with equation (19) yields a relation similar to equation (15), with

$$R_i = \begin{bmatrix} 0 & \pm 1/\sqrt{\hat{P}_i^2 - (\hat{y}_i - f_i)^2} \\ 0 & 1 \end{bmatrix}. \tag{20}$$

Based on the above analysis, if the motion examined is periodic with $k$ solution pieces, the error at the end of the first period will be related to the original error by

$$\epsilon_k = \Pi \epsilon_0, \tag{21}$$

after leaving out the contribution of the second and higher order terms. The matrix $\Pi$ is defined as a product of $k$ matrices, resulting from each individual time interval of the periodic solution: i.e.,

$$\Pi \equiv R_k \cdots R_1. \tag{22}$$

Furthermore, upon proceeding in a similar manner, the error in an $n$-periodic solution after $m \times n$ forcing cycles will be related to the original error by $\epsilon_{mk} = \Pi^m \epsilon_0$. This equation implies that the solution examined is asymptotically stable when $\lim_{m\to\infty} \Pi^m = 0$. This requires that the eigenvalues of matrix $\Pi$ have modulus less than one. On the other hand, when at least one eigenvalue of $\Pi$ has modulus greater than one, the periodic solution is unstable.

## 4. BIFURCATION ANALYSIS

From the analysis presented in the previous section, it is obvious that all the elements of matrix $\Pi$ are known functions of the system parameters. Therefore, a change in a system parameter implies a change in the eigenvalues of matrix $\Pi$. As a consequence, if $\lambda$ is the eigenvalue of matrix $\Pi$ with the largest modulus, then some parameter combinations may lead to $|\lambda| = 1$, which in turn signals important qualitative changes in the dynamics of the system examined [14]. Complete examination and analysis of these changes requires the inclusion and study of higher order (non-linear) terms in the relations obtained through the Taylor expansions of equations (12) and (18).

For a general periodic solution, the condition $|\lambda| = 1$ is usually met when $\lambda = 1$ (corresponding to saddle-node, pitchfork or transcritical bifurcation) or $\lambda = -1$ (corresponding to period-doubling bifurcation) and the other eigenvalue is also real and has modulus less than one. The same condition is also satisfied when the eigenvalues of matrix $\Pi$ are complex conjugate with unit modulus (corresponding to Hopf bifurcation). Derivation of conditions between the system parameters leading to $|\lambda| = 1$ are useful, since they establish the stability boundaries of the system. In addition, they provide information on the types of motion expected to arise near the bifurcation values. However, this derivation is a difficult task to accomplish for a general periodic solution because it involves the evaluation of both the determinant and the trace of matrix $\Pi$, which is a product of $k$ matrices. In fact, the main difficulty is coming from developing simple formulae for the trace of matrix $\Pi$, which is possible in some relatively simple forms of periodic solutions (e.g. [8]). Nevertheless, some quite general and useful results can be obtained by evaluating the determinant of matrix $\Pi$ only, as explained next.

First, for a non-sticking interval of the periodic solution, use of equation (16) leads to the formula

$$|R_i| = \frac{1}{\hat{a}_i}\left[(\hat{a}_i - r_{i6})r_{i2} + r_{i3}r_{i5}\right]$$

for the determinant of matrix $R_i$. Substituting the expressions of the parameters $r_{ij}$—which are presented in the previous section—and performing lengthy algebraic manipulations, finally leads to

$$|R_i| = e_i^2 \frac{\hat{a}_{i0}}{\hat{a}_i}, \tag{23}$$

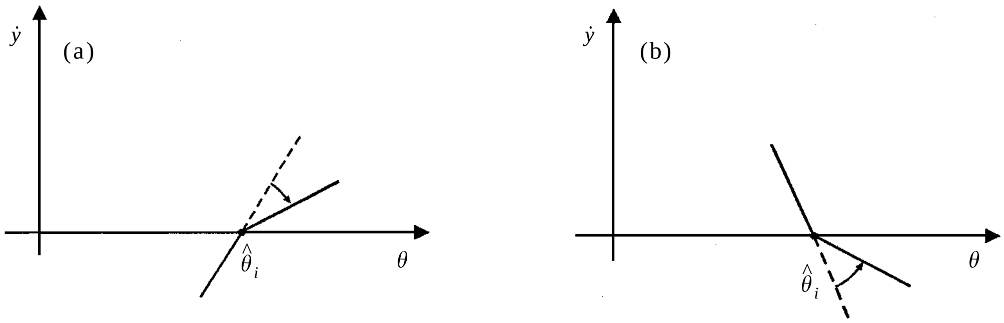


Figure 4. Velocity at a crossing time of a non-sticking periodic motion. (a) Positive acceleration; (b) negative acceleration.

where the symbol $\hat{a}_{i0}$ represents the acceleration at the beginning of the $i$th motion interval, namely

$$\hat{a}_{i0} \equiv a_i(0) = (\delta_i^2 - \eta_i^2)B_i - 2\delta_i\eta_i A_i - P_i \sin\alpha_i .$$

Therefore, if all the intervals of the motion considered are non-sticking, from equations (22) and (23) it is concluded that

$$|\Pi| = |R_k| \cdots |R_1| = A \exp\left(-2 \sum_{i=1}^{k} \delta_i \theta_{ic}\right), \tag{24}$$

with

$$A = \frac{\hat{a}_{1o}\hat{a}_{2o} \cdots \hat{a}_{ko}}{\hat{a}_1\hat{a}_2 \cdots \hat{a}_k} . \tag{25}$$

Next, consider the special but important case of systems with dry friction and constant stiffness (i.e., $\omega_1 = \omega_2 \equiv \hat{\omega}$) [7–9]. With reference to the velocity diagram of Figure 4(a)—drawn at the $i$th crossing point and for positive acceleration at crossing—the equation of motion (3) just before the crossing, when the velocity is negative, can be put in the form $\hat{a}_{i-1} = \hat{z}_i + \hat{\omega}^2\mu$, where $\mu$ is the coefficient of friction and $\hat{z}_i$ represents the sum of the external force and the spring force on the oscillator at the $i$th crossing point. Similarly, the equation of motion just after the $i$th crossing yields $\hat{a}_{io} = \hat{z}_i - \hat{\omega}^2\mu$. Combination of these last two equations leads to the result

$$0 < \hat{a}_{io} = \hat{a}_{i-1} - 2\hat{\omega}^2\mu < \hat{a}_{i-1} \Rightarrow 0 < \frac{\hat{a}_{io}}{\hat{a}_{i-1}} < 1.$$

By proceeding along the same lines, it can be shown that an identical result is obtained for the other possible case of crossing, with negative accelerations at crossing [Figure 4(b)]. Upon taking into account the periodicity conditions and equation (25), this means that in the case examined $0 < A < 1$. Therefore, if the system possesses non-negative damping, it is obvious from equation (24) that $|\Pi| < 1$. This conclusion is similar to a result presented for the class of systems examined in reference [4] and shows that only $\lambda = 1$ or $\lambda = -1$ bifurcations are possible (or, alternatively, that no Hopf bifurcation is possible) in the case considered.

It is noted that the above result holds for systems with constant stiffness and $h_i$, which corresponds to oscillators with variable viscous damping coefficients under constant external load [11, 12]. In this case, there is no discontinuity in the acceleration between subsequent intervals of a periodic motion, since

$$a_i(0) = a_{i-1}(\theta_{(i-1)c}) \Rightarrow \hat{a}_{io} = \hat{a}_{i-1}, \qquad i = 1, \ldots, k.$$

In such a case, it turns out from equation (24) that

$$|\Pi| = \exp\left(-2\sum_{i=1}^{k} \delta_i \theta_{ic}\right)$$

which excludes Hopf bifurcation, provided that all the damping coefficients of the system are positive.

Finally, for periodic motions involving even a single sticking interval, it is obvious from equation (20) that the determinant of the corresponding matrix $R_i$ is equal to zero, which implies that the determinant of $\Pi$ is also equal to zero. Therefore, one eigenvalue of $\Pi$ is equal to zero while the other is also real and equal to the trace of matrix $\Pi$. This implies that in this case only $\lambda = 1$ or $\lambda = -1$ bifurcations are possible excluding, once more, the possibility of a Hopf bifurcation.

## 5. MECHANICAL EXAMPLE

The analytical results presented in the previous sections are directly applicable to mechanical systems involving variable viscous damping and dry friction. In fact, some of these results are supported by stability and bifurcation findings reported in earlier studies for some special periodic motions of oscillators with Coulomb friction and constant viscous damping (e.g., [8]). In the present section, an example system with

$$g(\dot{x}, x) = \begin{cases} c_1\dot{x} + kx, & \dot{x} \geqslant 0 \\ c_2\dot{x} + kx, & \dot{x} < 0 \end{cases},$$

is considered, in order to confirm numerically that similar behaviour is also exhibited by oscillators with variable viscous damping. For instance, equation (1) can represent the equation of motion of a quarter-car model with dual-rate dampers [11]. In this particular example, the stiffness remains constant and if the road profile is harmonic, with amplitude $\hat{s}$ and wavelength $l$, the excitation term can be put in the form

$$f(t) = m\hat{s}\Omega^2 \sin(\Omega t + \varphi)$$

with $\Omega = 2\pi v/l$, where $v$ is the constant horizontal speed of the vehicle.

After introducing the frequency and forcing parameters $\bar{\omega} = \sqrt{k/m}$, $\omega = \Omega/\bar{\omega}$, $p = \hat{s}/x_c$, and applying the normalization presented in section 2, periodic steady state solutions of the mechanical model examined are sought, with form similar to that shown in Figure 5. According to the material presented in section 2, determination of such solutions requires the evaluation of six parameters (the crossing time $\hat{\theta}_1$, the phase $\varphi$ and the four constants $A_1$, $B_1$, $A_2$ and $B_2$ of the homogeneous solutions in the two discrete time intervals of the periodic solution where the damping properties remain constant). As usual, these constants can be determined by imposing an appropriate set of periodicity and matching conditions. In the present case, the corresponding set is

$$\dot{y}_1(0) = \dot{y}_1(\hat{\theta}_1) = \dot{y}_2(0) = \dot{y}_2(\hat{\theta}_2) = 0, \tag{26}$$

$$y_1(0) = y_2(\hat{\theta}_2), \qquad y_1(\hat{\theta}_1) = y_2(0), \tag{27}$$

with $\hat{\theta}_2 = 2\pi n - \hat{\theta}_1$. Originally, direct application of these conditions leads to a system of six transcendental equations for the six unknowns of the problem. However, this system can effectively be reduced to a single transcendental equation. Namely, application of the velocity conditions (26) yields first the four constants of the homogeneous solutions in the form

$$A_n = A_{ns} \sin\varphi + A_{nc} \cos\varphi, \qquad B_n = B_{ns} \sin\varphi + B_{nc} \cos\varphi, \qquad n = 1, 2, \tag{28}$$

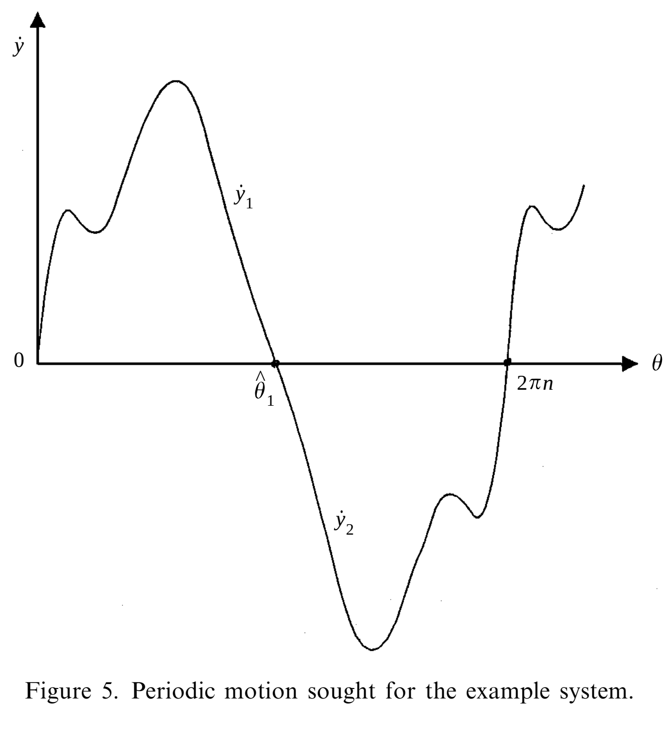


Figure 5. Periodic motion sought for the example system.

where the constants appearing on the right-hand sides of these equations are known functions of the system parameters and the crossing time $\hat{\theta}_1$. Then, upon substitution of the last expressions in the displacement conditions (27) leads to two equations with form

$$\begin{bmatrix} E_{1s} & E_{1c} \\ E_{2s} & E_{2c} \end{bmatrix} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} = 0, \tag{29}$$

where the constants $E_{ns}$ and $E_{nc}$ are known functions of the crossing time $\hat{\theta}_1$. Equation (29) represents a linear homogeneous system in $\sin\varphi$ and $\cos\varphi$. In order for this algebraic system to possess a non-trivial solution, its determinant must vanish. This implies that

$$f(\hat{\theta}_1) \equiv E_{1s}E_{2c} - E_{1c}E_{2s} = 0. \tag{30}$$

Therefore, numerical solution of the last condition determines the crossing time $\hat{\theta}_1$. Then, the process of capturing the periodic solution is completed by evaluating the corresponding phase $\varphi$ by equation (29) and the constants $A_1$, $B_1$, $A_2$ and $B_2$ from equation (28), by simple back substitutions.

Next, numerical results are presented for the example mechanical system. First, Figure 6 shows typical response diagrams for $n = 1$ solutions, obtained for $\hat{p} = p\omega^2 = 1$ and four different combinations of the damping parameters $(\zeta_1, \zeta_2)$. In the cases with $\zeta_1 = \zeta_2$, the system is linear and its behaviour is the expected one. Moreover, it is clear that the most favourable case in terms of maximum response

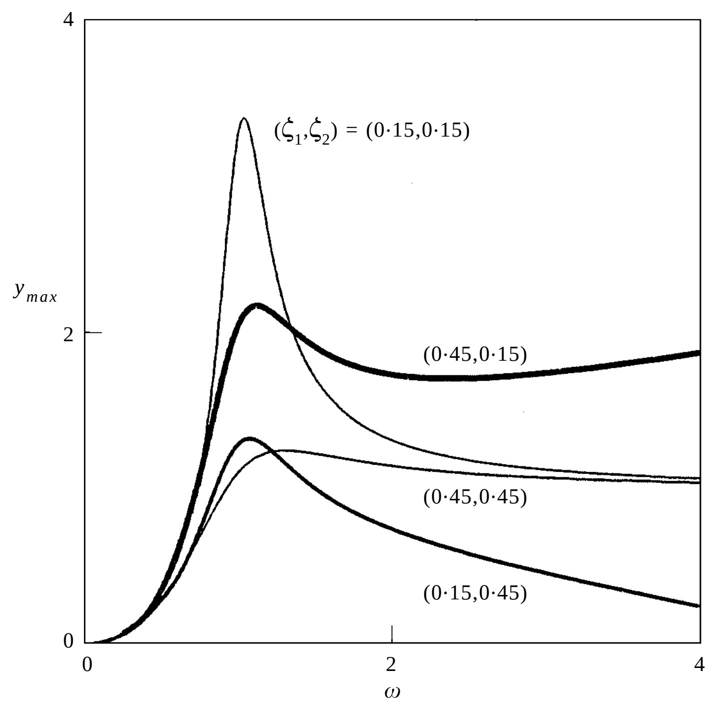


Figure 6. Response diagrams for $\hat{p} = 1$ and different damping combinations.

amplitude is obtained for $(\zeta_1, \zeta_2) = (0{\cdot}15, 0{\cdot}45)$. This corresponds to the optimum case, where the vehicle shock absorbers are dual-rate with about a three-to-one ratio between the rebound and jounce (compression) damping coefficient, according to usual design considerations of vehicle dynamics [11].

All the periodic solutions presented in Figure 6 are harmonic ($n = 1$) and stable. Next, Figure 7 shows response diagrams obtained for $p = 1$, $\zeta_2 = 0{\cdot}55$ and several values of the damping ratio $\zeta_1$. By gradually decreasing the value of $\zeta_1$, apart from an increase of the response amplitude near resonance, unstable solutions are also observed. The first unstable solutions are captured for $\zeta_1 = -0{\cdot}12$, at about

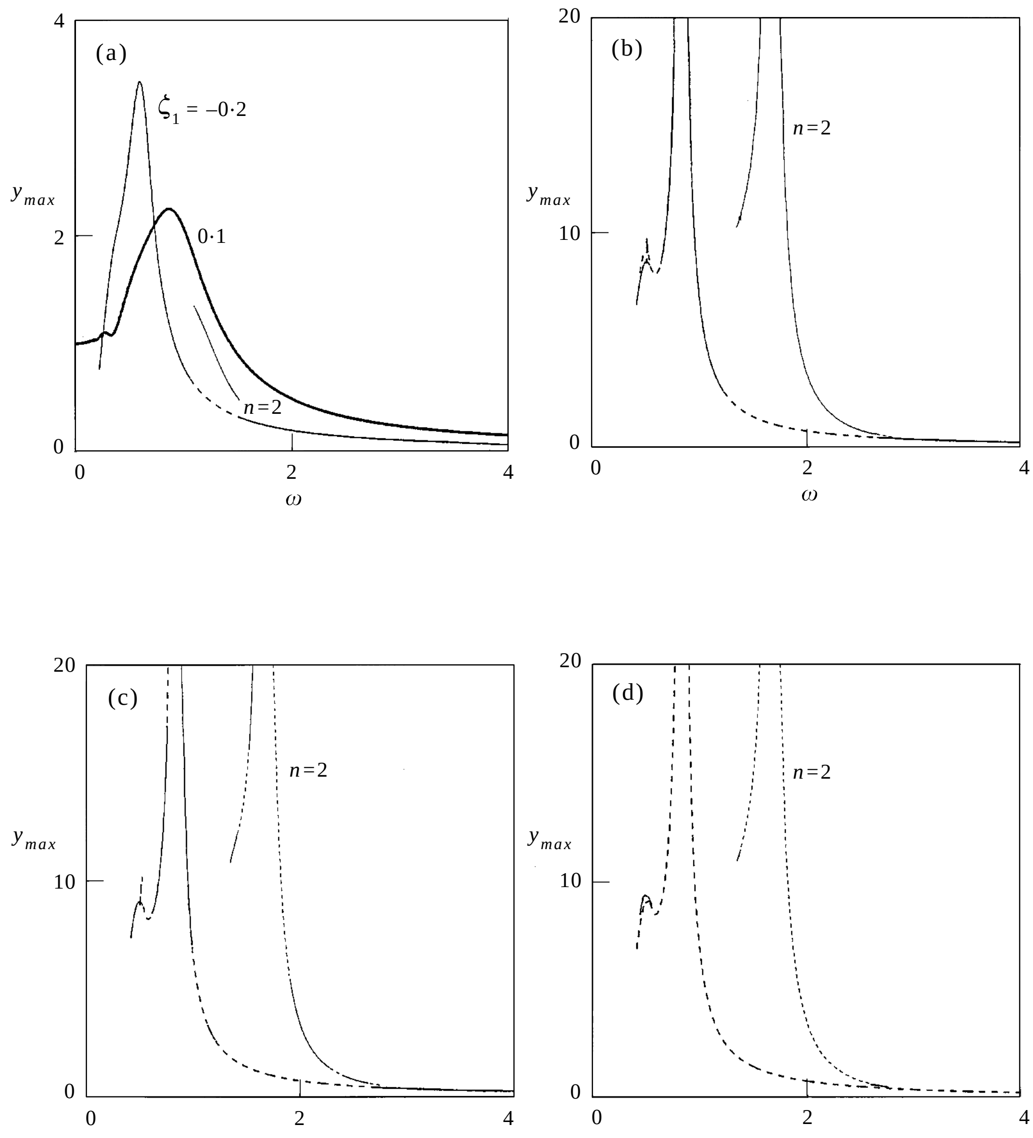


Figure 7. Response diagrams for $p = 1$, $\zeta_2 = 0{\cdot}55$ and (a) $\zeta_1 = 0{\cdot}1$ and $-0{\cdot}2$, (b) $\zeta_1 = -0{\cdot}54$, (c) $\zeta_1 = -0{\cdot}55$ and (d) $\zeta_1 = 0{\cdot}56$.

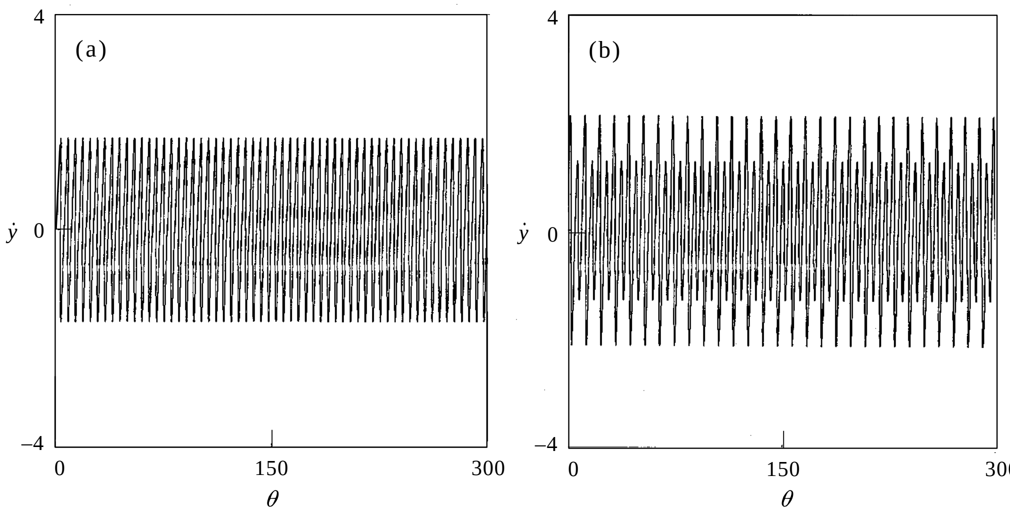


Figure 8. Numerical verification of a period-doubling bifurcation. (a) $\omega = 1{\cdot}22$; (b) $\omega = 1{\cdot}23$.

$\omega = 1{\cdot}87$. Further decrease in the value of $\zeta_1$ leads to a gradual enlargement of these solution branches, which are represented by broken lines in Figure 7. More specifically, these branches of unstable solutions are generated through period doubling bifurcations ($\lambda = -1$) at both of their ends. As a result, this gives rise to new branches of $n = 2$ periodic solutions, which are also shown in Figure 7.

For the parameter combinations leading to the response diagrams sequence presented in Figure 7, the most complex situation arose for the system with $\zeta_1 = -0{\cdot}55$ [Figure 7(c)]. In that case, a new branch of unstable periodic solutions is captured between $\omega = 0{\cdot}99$ and $1{\cdot}14$. This branch is generated via Hopf bifurcation (complex $\lambda$ with $|\lambda| = 1$) occurring at both of its ends. Moreover, for $\omega > 2{\cdot}67$ the stability properties of the periodic solutions are continuously

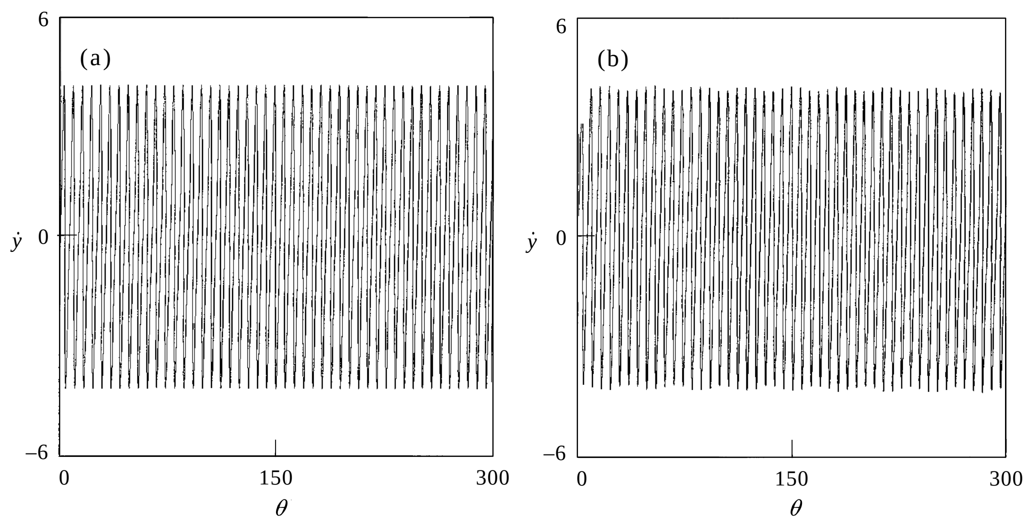


Figure 9. Numerical verification of a Hopf bifurcation. (a) $\omega = 0{\cdot}99$; (b) $\omega = 0{\cdot}992$.

interchanging over relatively small forcing frequency intervals. Finally, by decreasing the value of $\zeta_1$ below $-0{\cdot}55$, all the solutions occupying the branch lying at the right of the primary resonance become unstable [Figure 7(d)].

In the last part of this section, the stability and bifurcation results associated with the solutions of the previous figure are verified by numerical integration of the equation of motion (3). First, Figure 8 presents the periodic solutions obtained for $\zeta_1 = 0{\cdot}54$, at $\omega = 1{\cdot}22$ and $1{\cdot}23$, respectively. For this system, the analysis predicts a period doubling bifurcation at about $\omega = 1{\cdot}225$. As a consequence, the stable harmonic solution at $\omega = 1{\cdot}22$ becomes unstable and gives its place to an $n = 2$ periodic solution at $\omega = 1{\cdot}23$, as expected. Likewise, Figure 9 shows the solutions obtained by direct integration of the equation of motion (3) for $\zeta_1 = -0{\cdot}55$, at $\omega = 0{\cdot}99$ and $0{\cdot}992$, respectively. For this system, the originally stable harmonic solution loses stability via a Hopf bifurcation and is replaced by a quasiperiodic solution.

## 6. SYNOPSIS AND CONCLUSIONS

A stability analysis has been developed for periodic motions of a general class of single degree of freedom oscillators with piecewise linear characteristics. More specifically, the oscillators examined possess different viscous and constant damping parameters for positive and for negative velocity values. These oscillators are strongly non-linear and their response involves an arbitrary number of motion intervals. In addition, they may exhibit several sticking intervals over a response period. The analysis was based on the exact solution form of the response within each time interval included between two consecutive zero velocity states. The asymptotic stability properties of a periodic solution were identified through the construction of an appropriate matrix, which determines the propagation of small perturbations in the periodic solution over a response period. Eventually, simple analytic expressions were derived for the determinant of that matrix and it was shown that no Hopf bifurcations are possible in two special but important cases. Namely, when the periodic motion involves at least one sticking interval or when the motion is non-sticking and the damping coefficients are positive. Finally, some of these analytical predictions were verified by examining the response of an example mechanical oscillator with piecewise linear damping. In closing, it is noted that the present analysis can be extended to cover wider classes of dynamical systems, like mechanical oscillators with similar damping characteristics and many degrees of freedom.

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