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FREE VIBRATION OF ELASTICALLY SUPPORTED THIN CYLINDERS INCLUDING GYROSCOPIC EFFECTS

P. W. LOVEDAY

Sensor Systems, Division of Materials Science and Technology, CSIR, P.O. Box 395, Pretoria 0001, South Africa

AND

C. A. ROGERS

College of Engineering, University of South Carolina, Columbia, SC 29208, U.S.A.

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The free vibration of thin cylinders with elastic boundary conditions was analysed by application of the exact solution of the Flügge shell theory equations of motion. The elastic boundaries were represented by distributed linear springs. By varying the eight spring constants any elastic or ideal boundary conditions can be simulated. The effect of flexibility in the boundary conditions for a cylinder supported at both ends and a cylinder supported at one end was determined for the lowest frequency vibration mode with two circumferential wavelengths as this mode is used in vibratory gyroscopes. It was found that the tangential stiffness has the greatest effect on the natural frequency of the cylinder supported at both ends while the axial boundary stiffness has the greatest influence on the natural frequency of the cylinder supported at one end. The effect of low rotation rates was also determined for these boundary conditions. It was found that the tangential stiffness of the boundaries has the largest effect on the sensitivity to rotation. The lowest frequency vibration mode with four circumferential wavelengths did not show the same trends. A range of boundary stiffnesses for which analysis as an elastic boundary is essential was determined. Boundary stiffnesses out of this range may be regarded as zero (free) or infinite (rigid).

1. INTRODUCTION

The vibration of thin elastic shells has been studied by many researchers. The results of many of these studies have been summarised by Leissa [1] and Blevins [2]. The literature contains numerous analyses of thin cylindrical shells with ideal boundary conditions classed, for example, as clamped, free, simply supported with axial constraint and simply supported without axial constraint. In reality, the perfect clamped boundary condition cannot be achieved as there will always be some flexibility in the support. The effects of this flexibility are investigated in this paper.

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The effect of rotation on the vibration of a cylinder was first analysed by Bryan [3] in 1890. Bryan showed that rotation causes the nodes of a standing wave pattern to rotate relative to the cylinder. The vibrating pattern lags behind the rotation of the cylinder. This effect, which is referred to as the Bryan effect, is used today in a class of angular rotation rate sensors which are based on vibrating cylinders [4–7]. The influence of boundary conditions on the natural frequencies and sensitivity to rotation is of importance in the design of these sensors.

Three methods are generally applied to the analysis of thin cylindrical shell vibration. Exact solution of waves propagating in infinite, hollow cylinders, based on the three-dimensional theory of elasticity, was described by Armenàkas *et al*. [8]. This solution is also valid for simply supported shells and serves as a benchmark against which results from analyses based on shell theories can be evaluated. Approximate analyses based on various shell theories have been performed using the Rayleigh–Ritz method [1, 2]. Exact solution of the Flügge shell theory equations of motion has been performed by various authors [9–12] in which different ideal boundary conditions were analysed. The solutions achieved by this approach are accurate within the limits of the shell theory used.

In this work the general analysis procedure presented by Warburton [10] is adopted and extended to include elastic boundary conditions and rotation of the cylindrical shell. The elastic boundary conditions are represented by distributed springs along the edges of the cylinder. By varying the stiffness coefficients of these springs it is possible to represent any of the ideal boundary conditions and also to investigate the effect of departures from the ideal conditions. Using this method the effect of elastic boundary conditions on the free vibrations of cylindrical shells can be quantified.

2. THEORETICAL FORMULATION

The derivation presented here follows that of Warburton [10] but includes the elastic supports in the potential energy and the rotation rate in the kinetic energy. The effect of centrifugal forces producing a deformed equilibrium position with stresses which contribute to the potential energy has been neglected. Analysis of high rotation rates requires the calculation of the stresses in the equilibrium position which is beyond the scope of the present paper. Centrifugal forces arising from the kinetic energy which are proportional to the square of the rotation rate, have therefore also been neglected. The analysis is therefore only valid for rotation rates well below the natural frequencies of interest.

2.1. KINETIC AND POTENTIAL ENERGY EXPRESSIONS

The kinetic energy of the rotating cylinder may be written as follows (nomenclature is listed in Appendix B):

$$T = \frac{\rho h a}{2} \int_0^{2\pi} \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t} - \Omega(a + w)\right)^2 + \left(\frac{\partial w}{\partial t} + \Omega v\right)^2 \mathrm{d}x \, \mathrm{d}\phi. \qquad (1)$$

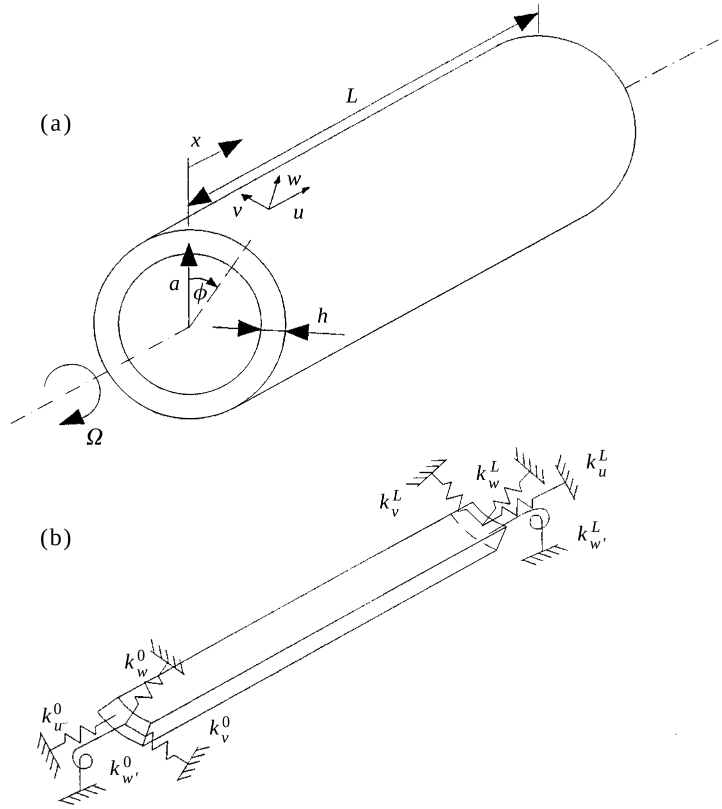

Figure 1. (a) Definition of co-ordinates and dimensions. (b) Elastic boundary conditions shown on a segment of the cylinder.

The potential energy of the system is the sum of the strain energy of the cylindrical shell and the strain energy stored in the elastic boundaries. Flügge shell theory is used, therefore four distributed springs are required to represent the elastic boundary conditions at each end of the cylinder as shown in Figure 1.

The potential energy may be written as follows (note that the radial displacement is defined as positive outwards while in [10] it was positive inwards):

$$S = \frac{Eah}{2(1-\nu^2)} \int_0^{2\pi} \int_0^L \left[\left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{a^2} \left(\frac{\partial v}{\partial \phi} \right)^2 + \frac{w^2}{a^2} + \frac{2w}{a^2} \frac{\partial v}{\partial \phi} + \frac{2\nu}{a} \frac{\partial u}{\partial x} \frac{\partial v}{\partial \phi} \right.$$

$$+ \frac{2\nu}{a} \frac{\partial u}{\partial x} w + \frac{1}{2a^2} (1-\nu) \left(\frac{\partial u}{\partial \phi} \right)^2 + \tfrac{1}{2}(1-\nu) \left(\frac{\partial v}{\partial x} \right)^2 + \frac{1}{a} (1-\nu) \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial x}$$

$$+ \beta \left\{ a^2 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{a^2} \left(\frac{\partial^2 w}{\partial \phi^2} \right)^2 + \frac{w^2}{a^2} + \frac{2w}{a^2} \frac{\partial^2 w}{\partial \phi^2} + 2\nu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial \phi^2} \right.$$

$$+2(1-v)\left(\frac{\partial^2 w}{\partial x\,\partial\phi}\right)^2+\frac{1}{2a^2}(1-v)\left(\frac{\partial u}{\partial\phi}\right)^2+\frac{1}{a}(1-v)\frac{\partial u}{\partial\phi}\frac{\partial^2 w}{\partial x\,\partial\phi}$$

$$-2a\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\tfrac{3}{2}(1-v)\left(\frac{\partial v}{\partial x}\right)^2-2v\frac{\partial v}{\partial\phi}\frac{\partial^2 w}{\partial x^2}-3(1-v)\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x\,\partial\phi}\Bigg\}\Bigg]\mathrm{d}\phi\,\mathrm{d}x$$

$$+\tfrac{1}{2}a\int_0^{2\pi}\Bigg[\Bigg[k_u^0u^2+k_v^0v^2+k_w^0w^2+k_{w'}^0\left(\frac{\partial w}{\partial x}\right)^2\Bigg]_{x=0}$$

$$+\Bigg[k_u^Lu^2+k_v^Lv^2+k_w^Lw^2+k_{w'}^L\left(\frac{\partial w}{\partial x}\right)^2\Bigg]_{x=L}\Bigg]\mathrm{d}\phi. \tag{2}$$

2.2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The following equations of motion and boundary conditions were derived by application of Hamilton's principle using the above expressions for the potential and kinetic energies. Terms proportional to the square of the rotation rate were omitted while the Coriolis forces were retained. The equations of motion in the axial, tangential and radial directions are presented in equation (3).

$$a\frac{\partial^2 u}{\partial x^2}+\frac{(1-v)}{2a}\frac{\partial^2 u}{\partial\phi^2}+\frac{(1+v)}{2}\frac{\partial^2 v}{\partial x\,\partial\phi}+v\frac{\partial w}{\partial x}$$

$$+\frac{h^2}{12a^2}\left(\frac{(1-v)}{2a}\frac{\partial^2 u}{\partial\phi^2}-a^2\frac{\partial^3 w}{\partial x^3}+\frac{(1-v)}{2}\frac{\partial^3 w}{\partial x\,\partial^2\phi}\right)=\frac{\rho a(1-v^2)}{E}\left[\frac{\partial^2 u}{\partial t^2}\right]$$

$$\frac{(1+v)}{2}\frac{\partial^2 u}{\partial x\,\partial\phi}+\frac{1}{a}\frac{\partial^2 v}{\partial\phi^2}+\frac{a(1-v)}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1}{a}\frac{\partial w}{\partial\phi}$$

$$+\frac{h^2}{12a^2}\left[\frac{3a(1-v)}{2}\frac{\partial^2 v}{\partial x^2}-\frac{a(3-v)}{2}\frac{\partial^3 w}{\partial x^2\,\partial\phi}\right]=\frac{\rho a(1-v^2)}{E}\left[\frac{\partial^2 v}{\partial t^2}-2\Omega\frac{\partial w}{\partial t}\right]$$

$$v\frac{\partial u}{\partial x}+\frac{1}{a}\frac{\partial v}{\partial\phi}+\frac{1}{a}w+\frac{h^2}{12a^2}\left[\frac{(1-v)}{2}\frac{\partial^3 u}{\partial x\,\partial\phi^2}-a^2\frac{\partial^3 u}{\partial x^3}-\frac{a(3-v)}{2}\frac{\partial^3 v}{\partial x^2\,\partial\phi}+a^3\frac{\partial^4 w}{\partial x^4}\right.$$

$$\left.+2a\frac{\partial^4 w}{\partial x^2\,\partial^2\phi}+\frac{1}{a}\frac{\partial^4 w}{\partial\phi^4}+\frac{2}{a}\frac{\partial^2 w}{\partial\phi^2}+\frac{1}{a}w\right]=\frac{\rho a(1-v^2)}{E}\left[\frac{\partial^2 w}{\partial t^2}+2\Omega\frac{\partial v}{\partial t}\right]. \tag{3}$$

At $x = 0$ the boundary conditions are:

$$\frac{Eh}{(1-v^2)}\left[\frac{\partial u}{\partial x} + \frac{v}{a}\left(\frac{\partial v}{\partial \phi} + w\right) - \beta a\frac{\partial^2 w}{\partial x^2}\right] - k_u^0 u = 0$$

$$\frac{Eh}{2(1+v)}\left[\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \phi} + 3\beta\left(\frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x\,\partial \phi}\right)\right] - k_v^0 v = 0$$

$$\frac{Eh^3}{12(1-v^2)}\left[\frac{\partial^3 w}{\partial x^3} - \frac{1}{a}\frac{\partial^2 u}{\partial x^2} + \frac{(1-v)}{2a^3}\frac{\partial^2 u}{\partial \phi^2} + \frac{(2-v)}{a^2}\frac{\partial^3 w}{\partial x\,\partial \phi^2} - \frac{(3-v)}{2a^3}\frac{\partial^2 v}{\partial x\,\partial \phi}\right]$$

$$+ k_w^0 w = 0$$

$$\frac{Eh^3}{12(1-v^2)}\left[-\frac{\partial^2 w}{\partial x^2} - \frac{v}{a^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{a}\frac{\partial u}{\partial x} + \frac{v}{a^2}\frac{\partial v}{\partial \phi}\right] + k_{w'}^0\frac{\partial w}{\partial x} = 0. \qquad (4a)$$

At $x = L$ the boundary conditions are:

$$\frac{Eh}{(1-v^2)}\left[\frac{\partial u}{\partial x} + \frac{v}{a}\left(\frac{\partial v}{\partial \phi} + w\right) - \beta a\frac{\partial^2 w}{\partial x^2}\right] + k_u^L u = 0$$

$$\frac{Eh}{2(1+v)}\left[\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \phi} + 3\beta\left(\frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x\,\partial \phi}\right)\right] + k_v^L v = 0$$

$$\frac{Eh^3}{12(1-v^2)}\left[\frac{\partial^3 w}{\partial x^3} - \frac{1}{a}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2a^3}(1-v)\frac{\partial^2 u}{\partial \phi^2} + \frac{(2-v)}{a^2}\frac{\partial^3 w}{\partial x\,\partial \phi^2} - \frac{(3-v)}{2a^3}\frac{\partial^2 v}{\partial x\,\partial \phi}\right]$$

$$- k_w^L w = 0$$

$$\frac{Eh^3}{12(1-v^2)}\left[-\frac{\partial^2 w}{\partial x^2} - \frac{v}{a^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{1}{a}\frac{\partial u}{\partial x} + \frac{v}{a^2}\frac{\partial v}{\partial \phi}\right] - k_{w'}^L\frac{\partial w}{\partial x} = 0. \qquad (4b)$$

The boundary conditions at the ends of the cylinder provide the conditions for force equilibrium. The boundary conditions at each end of the cylinder are identical except for the sign difference in the distributed spring forces. Note that in the second boundary condition at each end of the cylinder there is a factor 3 in the term dependent on β. This factor was not present in [10] as the actual shear force instead of the effective shear force was used in that work [11]. This difference has only small influence on the resulting natural frequencies.

2.3. SOLUTION OF THE EQUATIONS OF MOTION

The general solution, used by Warburton, represents a standing wave and is applicable for the non-rotating cylinder. For the rotation cylinder it is necessary to use the more general travelling wave solution listed below.

$$u = U_0\, e^{\alpha x/a} \cos(n\phi + \omega t)$$

$$v = V_0\, e^{\alpha x/a} \sin(n\phi + \omega t)$$

$$w = W_0\, e^{\alpha x/a} \cos(n\phi + \omega t). \tag{5}$$

Substitution of this general solution into the equations of motion (3) yields the following system of equations:

$$\left[\alpha^2 - \frac{(1-\nu)}{2} n^2 - \frac{(1-\nu)}{2} n^2\beta + \Delta^2\right] U_0 + \left[\frac{(1+\nu)}{2}\alpha n\right] V_0 + \left[\nu\alpha + \beta\left\{-\alpha^3 - \frac{(1-\nu)}{2}\alpha n^2\right\}\right] W_0 = 0$$

$$\left[-\frac{(1+\nu)}{2}\alpha n\right] U_0 + \left[-n^2 + \frac{(1-\nu)}{2}\alpha^2 + \beta\frac{3(1-\nu)}{2}\alpha^2 + \Delta^2\right] V_0 + \left[-n + \beta\frac{(3-\nu)}{2}\alpha^2 n - \frac{2\Omega}{\omega}\Delta^2\right] W_0 = 0$$

$$\left[\nu\alpha + \beta\left\{-\alpha^3 - \frac{(1-\nu)}{2}\alpha n^2\right\}\right] U_0 + \left[n - \beta\frac{(3-\nu)}{2}\alpha^2 n - \frac{2\Omega}{\omega}\Delta^2\right] V_0 + [1 + \beta\{\alpha^4 - 2\alpha^2 n^2 + n^4 - 2n^2 + 1\} + \Delta^2] W_0 = 0 \tag{6}$$

where, the frequency factor, $\Delta = \omega a\sqrt{\rho(1-\nu^2)/E}$.

Non-trivial solutions of this system of equations are found by equating the determinant to zero. For a given cylinder, if the frequency factor (Δ), the rotation rate (Ω) and the number of circumferential waves (n) are specified the determinant can be written as a quartic in α^2. In the calculation of the determinant, squares and higher powers of β were neglected. The roots of this equation yield the values of α that satisfy the equations of motion and are the admissible axial wave numbers.

The frequency factor for the infinite cylinder is given by $\Delta_R^2 = \beta\{[n^2(n^2-1)^2]/[(n^2+1)]\}$. At frequencies greater than the natural frequency of an infinite cylinder the roots have the form $\pm\alpha_1$, $\pm i\gamma_2$, $\pm(p \pm iq)$, where α_1, γ_2, p and q are real and positive [10]. Once the roots have been determined it is possible to use equation (6) to calculate the amplitude ratios as was done in reference [10]. The equations required for this procedure, including the gyroscopic terms, are included in Appendix A. The displacement functions can then be written as follows:

$$W(x) = C_1 \cosh\frac{\alpha_1 x}{a} + C_2 \sinh\frac{\alpha_1 x}{a} + C_3 \cos\frac{\gamma_2 x}{a} + C_4 \sin\frac{\gamma_2 x}{a} + e^{px/a}\left(C_5 \cos\frac{qx}{a} + C_6 \sin\frac{qx}{a}\right) + e^{-px/a}\left(C_7 \cos\frac{qx}{a} + C_8 \sin\frac{qx}{a}\right) \quad (7a)$$

$$V(x) = A_1 C_1 \cosh\frac{\alpha_1 x}{a} + A_1 C_2 \sinh\frac{\alpha_1 x}{a} + A_3 C_3 \cos\frac{\gamma_2 x}{a} + A_3 C_4 \sin\frac{\gamma_2 x}{a} + e^{px/a}\left[(A_5 C_5 + A_6 C_6)\cos\frac{qx}{a} + (A_5 C_6 - A_6 C_5)\sin\frac{qx}{a}\right] + e^{-px/a}\left[(A_5 C_7 - A_6 C_8)\cos\frac{qx}{a} + (A_5 C_8 + A_6 C_7)\sin\frac{qx}{a}\right] \quad (7b)$$

$$U(x) = A_2 C_2 \cosh\frac{\alpha_1 x}{a} + A_2 C_1 \sinh\frac{\alpha_1 x}{a} + A_4 C_4 \cos\frac{\gamma_2 x}{a} - A_4 C_3 \sin\frac{\gamma_2 x}{a} + e^{px/a}\left[(A_7 C_5 + A_8 C_6)\cos\frac{qx}{a} + (A_7 C_6 - A_8 C_5)\sin\frac{qx}{a}\right] + e^{-px/a}\left[(-A_7 C_7 + A_8 C_8)\cos\frac{qx}{a} + (-A_7 C_8 - A_8 C_7)\sin\frac{qx}{a}\right] \quad (7c)$$

where the amplitude ratios ($A_1, \ldots, A_8$) are real constants, defined in Appendix A, and the unknown coefficients ($C_1, \ldots, C_8$) depend on the boundary conditions.

Substitution of the displacement functions (7) into the boundary conditions (4) produces a homogeneous system of equations in the unknown displacement coefficients ($C_1, \ldots, C_8$). This system of equations may be written in matrix form and the determinant of the matrix must once again be zero for non-trivial solutions. For this determinant to be zero the correct frequency must be chosen at the beginning of the procedure and it is necessary to iteratively search for the frequencies which produce non-trivial solutions. These frequencies correspond to the natural frequencies of the cylinder for the selected number of circumferential waves. The index m is commonly used to number the axial modes starting from the lowest frequency mode ($m = 1$). During each iteration the roots of the characteristic equation, given by the determinant of equation (6), and the amplitude ratios, defined in Appendix A, have to be computed.

2.4. THE INFLUENCE OF ROTATION

If the cylinder is not rotating and a travelling wave solution with frequency ω is found there will also be a solution with frequency $-\omega$ because the travelling wave can travel in either direction around the cylinder. When a rotation rate is applied the positive and negative travelling waves no longer have the same magnitude of frequency. For low rotation rates the two travelling wave solutions have almost identical amplitude ratios and may be combined and represented as a "standing wave" which rotates relative to the cylinder [13]. The rotation rate of the "standing wave" may be related to the positive and negative frequencies (ω_p and ω_q) by considering the combination of the two travelling waves as follows [13]:

$$\cos(n\phi + \omega_p t) + \cos(n\phi + \omega_q t) = 2\cos[n(\phi + \tfrac{1}{2}(\omega_p + \omega_q)t/n)]\cos\tfrac{1}{2}(\omega_p - \omega_q)t. \tag{8}$$

This equation shows that the two travelling waves may be represented as a "standing wave" with frequency $\frac{1}{2}(\omega_p - \omega_q)$ which rotates relative to the cylinder at a rate of $\frac{1}{2}(\omega_p + \omega_q)/n$. The ratio of this precession of the "standing wave" (relative to the cylinder) to the angular rotation rate applied to the cylinder is called the Bryan Factor and gives a measure of the sensitivity of the mode of vibration of the cylinder to rotation. The Bryan Factor is defined as follows:

$$BF = \frac{1}{2n\Omega}(\omega_p + w_q). \tag{9}$$

Bryan showed that for rings and infinite cylinders performing inextensional oscillations this factor is equal to $2/(n^2 + 1)$. Therefore, the vibrating pattern will lag the applied rotation rate, Ω by $0{\cdot}4\Omega$, $0{\cdot}2\Omega$ and $0{\cdot}118\Omega$ for $n = 2$, 3 and 4 respectively. The decrease in Bryan Factor with increasing number of circumferential waves is one reason that the $n = 2$ mode is generally selected for use in vibratory gyroscopes.

For the low rotation rates being considered it is possible to calculate the Bryan Factor from the displacement functions of the non-rotating cylinder as follows [14]:

$$BF = \frac{2}{n}\frac{\displaystyle\int_0^L v(x)w(x)\,\mathrm{d}x}{\displaystyle\int_0^L (u(x)^2 + v(x)^2 + w(x)^2)\,\mathrm{d}x}. \tag{10}$$

For low rotation rates equations (9) and (10) give the same result.

3. RESULTS

To verify the analysis, various non-rotating cylinders with idealised boundary conditions were analysed and compared to published results. These results, for a

TABLE 1

Comparison with published results

L/a	h/a	n	m	Boundary conditions	$\Delta_{\text{literature}}$	Δ
1·14	0·05	2	1	Clamped–free	0·308[11]	0·308
1·37	0·05	4	1	Clamped–free	0·245[11]	0·245
1	0·05	4	1	Simply supported–simply supported	0·492[8]	0·496
1	0·10	4	1	Simply supported–simply supported	0·752[8]	0·779
1	0·05	2	1	Simply supported–simply supported	0·675[8]	0·677
1	0·10	2	1	Simply supported–simply supported	0·738[8]	0·747

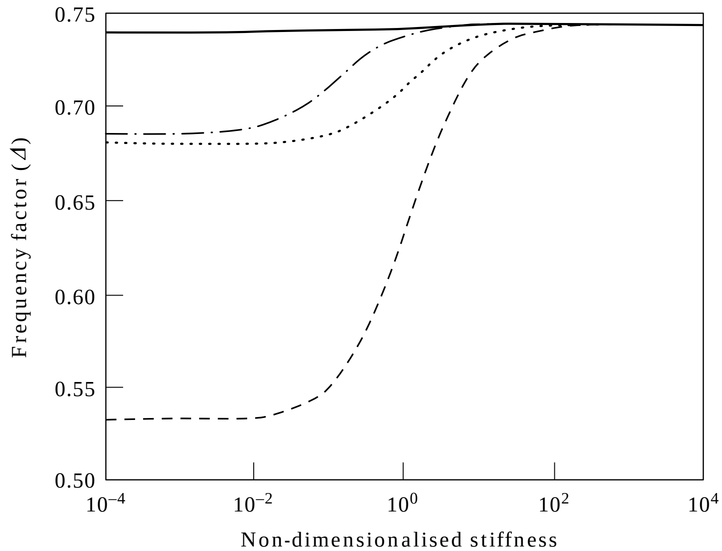

Figure 2. Influence of boundary stiffness on frequency factor of a steel cylinder supported at both ends. ——, varying k_u^*; - - -, varying k_v^*; - · - · -, varying k_w^*; . . . ., varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}05$, $a = 6{\cdot}25$ mm, $n = 2$.)

selection of geometries and circumferential wave numbers, are isted in Table 1. The results for the clamped–free boundary conditions agree with the results presented in reference [11]. This indicates that the numerical implementation of the algorithm is correct. The comparison with results from the exact solution of the three

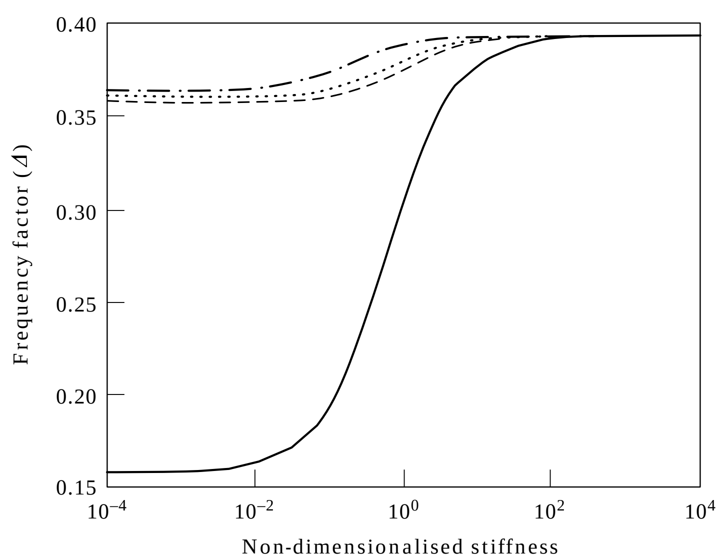

Figure 3. Influence of boundary stiffness on frequency factor of a steel cylinder supported at one end. ——, varying k_u^*; - - -, varying k_v^*; - · - · -, varying k_w^*; . . . ., varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}1$, $a = 6{\cdot}25$ mm, $n = 2$.)

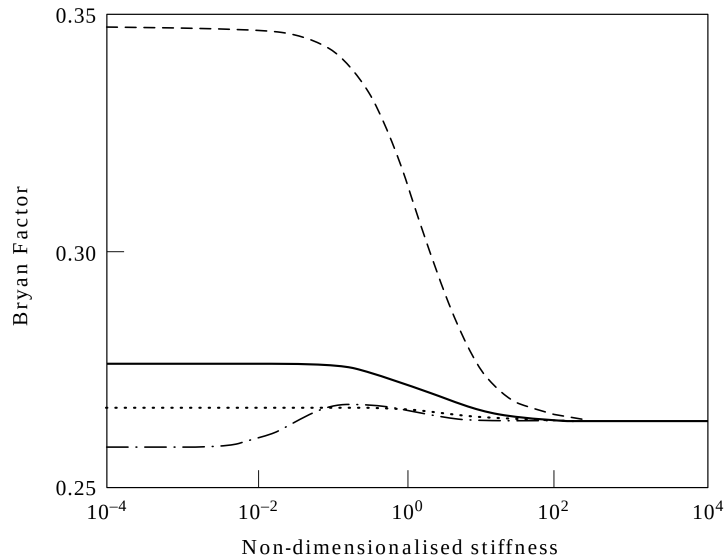

Figure 4. Influence of boundary stiffness on Bryan Factor of a steel cylinder supported at both ends. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}05$, $a = 6{\cdot}25$ mm, $n = 2$.)

dimensional elasticity problem [8] shows the accuracy of the method, based on Flügge shell theory, for calculating natural frequencies of simply supported cylinders without axial constraint. It is noted that the accuracy of the method decreases as the thickness to radius ratio increases and also as the number of waves around the circumference increases.

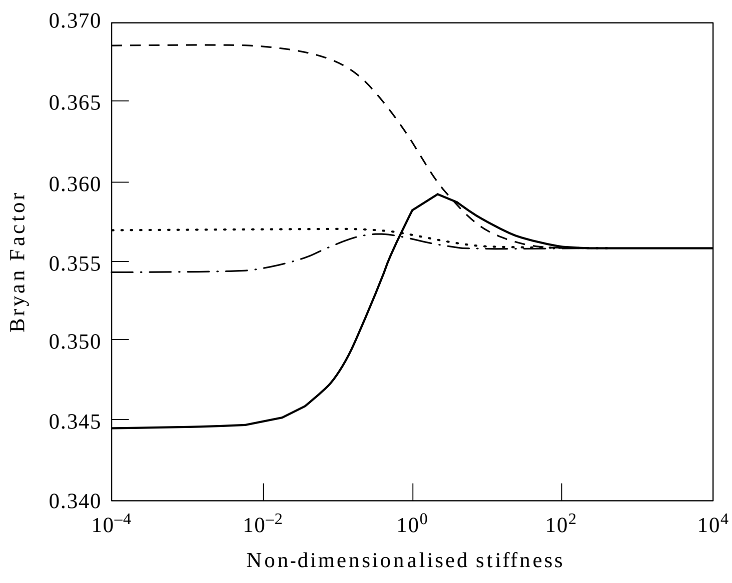

Figure 5. Influence of boundary stiffness on Bryan Factor of a steel cylinder supported at one end. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}1$, $a = 6{\cdot}25$ mm, $n = 2$.)

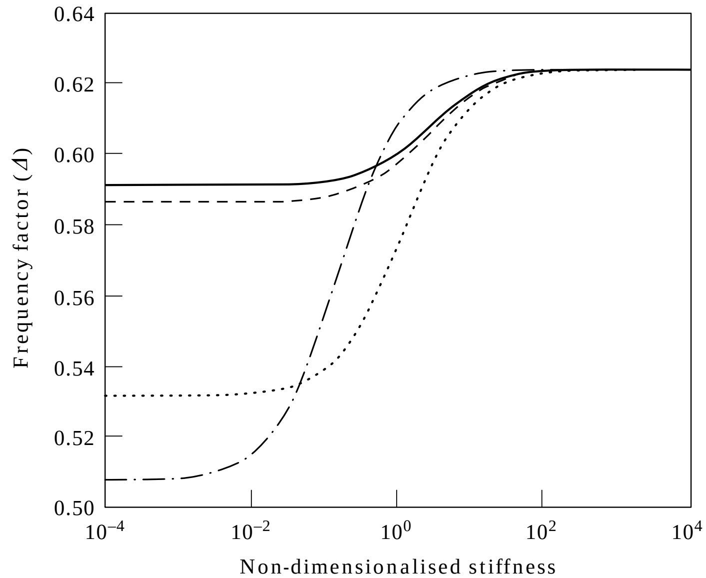

Figure 6. Influence of boundary stiffness on frequency factor of a steel cylinder supported at both ends. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}05$, $a = 6{\cdot}25$ mm, $n = 4$.)

The influence of boundary stiffnesses, on the natural frequencies, for the case of two circumferential waves was investigated as this vibration mode is generally used in vibrating cylinder gyroscopes. Firstly, a cylinder supported at both ends, with two circumferential waves, was considered. The boundary stiffness in one

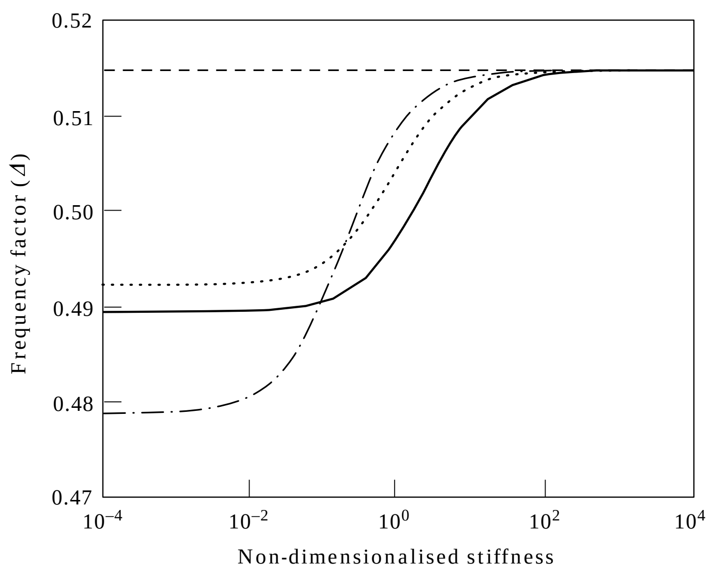

Figure 7. Influence of boundary stiffness on frequency factor of a steel cylinder supported at one end. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a = 1$, $h/a = 0{\cdot}1$, $a = 6{\cdot}25$ mm, $n = 4$.)

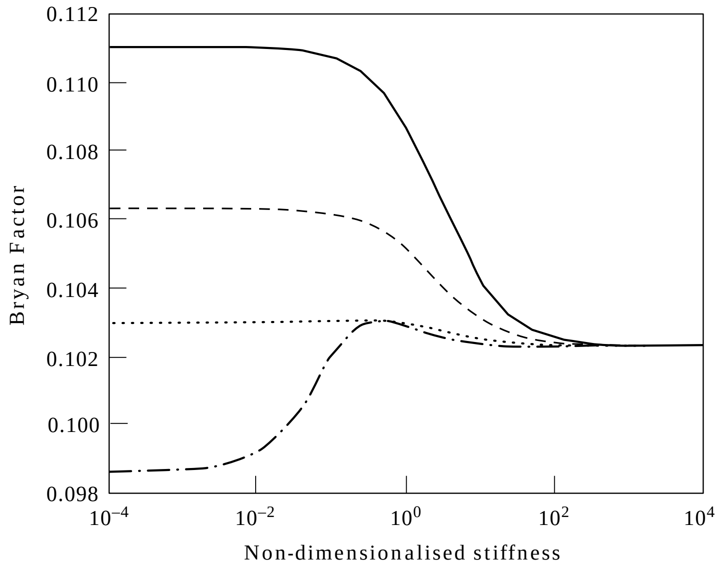

Figure 8. Influence of boundary stiffness on Bryan Factor of a steel cylinder supported at both ends. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a=1$, $h/a = 0{\cdot}05$, $a = 6{\cdot}25$ mm, $n = 4$.)

direction was then varied at both ends while the other stiffnesses had high values representing rigid boundaries. The natural frequency was calculated for each combination of boundary stiffnesses. For example, the value of k_u (the axial stiffness) was varied at both ends of the cylinder while the other stiffnesses were

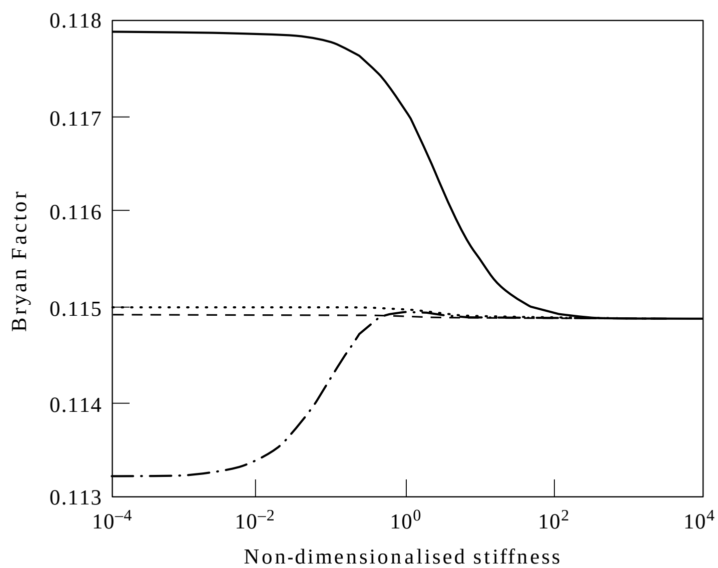

Figure 9. Influence of boundary stiffness on Bryan Factor of a steel cylinder supported at one end. ——, varying k_u^*; - - - , varying k_v^*; - · - · - , varying k_w^*; . . . . , varying $k_{w'}^*$. ($L/a=1$, $h/a = 0{\cdot}1$, $a = 6{\cdot}25$ mm, $n = 4$.)

set to a large constant value. Secondly, a cylinder supported at one end and free at the other end was considered. At the free end the spring constants were set to zero while at the supported end one spring constant was varied while the others were kept high. The spring constants may be non-dimensionalised as follows:

$$k_u^* = \frac{k_u}{Eh/L}; \quad k_v^* = \frac{k_v}{Eh/L}; \quad k_w^* = \frac{k_w}{Eh/L}; \quad k_{w'}^* = \frac{k_{w'}}{Eh^3/L}.$$

The results of the analysis of the cylinder with equal boundary conditions at either end are shown in Figure 2. In the figure the variation of the frequency factor with boundary stiffness for the case of two waves around the circumference is illustrated. It is seen that the tangential stiffness has a large effect on the natural frequency. The radial and radial bending stiffnesses have a smaller influence while reducing the axial stiffness has a negligible effect. The case of a cylinder supported at one end and free at the other is illustrated in Figure 3. Here it is found that the axial stiffness has an extremely large influence on natural frequency while the other stiffnesses have smaller influence. Removing the axial constraint reduced the natural frequency of this mode by 75%. In all cases increasing the boundary stiffnesses increased the natural frequency as expected.

The influence of boundary stiffness on Bryan Factor is presented in Figures 4 and 5 for the cylinder supported at both ends and the cylinder supported only at one end respectively. It is evident that the tangential stiffness tends to decrease the Bryan Factor significantly in the cylinder supported at both ends and to a lesser extent in the cylinder supported at one end. The axial restraint increased the Bryan Factor of the cylinder supported at one end but decreased the Bryan Factor of the cylinder supported at both ends. In general the Bryan Factor is greater and also less sensitive to boundary stiffness variations when the cylinder is supported at only one end. The Bryan Factor of the short cylinder ($L/a = 1$) clamped at one end was found to be 0·3556 which is only 11% lower than the Bryan Factor of a ring or infinite cylinder.

Equation (10) shows that the Bryan Factor is dependent on the product of the tangential and radial displacements integrated over the length of the cylinder. It appears that restraining the tangential motion of the ends of the cylinder (for the $n = 2$ case) tends to reduce the tangential motion and therefore reduces the Bryan Factor. The axial displacement reduces the Bryan Factor and it is expected that the Bryan Factor of finite cylinders will be less than that of rings or infinite cylinders, performing inextensional vibrations, where the axial displacement is zero.

Finally, the case of $n = 4$ was analysed following the same procedure as was used in the $n = 2$ investigation. Although this mode is not generally used in vibratory gyroscopes it was studied to determine if the trends for the $n = 2$ case are repeated. The results of this analysis are shown in Figures 6–9. It was found that in the cylinder supported at both ends the radial and the radial bending stiffnesses had the largest influence on the natural frequency while in the cylinder supported at only one end the radial stiffness had the largest influence followed by the axial and rotational stiffnesses. The Bryan Factor was decreased by the axial restraint when both ends and when one end was supported. The Bryan Factor was not influenced by the tangential stiffness when one end was supported as it was in the $n = 2$ case. The Bryan factor was close to that of a ring (0·118) for both cases and was not greatly reduced by supporting both ends as it was in the $n = 2$ case.

The different trends observed in the results for $n = 2$ and $n = 4$ make it impossible to make general statements about the effects of the various boundary stiffnesses on the frequency factors and Bryan Factors. It is therefore necessary to examine each mode of interest and to draw conclusions for each mode separately.

It was observed that the changes in natural frequency and Bryan Factor occur when the non-dimensionalised stiffness is between 10^{-2} and 10^{2}. This indicates that it is only necessary to consider the boundaries as elastic when they have stiffness within this range. If the boundaries have stiffnesses outside this range it is possible to treat the boundary stiffnesses as either zero or infinite as is done in the idealised boundary conditions.

A cylinder which is to be used in a vibratory gyroscope, is required to have a constant Bryan Factor, for example in the presence of temperature changes. It is therefore desirable to design the cylinder and supporting means to ensure that the boundary stiffnesses are out of the sensitive range.

4. CONCLUSIONS

The free vibrations of elastically supported cylinders including gyroscopic effects may be calculated by the exact solution of the Flügge shell theory equations of motion. The accuracy of Flügge shell theory, was confirmed for a non-rotating cylinder with simply supported boundary conditions, by comparison with an exact solution of the three-dimensional elasticity problem.

Departures from ideal clamped boundary conditions were investigated for a cylinder supported at both ends and for a cylinder supported at one end only. The results for the lowest frequency $n = 2$ vibration mode are applicable to the design of vibrating cylinder gyroscopes. The natural frequency, of this mode, was most sensitive to changes in the tangential stiffness of the boundaries when both ends were supported and to changes in the axial stiffness of the boundary when only one end was supported. The Bryan Factor was decreased by increasing the tangential stiffness of the boundaries. In general the Bryan Factor is higher and also less sensitive to boundary stiffness variations when only one end is supported.

The results for the lowest frequency $n = 4$ vibration mode indicate that these trends do not apply to all vibration modes making it necessary to analyse all the modes of interest in a particular cylinder with elastic boundary conditions.

If the non-dimensionalised stiffnesses of the boundaries are in the range 10^{-2} to 10^{2} it is necessary to consider the boundaries to be elastic. Stiffnesses out of this range may be considered to be zero (free) or infinite (rigid). If the boundary conditions are designed to have stiffnesses out of this range the free vibration of the cylinder will be practically insensitive to variatons in the boundary stiffnesses.

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APPENDIX A: CALCULATION OF AMPLITUDE RATIOS

Equations (6) can be used to write the ratios of the axial and tangential displacements to the radial displacements. In the following equations the small terms containing β^2 have been neglected and ν is assumed to be 0·3.

$$\frac{U_0}{W_0} = \frac{\alpha n\beta\left[0{\cdot}65(\alpha^2 - n^2)^2 + 1{\cdot}405\alpha^2 - 0{\cdot}95n^2 + 0{\cdot}65 - \Delta\frac{\Omega}{\omega}\left(0{\cdot}7n + 2\frac{\alpha^2}{n}\right)\right] + 0{\cdot}35\alpha n + 0{\cdot}65\alpha n\Delta + 0{\cdot}6\frac{\Omega}{\omega}\Delta\alpha}{n\beta\left[-0{\cdot}7\alpha^4 + 0{\cdot}7\alpha^2n^2 - 0{\cdot}35n^2 - 1{\cdot}35\alpha^2\Delta + 0{\cdot}7\frac{\Omega}{\omega}\Delta n\right] - 0{\cdot}35n^3 + 0{\cdot}805n\alpha^2 + n\Delta + \frac{\Omega}{\omega}\Delta[0{\cdot}7n^2 - 2(\alpha^2 + \Delta^2)]} \qquad \text{(A1)}$$

$$\frac{V_0}{W_0}=\frac{n\beta\left[0{\cdot}65(\alpha^2-n^2)^2+1{\cdot}405\alpha^2-0{\cdot}95n^2+0{\cdot}65+\Delta\dfrac{\Omega}{\omega}\left(0{\cdot}7n+2\dfrac{\alpha^2}{n}\right)\right]+0{\cdot}35n+0{\cdot}65n\Delta-0{\cdot}6\dfrac{\Omega}{\omega}\Delta}{\beta[0{\cdot}35\alpha^4-0{\cdot}35n^4+\alpha^2\Delta-0{\cdot}35\alpha^2+0{\cdot}35n^2\Delta]-0{\cdot}35n^2-0{\cdot}105\alpha^2+1{\cdot}3n\dfrac{\Omega}{\omega}\Delta-0{\cdot}3\Delta}. \tag{A2}$$

The amplitude ratios are then calculated by substituting the roots ($\pm\alpha_1$, $\pm i\gamma_2$, $\pm(p\pm iq)$) in equations (A1) and (A2), for α, as follows:

$$A_1=(V_0/W_0)\quad\text{with }\alpha=\alpha_1$$

$$A_2=(U_0/W_0)\quad\text{with }\alpha=\alpha_1$$

$$A_3=(V_0/W_0)\quad\text{with }\alpha=i\gamma_2$$

$$iA_4=(U_0/W_0)\quad\text{with }\alpha=i\gamma_2$$

$$A_5+iA_6=(V_0/W_0)\quad\text{with }\alpha=p+iq$$

$$A_7+iA_8=(U_0/W_0)\quad\text{with }\alpha=p+iq.$$

APPENDIX B: NOMENCLATURE

a	mean radius of cylinder
$A_1, \ldots, A_8$	amplitude ratios defined in Appendix A
$C_1, \ldots, C_8$	displacement coefficients
E	Young's modulus
h	cylinder wall thickness
$k_u^0, k_v^0, k_w^0, k_{w'}^0$	axial, tangential, radial and rotational boundary stiffnesses at $x = 0$
$k_u^L, k_v^L, k_w^L, k_{w'}^L$	axial, tangential, radial and rotational boundary stiffnesses at $x = L$
$k_u^*, k_v^*, k_w^*, k_{w'}^*$	non-dimensionalised axial, tangential, radial and rotational boundary stiffnesses
L	cylinder length
m	axial mode number
n	number of circumferential waves
S	potential energy of the cylinder and boundaries
t	time
T	kinetic energy of the cylinder
u, v, w	components of displacement in the axial, tangential and radial directions
U_0, V_0, W_0	axial, tangential and radial displacement amplitudes
x, ϕ	axial and angular coordinates
α	axial wave number
β	dimensionless parameter, $\beta = h^2/12a^2$
Δ	frequency factor, $\Delta = \omega a\sqrt{\rho(1-\nu^2)/E}$
ν	Poisson's ratio
ρ	mass density
ω	circular frequency
Ω	rate of angular rotation of cylinder.