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LETTERS TO THE EDITOR

COMMENTS ON THE NATURAL FREQUENCIES OF RECTANGULAR PLATES DERIVED FROM THE RAYLEIGH–RITZ METHOD

R. S. Groves and R. L. Clark

Department of Mechanical Engineering and Materials Science, *Duke University*, *Durham*, *North Carolina* 27708-0300, *U.S.A.*

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1. INTRODUCTION

The study of the transverse vibrations of rectangular plates is among the most widely studied topics in structural dynamics, and the application of the assumed modes/Rayleigh–Ritz method to derive models of this vibration for various sets of boundary conditions has been employed for nearly the entire century. The shape functions employed in the assumed modes method need not be eigenfunctions of the governing equations of motion, but instead must form an *admissible set*, in part by satisfying the geometric boundary conditions of the system, as shown by Meirovitch [1]. Given the extreme difficulty in solving for closed form, analytic solutions to the plate vibration equations, study of these vibrations has often relied upon admissible function sets taken from other sources. In particular, for free vibration of a beam, closed form solutions are easily written for multiple boundary condition pairs, and such solutions are often employed as admissible sets in the rectangular plate vibration problem. Specifically, shape functions satisfying similar beam boundary conditions have been employed in the assumed modes method for rectangular plates by Leissa [2, 3], Warburton [4] and Young [5]. Relatively recently, Bhat and Mundkur [6] offered an excellent collection of natural frequencies of the freely vibrating plate. These frequencies were obtained by using "plate characteristic functions" obtained from reduction of the plate vibration equations in the Rayleigh–Ritz method. Specifically, Bhat and Mundkur offer the first 36 frequencies for 11 cases, with variation of plate aspect ratio and boundary conditions. They demonstrate the validity of the plate functions by comparing the frequencies generated by the plate functions with previously published data. However, in the results presented for asymmetric plates, several vibration modes/frequencies were omitted from the sorted sets tabulated. As such, data presented herein are intended to supplement the work of Bhat and Mundkur [6], tabulating modal indices and natural frequencies for omitted modes.

2. ASSUMED MODES METHOD

The authors have constructed a numerical method for implementing the assumed modes/Rayleigh–Ritz method for the rectangular plate, employing as assumed modes a set of beam functions, as first presented by Warburton [4] and later by Blevins [7]. (Note that separate beam functions are used in each of the plate's two directions, and the convolution of the two, according to principles of

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separability, produces the actual mode shape.) In his study of the rectangular plate, Warburton presents approximate natural frequencies for the plate by using asymptotic behavior of the beam functions in the assumed modes method. That is, he gives a means to determine vibration frequencies for particular beam function pairs, e.g. the fundamental vibration mode in the x direction and the second vibration mode in the y direction.

In the current method, the authors employ these approximate frequencies to determine the set of beam functions which correspond, approximately, to the lowest occurring plate frequencies. This sorted set of lowest frequency mode shapes is then utilized in the assumed modes method. While studies of small numbers of vibration modes generally allow for intuitive ordering of modes, in larger studies, the ordering process is useful, as will be seen.

In particular, a state-space model of the rectangular plate is created, and the natural frequencies of the system are found as the eigenvalues of the state matrix. For comparison and validation, data from the present method are compared with the data published by Bhat and Mundkur in the aforementioned article for the various aspect ratios and boundary conditions.

3. RESULTS

For square plates ($\alpha = 1{\cdot}0$) of all boundary conditions, the present method yields excellent agreement with Bhat and Mundkur's results, generating natural frequencies which are generally only slightly greater than those published. Data for these cases are not presented in this letter.

TABLE 1

Natural frequencies of CCCC plate, $\Omega = \omega a^2\sqrt{m_s''/D}$, $\alpha = 0{\cdot}5$, $N = 36$ *modes. PF, plate functions* (*Bhat and Mundkur* [6]); *BF, beam functions*

i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)	i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)
1	24·5789	24·5898	2/2	19	–	186·4452	3/8
2	31·8298	31·8523	2/3	20	–	189·9961	2/9
3	44·7796	44·8124	2/4	21	202·2706	202·3528	5/2
4	63·3473	63·3986	2/5	22	209·4899	209·6852	4/7
5	63·9916	64·0282	3/2	23	209·7974	210·2058	5/3
6	71·0982	71·1728	3/3	24	221·7169	221·8397	5/4
7	83·3386	83·3951	3/4	25	–	225·3275	3/9
8	87·2815	87·3554	2/6	26	–	234·9169	2/10
9	100·8726	100·9500	3/5	27	238·7859	238·9466	5/5
10	116·3693	116·4996	2/7	28	–	243·5799	4/8
11	123·2723	123·3282	4/2	29	261·2743	261·7169	5/6
12	123·8458	123·9094	3/6	30	–	269·3846	3/10
13	130·4184	130·5373	4/3	31	–	281·6776	4/9
14	142·5743	142·7105	4/4	32	–	284·3529	2/11
15	–	150·7328	2/8	33	288·3459	289·0325	5/7
16	151·9839	152·1821	3/7	34	300·9999	301·3160	6/2
17	159·7423	159·9558	4/5	35	308·2931	308·9632	6/3
18	182·3265	182·4400	4/6	36	–	318·4034	3/11

TABLE 2

Natural frequencies of CPCP plate, $\Omega = \omega a^2\sqrt{m_s''/D}$, $\alpha = 0{\cdot}5$, $N = 36$ *modes*. *PF*, *plate functions* (*Bhat and Mundkur* [6]); *BF*, *beam functions*

i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)	i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)
1	23·8156	23·8160	2/2	19	–	171·9190	3/8
2	28·9516	28·9566	2/3	20	174·8917	175·1643	4/6
3	39·0933	39·1114	2/4	21	199·9773	200·4481	4/7
4	54·7541	54·7937	2/5	22	201·9816	201·9822	5/2
5	63·5345	63·5349	3/2	23	–	208·0484	3/9
6	69·3279	69·3324	3/3	24	208·3934	208·4013	5/3
7	75·8635	75·9303	2/6	25	–	212·2580	2/10
8	79·5307	79·5490	3/4	26	219·2186	219·2520	5/4
9	94·6026	94·6487	3/5	27	–	230·8852	4/8
10	102·2462	102·3491	2/7	28	234·6228	234·7123	5/5
11	114·8177	114·9062	3/6	29	–	249·3080	3/10
12	122·9296	122·9299	4/2	30	254·7748	254·9583	5/6
13	129·0978	129·1110	4/3	31	–	259·7159	2/11
14	–	133·9240	2/8	32	–	266·4915	4/9
15	139·6374	139·6919	4/4	33	279·7837	280·1332	5/7
16	140·2558	140·4158	3/7	34	–	295·6502	3/11
17	154·8229	154·9629	4/5	35	300·7397	300·7399	6/2
18	–	170·5763	2/9	36	307·3205	307·2524	4/10

TABLE 3

Natural frequencies of CPCP plate, $\Omega = \omega a^2\sqrt{m_s''/D}$, $\alpha = 2{\cdot}0$, $N = 36$ *modes*. *PF*, *plate functions* (*Bhat and Mundkur* [6]); *BF*, *beam functions*

i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)	i	$\Omega_{i(PF)}$	$\Omega_{i(BF)}$	(m/n)
1	54·7431	54·7871	2/2	19	642·9176	643·2697	2/5
2	94·5853	94·6382	3/2	20	676·3539	677·1699	3/5
3	154·7757	154·8164	4/2	21	–	702·6006	8/3
4	170·3819	170·5454	2/3	22	732·1520	732·4469	4/5
5	206·7756	207·0584	3/3	23	–	749·7553	9/2
6	234·5854	234·6168	5/2	24	753·4817	<u>*753·2002*</u>	7/4
7	265·4212	265·5382	4/3	25	808·6187	808·8001	5/5
8	333·9558	333·9802	6/2	26	–	861·5418	9/3
9	344·7531	344·9162	5/3	27	–	892·9465	8/4
10	366·9272	367·1765	2/4	28	907·4863	908·2635	6/5
11	401·3584	401·8606	3/4	29	–	927·6925	10/2
12	444·4839	444·3308	6/3	30	998·0407	998·4505	2/6
13	452·8081	452·9296	7/2	31	1024·4605	1025·6754	7/5
14	458·2396	458·4998	4/4	32	1030·9283	1033·8248	3/6
15	536·0290	536·2239	5/4	33	–	1039·8529	10/3
16	563·4211	563·5944	7/3	34	–	1051·4095	9/4
17	–	591·5122	8/2	35	1086·0900	1088·9955	4/6
18	635·3948	<u>*634·5721*</u>	6/4	36	–	1125·3052	11/2

For the non-square cases ($\alpha = 0{\cdot}5$), however, the present method yields numerous occurrences of appreciably smaller frequencies than those published. Table 1 lists non-dimensionalized frequency data for a completely clamped plate with $\alpha = 0{\cdot}5$. Careful comparison shows that for the first 36 frequencies, the present method contains all of the frequencies/modes listed by Bhat and Mundkur, as well as ten others. As such, it is concluded that the present method captures modes previously omitted.

Examination of the nature of these "omitted" modes shows that they correspond to highly asymmetric mode shapes; the beam function indices corresponding to each frequency are also presented for the clamped plate with $\alpha = 0{\cdot}5$ in Table 1. That the pattern of omission occurs in greater number in the non-square plates than the square plates is expected, for in these plates asymmetric vibration patterns occur at lower frequencies than in the square plate as a result of the asymmetric shape.

Data for the $\alpha = 0{\cdot}5$ and $\alpha = 2{\cdot}0$ cases of the CPCP and CCPF plates are presented in Tables 2–5. Bhat and Mundkur [6] considered both beam end conditions and plate edge conditions in constructing plate functions involving free edges, and data from both of these cases are given in Tables 4 and 5. (The standard convention has been used in naming plates according to boundary conditions, namely beginning with the $x = 0$ (left) edge, and proceeding counter-clockwise.)

Tables 3–5 demonstrate a few occurrences where the frequencies of the current method are actually lower than those of the plate function set, indicated by

Table 4

Natural frequencies of CCPF plate, $\Omega = \omega a^2\sqrt{m_s''/D}$, $\alpha = 0{\cdot}5$, $\nu = 0{\cdot}3$, $N = 36$ *modes. PF plate functions* (*Bhat and Mundkur* [6]); *BC beam boundary conditions*; *PC plate boundary conditions*; *BF*, *beam functions*

i	$\Omega_{i(PF-BC)}$	$\Omega_{i(PF-PC)}$	$\Omega_{i(BF)}$	(m/n)	i	$\Omega_{i(PF-BC)}$	$\Omega_{i(PF-PC)}$	$\Omega_{i(BF)}$	(m/n)
1	15·8713	15·8224	15·8563	2/1	19	–	–	150·9801	2/8
2	20·1727	20·1526	20·5302	2/2	20	152·3236	152·1441	152·7465	4/5
3	29·1861	29·1707	29·3917	2/3	21	178·1298	177·0861	*176·5741*	4/6
4	43·2528	43·2208	43·3855	2/4	22	178·7329	178·6274	178·9479	5/1
5	50·4234	50·3326	50·4475	3/1	23	–	–	183·9330	3/8
6	54·9507	54·9035	55·6948	3/2	24	183·4222	183·3634	184·3880	5/2
7	62·4517	62·3891	62·6252	2/5	25	–	–	190·1825	2/9
8	64·1730	64·1442	64·8704	3/3	26	192·9072	192·8707	193·9053	5/3
9	78·3461	78·2553	78·7281	3/4	27	–	–	205·6971	4/7
10	86·7645	86·6492	86·8753	2/6	28	207·5270	207·4227	208·3018	5/4
11	97·3794	97·2207	97·6572	3/5	29	–	–	222·8437	3/9
12	104·7084	104·6075	104·8381	4/1	30	226·8730	226·6956	227·5232	5/5
13	109·3366	109·2813	110·2598	4/2	31	–	–	234·9091	2/10
14	–	–	116·2805	2/7	32	–	–	238·1632	4/8
15	118·7109	118·6769	119·7002	3/6	33	253·8962	252·5287	*251·5016*	5/6
16	122·0101	121·4149	121·4953	4/3	34	–	–	268·0755	3/10
17	133·1380	133·0358	133·8311	4/4	35	272·4962	272·3885	272·7349	4/9
18	–	–	150·4869	3/7	36	277·2285	277·1677	*276·7913*	6/1

TABLE 5

Natural frequencies of CCPF plate, $\Omega = \omega a^2\sqrt{m_s''/D}$, $\alpha = 2{\cdot}0$, $\nu = 0{\cdot}3$, $N = 36$ *modes*. *PF*, *plate functions* (*Bhat and Mundkur* [6]); *BC*, *beam boundary conditions*; *PC*, *plate boundary conditions*; *BF*, *beam functions*

i	$\Omega_{i(PF\text{–}BC)}$	$\Omega_{i(PF\text{–}PC)}$	$\Omega_{i(BF)}$	(m/n)	i	$\Omega_{i(PF\text{–}BC)}$	$\Omega_{i(PF\text{–}PC)}$	$\Omega_{i(BF)}$	(m/n)
1	26·3361	26·3006	26·5945	2/1	19	528·7590	528·7333	529·3469	3/4
2	59·9958	59·8951	60·9949	3/1	20	–	–	530·3010	8/1
3	101·4408	101·4374	101·7462	2/2	21	584·2719	584·2179	584·6026	8/2
4	113·7162	113·5956	115·4156	4/1	22	–	–	618·7429	4/4
5	137·8506	137·8308	139·0161	3/2	23	634·6429	634·6098	638·8085	7/3
6	187·3360	187·2092	189·5492	4/2	24	660·6503	660·5690	661·0411	5/4
7	193·5639	193·5293	196·4864	5/1	25	–	–	683·3217	9/1
8	258·6185	258·6159	258·9862	5/2	26	757·1033	757·0005	*756·7958*	9/2
9	268·0493	268·0043	273·0526	2/3	27	–	–	772·6262	6/4
10	280·8205	280·6914	283·4052	6/1	28	–	–	774·6338	8/3
11	293·4216	293·4093	293·9701	3/3	29	810·5154	810·5100	811·0097	2/5
12	349·4463	349·4247	349·8267	6/2	30	843·6211	843·5938	844·4858	3/5
13	361·7837	361·7315	369·1263	4/3	31	–	–	856·0787	10/1
14	394·0984	393·9687	396·9792	7/1	32	873·0693	872·9515	874·2815	7/4
15	425·3304	425·3027	426·4601	5/3	33	898·4257	898·3667	899·0222	4/5
16	474·9500	474·8928	484·0216	7/2	34	–	–	929·4849	10/2
17	494·9540	494·9405	495·3763	2/4	35	–	–	946·1242	9/3
18	520·4235	520·3925	522·9345	6/3	36	974·4104	974·3156	975·8634	5/5

underlining. That the beam functions would yield superior frequencies to those from the plate functions indicates the degree in which the modal omission hinders the convergence of frequencies of existing vibration modes/frequencies. In short, modal omission results in errors even for some modes not omitted.

4. CONCLUSIONS

The data presented by Bhat and Mundkur [6] embody a thorough examination of the natural frequencies of the free vibration of a rectangular plate, as determined by the Rayleigh–Ritz method. A similar approach using beam functions as assumed modes is employed in the current method. The set of assumed modes is sorted according to lowest beam function frequencies, and then employed in the Rayleigh–Ritz process. In the non-square plates, the sorting shows the prior omission of numerous frequencies/vibration modes. It is concluded that sorting of assumed modes by approximate frequencies, according to aspect ratio and boundary conditions, is essential for accurate modelling in the Rayleigh–Ritz process.

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APPENDIX: NOMENCLATURE

Ω	non-dimensionalized frequency
ω	natural frequency
a, b	plate dimensions
$\alpha = a/b$	aspect ratio
m_s''	mass per unit area
D	flexural rigidity
ν	Poisson's ratio