Journal of Sound and Vibration (1998) **217**(3), 585–595
Article No. sv981796

ON THE ALTERNATIVE FORMULATIONS OF THE FREQUENCY EQUATION OF A BERNOULLI–EULER BEAM TO WHICH SEVERAL SPRING-MASS SYSTEMS ARE ATTACHED IN-SPAN

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(*Received* 16 *April* 1998, *and in final form* 19 *June* 1998)

1. INTRODUCTION

In two recent publications [1, 2], the present author investigated the eigencharacteristics of a clamped–free Bernoulli–Euler beam with a tip mass where a spring–mass system was attached to it. In reference [1] the secondary spring–mass system was attached at the tip of the beam, whereas in [2] the attachment point was in-span. The common property of both studies was that there was only one spring–mass system. Unlike these two investigations, it is assumed here that several secondary spring–mass systems are attached to the beam in-span. Hence, the present letter presents a generalization of the results and expressions given in the mentioned works. The fact that no tip mass will be considered here does not mean any restriction, as it can be included via a limiting process. The main subject of this study is to give two alternative formulations for the frequency equation of the mechanical system described above. Both formulations are based on the discretization of the elastic beam by its first n eigenfunctions, according to the assumed modes method.

Most of the work on similar subjects are cited in references [1, 2]. References [3, 4] can be cited as examples of some of the important studies. In reference [4], a general method for determining the exact undamped natural frequencies and natural modes of vibration, the orthogonality relation for the natural modes and the response to arbitrary excitation for both damped and undamped combined systems is given. The method is based upon Green's functions of the vibrating distributed subsystems. While the final results are exact, the approach may be quite complicated, as the Green's function for the elastic structure needs first to be determined, which can be both tedious and time consuming in certain cases. At the end, the eigenfrequencies of the system are obtained by equating the determinant of coefficients of a set of homogeneous equations to zero. In contrast to the above, both of two alternative methods given in this work are approximate methods. In the first method, eigenfrequencies are obtained by equating a determinant of coefficients to zero as in reference [4], while in the second method they are obtained as the eigenvalues of a matrix. The comparison of the two methods clearly shows that finding the eigenvalues of a matrix is by far a

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numerically less problematic procedure than finding the zeros of a determinant. On the other hand, the fact that the eigenfrequencies obtained by solving the eigenvalue problem are very close to the exact values for an example problem in [4] indicates the effectiveness of the proposed approximate method.

2. THEORY

The mechanical system to be dealt with in the present study is shown in Figure 1. The system consists of a clamped–free Bernoulli–Euler beam of bending rigidity EI, length L and mass per unit length m to which s additional secondary spring-mass systems are attached in-span. The jth spring–mass system consists of a spring of stiffness k_{e_j} and a mass m_{e_j}. The main aim of this letter is to give two alternative formulations of the frequency equation of the system described above. Both formulations of the frequency equation are based on the discretization of the beam by its first n bending eigenfunctions, according to the assumed modes method.

The kinetic and potential energies of the system in Figure 1 are

$$T = \frac{1}{2} m \int_0^L \dot{w}^2(x,\, t)\, \mathrm{d}x + \frac{1}{2} \sum_{j=1}^{s} m_{e_j} \dot{z}_j^2, \tag{1}$$

$$V = \frac{1}{2} EI \int_0^L w''^2(x,\, t)\, \mathrm{d}x + \frac{1}{2} \sum_{j=1}^{s} k_{e_j} (z_j - \delta_j)^2, \tag{2}$$

where dots and primes denote partial derivatives with respect to time t and the position co-ordinate x, respectively. Here, δ_j denotes the lateral displacement of the attachment point of the jth spring–mass to the beam while z_j represents the

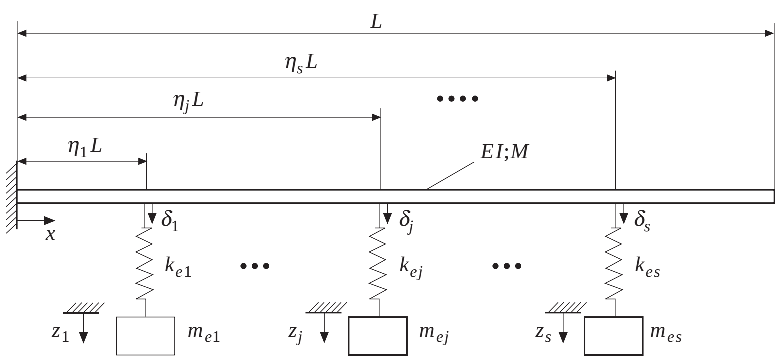

Figure 1. Clamped free Bernoulli–Euler beam to which s spring–mass systems are attached in-span.

displacement of the mass m_{e_j}. The lateral displacement of the beam at point x is assumed to be expressible in the form of a finite series

$$w(x, t) = \sum_{i=1}^{n} w_i(x)\eta_i(t), \tag{3}$$

where

$$w_i(x) = (1/\sqrt{mL})[\cosh \beta_i x - \cos \beta_i x - \bar{\eta}_i (\sinh \beta_i x - \sin \beta_i x)],$$

$$\bar{\eta}_i = (\cosh \beta_i L + \cos \beta_i L)/(\sinh \beta_i L + \sin \beta_i L). \tag{4}$$

Here, $w_i(x)$ are the orthonormalized eigenfunctions of the clamped–free beam and $\eta_i(t)$ $(i = 1, \ldots, n)$ are the generalized co-ordinates [5]. If the expression (3) is set into equations (1) and (2)

$$T = \frac{1}{2}\sum_{i=1}^{n} \dot{\eta}_i^2 + \frac{1}{2}\sum_{j=1}^{s} m_{e_j}\dot{z}_j^2, \qquad V = \frac{1}{2}\sum_{i=1}^{n} \omega_i^2\eta_i^2 + \frac{1}{2}\sum_{j=1}^{s} k_{e_j}(z_j - \delta_j)^2 \tag{5, 6}$$

are obtained for the kinetic and potential energies. Here, the following expressions have been used, which are due to the special normalization of the eigenfunctions $w_i(x)$ given in (4):

$$M_{ij} = \int_0^L m w_i(x) w_j(x)\, \mathrm{d}x = \delta_{ij}, \qquad K_{ij} = \int_0^L EI w_i''(x) w_j''(x)\, \mathrm{d}x = \omega_i^2 M_{ij}. \tag{7}$$

Here, δ_{ij} represents the Kronecker delta and ω_i is the ith eigenfrequency of the bare fixed–free Bernoulli–Euler beam.

The first alternative formulation uses the approach of Dowell [6] which is essentially based on the assumed modes method in conjunction with the Lagrange multipliers method. The result is a determinantal equation for the frequency equation of the system. Hence, the eigenfrequency parameters of the system are determined by solving this equation numerically.

The second formulation of the frequency equation follows directly from the formulation of the Lagrange equations where the displacements of the attachment points of the secondary spring–mass systems to the beam are expressed in terms of the generalized co-ordinates [7]. The formulation leads to a standard eigenvalue problem the solution of which gives all of the eigenfrequency parameters of the system simultaneously.

3. FIRST FORM OF THE FREQUENCY EQUATION

In order to derive the first alternative form of the frequency equation of the system, it is necessary to formulate first the equations of the motion of the system. To this end the approach developed in reference [6] will be employed here, which was also used in the earlier works of the present author [1, 2].

According to the Dowell's method, the equations of motion of the system in Figure 1 will be obtained by means of the Lagrange's multipliers, in connection with Lagrange's multipliers which when considered for a system with n degrees of freedom where v redundant co-ordinates are used, are as follows [8]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{v} \lambda_l \frac{\partial f_l}{\partial q_k}, \qquad k = 1, \ldots, n + v, \tag{8}$$

with the kinetic potential

$$\mathrm{L} = \mathrm{T} - \mathrm{V} \tag{9}$$

and v constraint equations

$$f_l(t; q_1, \ldots, q_{n+v}) = 0, \qquad l = 1, \ldots, v. \tag{10}$$

Here, λ_l denotes the lth Langrangian multiplier. In the present case, there are s constraint equations

$$f_j := w(\eta_j L, t) - \delta_j(t) = 0, \tag{11}$$

which, when considered the expansion in (3), can be written as

$$f_j = \sum_{k=1}^{n} w_k(\eta_j L)\eta_k(t) - \delta_j(t) = 0, \qquad j = 1, \ldots, s. \tag{12}$$

Upon taking into account the correspondence

$$\begin{aligned} &[q_1, \ldots, q_n; q_{n+1}, \ldots, q_{n+s}; q_{n+s+1}, \ldots, q_{n+2s}] \\ &\quad \widehat{=} [\eta_1, \ldots, \eta_n; \delta_1, \ldots, \delta_s; z_1, \ldots, z_s] \end{aligned} \tag{13}$$

the following equations are obtained from equation (8) together with equations (5), (6), (9) and (12):

$$\ddot{\eta}_k + \omega_k^2 \eta_k = \sum_{l=1}^{s} \lambda_l w_k(\eta_l L); \qquad k = 1, \ldots, n, \tag{14}$$

$$k_{e_j}(z_j - \delta_j) = \lambda_j, \qquad m_{e_j}\ddot{z}_j + k_{e_j}(z_j - \delta_j) = 0, \qquad j = 1, \ldots, s \tag{15, 16}$$

The substitution of the harmonic solutions

$$\eta_k = \bar{\eta}_k\, e^{i\omega t}(k = 1, \ldots, n); \qquad \delta_j = \overline{\delta}_j\, e^{i\omega t}, \qquad z_j = \bar{z}_j\, e^{i\omega t},$$

$$\lambda_j = \overline{\lambda}_j\, e^{i\omega t}, \qquad (j = 1, \ldots, s) \tag{17}$$

into equations (14)–(16) and (12), after rearrangement results in

$$\bar{\eta}_k = \sum_{l=1}^{s} \overline{\lambda}_l w_k(\eta_l L)/(\omega_k^2 - \omega^2), \qquad k = 1, \ldots, n, \tag{18}$$

$$k_{e_j}(\bar{z}_j - \overline{\delta}_j) = \overline{\lambda}_j, \qquad -m_{e_j}\bar{z}_j\omega^2 + k_{e_j}(\bar{z}_j - \overline{\delta}_j) = 0 \tag{19, 20}$$

and

$$f_j = \sum_{k=1}^{n} w_k(\eta_j L)\bar{\eta}_k - \overline{\delta}_j = 0, \qquad j = 1, \ldots, s. \tag{21}$$

From (19)

$$\bar{z}_j = \overline{\lambda}_j/k_{e_j} + \overline{\delta}_j \tag{22}$$

can be obtained which, when put into equation (20), yields

$$\overline{\delta}_j = (1/m_{e_j}\omega^2 - 1/k_{e_j})\overline{\lambda}_j, \qquad j = 1, \ldots, s. \tag{23}$$

If the above equations and equations (18) are substituted into the constraint equations (21), the following set of s homogeneous equations for $\overline{\lambda}_j$ are obtained

$$\sum_{k=1}^{n} w_k(\eta_j L)\left[\sum_{l=1}^{s} \overline{\lambda}_l w_k(\eta_l L)\right]\Big/(\omega_k^2 - \omega^2) - (1/m_{e_j}\omega^2 - 1/k_{e_j})\overline{\lambda}_j = 0,$$

$$j = 1, \ldots, s. \tag{24}$$

A non-trivial solution of this set is possible only if the determinant of the coefficients vanish. This in turn leads to the following frequency equation of the mechanical system shown in Figure 1; which is written explicitly in order to reflect the symmetry properties better.

$$\begin{vmatrix} \sum_{k=1}^{n} \frac{w_k^2(\eta_1 L)}{\omega_k^2 - \omega^2} - \left(\frac{k_{e1} - m_{e_1}\omega^2}{k_{e1} m_{e1}\omega^2}\right) & \sum_{k=1}^{n} \frac{w_k(\eta_1 L) w_k(\eta_2 L)}{\omega_k^2 - \omega^2} & \cdots & \sum_{k=1}^{n} \frac{w_k(\eta_1 L) w_k(\eta_s L)}{\omega_k^2 - \omega^2} \\ \sum_{k=1}^{n} \frac{w_k(\eta_2 L) w_k(\eta_1 L)}{\omega_k^2 - \omega^2} & \sum_{k=1}^{n} \frac{w_k^2(\eta_2 L)}{\omega_k^2 - \omega^2} - \left(\frac{k_{e2} - m_{e2}\omega^2}{k_{e2} m_{e2}\omega^2}\right) & \cdots & \sum_{k=1}^{n} \frac{w_k(\eta_2 L) w_k(\eta_s L)}{\omega_k^2 - \omega^2} \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} \frac{w_k(\eta_s L) w_k(\eta_1 L)}{\omega_k^2 - \omega^2} & \sum_{k=1}^{n} \frac{w_k(\eta_s L) w_k(\eta_2 L)}{\omega_k^2 - \omega^2} & \cdots & \sum_{k=1}^{n} \frac{w_k^2(\eta_s L)}{\omega_k^2 - \omega^2} - \left(\frac{k_{es} - m_{es}\omega^2}{k_{es} m_{es}\omega^2}\right) \end{vmatrix} = 0. \tag{25}$$

For further frequency investigations, it is more suitable to rewrite the frequency equation above in terms of non-dimensional quantities as

$$\begin{vmatrix} \sum_{k=1}^{n} \frac{a_{k1}^2}{\lambda_k - \omega^{*2}} + \frac{\alpha_{me1}\omega^{*2} - \alpha_{ke1}}{\alpha_{me1}\alpha_{ke1}\omega^{*2}} & \sum_{k=1}^{n} \frac{a_{k1} a_{k2}}{\lambda_k - \omega^{*2}} & \cdots & \sum_{k=1}^{n} \frac{a_{k1} a_{ks}}{\lambda_k - \omega^{*2}} \\ \sum_{k=1}^{n} \frac{a_{k2} a_{k1}}{\lambda_k - \omega^{*2}} & \sum_{k=1}^{n} \frac{a_{k2}^2}{\lambda_k - \omega^{*2}} + \frac{\alpha_{me2}\omega^{*2} - \alpha_{ke2}}{\alpha_{me2}\alpha_{ke2}\omega^{*2}} & \cdots & \sum_{k=1}^{n} \frac{a_{k2} a_{ks}}{\lambda_k - \omega^{*2}} \\ \vdots & \vdots & & \vdots \\ \sum_{k=1}^{n} \frac{a_{ks} a_{k1}}{\lambda_k - \omega^{*2}} & \sum_{k=1}^{n} \frac{a_{ks} a_{k2}}{\lambda_k - \omega^{*2}} & \cdots & \sum_{k=1}^{n} \frac{a_{ks}^2}{\lambda_k - \omega^{*2}} + \frac{\alpha_{mes}\omega^{*2} - \alpha_{kes}}{\alpha_{mes}\alpha_{kes}\omega^{*2}} \end{vmatrix} = 0 \tag{26}$$

Here, the following abbreviations are used:

$$\bar{\beta}_k = \beta_k L, \qquad \lambda_k = \bar{\beta}_k^4, \qquad \omega_k^2 = \lambda_k \omega_0^2, \qquad \omega_0^2 = EI/mL^4,$$
$$w_k(\eta_j L) = (1/\sqrt{mL})a_{k_j},$$
$$a_{k_j} = \cosh \bar{\beta}_k \eta_j - \cos \bar{\beta}_k \eta_j - \bar{\eta}_k(\sinh \bar{\beta}_k \eta_j - \sin \bar{\beta}_k \eta_j),$$
$$\bar{\eta}_k = (\cosh \bar{\beta}_k + \cos \bar{\beta}_k)/(\sinh \bar{\beta}_k + \sin \bar{\beta}_k),$$
$$\omega^* = \omega/\omega_0, \qquad \alpha_{m_{e_j}} = m_{e_j}/mL, \qquad \alpha_{k_{e_j}} = k_{e_j}/(EI/L^3). \tag{27}$$

4. ALTERNATIVE FORM OF THE FREQUENCY EQUATION

The kinetic and potential energies of the mechanical system, i.e., expressions (5) and (6) can be written in matrix notation as

$$T = \frac{1}{2}\dot{\eta}^{\mathrm{T}}\mathbf{I}_n\dot{\boldsymbol{\eta}} + \frac{1}{2}\sum_{j=1}^{s} m_{e_j}\dot{z}_j^2, \qquad V = \frac{1}{2}\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\Omega}^2\boldsymbol{\eta} + \frac{1}{2}\sum_{j=1}^{s} k_{e_j}(z_j - \delta_j)^2, \tag{28, 29}$$

where

$$\boldsymbol{\eta}^{\mathrm{T}}(t) = [\eta_1(t), \ldots, \eta_n(t)], \qquad \boldsymbol{\Omega}^2 = \mathbf{diag}(\omega_i^2), \qquad i = 1, \ldots, n \tag{30}$$

$\mathbf{I}_n n \times n$ identity matrix.

The idea behind this approach is to express the displacements of the spring attachment points on to the beam, i.e., $\delta_j(t)$ $(j = 1, \ldots, s)$ in terms of the generalized co-ordinate vector $\boldsymbol{\eta}(t)$:

$$\delta_j(t) = w(\eta_j L, t) = \mathbf{I}_j^{\mathrm{T}}\boldsymbol{\eta}, \qquad j = 1, \ldots, s, \tag{31}$$

where

$$\mathbf{I}_j = [w_1(\eta_j L), \ldots, w_n(\eta_j L)]^{\mathrm{T}}. \tag{32}$$

Starting with the energy expressions (28) and (29), along with equations (30)–(32) the following matrix differential equation is obtained, by using the Lagrange equation formalism:

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{m}_e \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \ddot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega}^2 + \mathbf{I}\mathbf{k}_e\mathbf{I}^{\mathrm{T}} & -\mathbf{I}\mathbf{k}_e \\ -(\mathbf{I}\mathbf{k}_e)^{\mathrm{T}} & \mathbf{k}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \mathbf{0}, \tag{33}$$

where the following matrices and vectors are introduced

$$\mathbf{I} = [\mathbf{I}_1, \ldots, \mathbf{I}_s], \qquad \mathbf{z} = [z_1, \ldots, z_s]^{\mathrm{T}}, \qquad \mathbf{k}_e = \mathbf{diag}(k_{e_i}), \qquad \mathbf{m}_e = \mathbf{diag}(m_{e_i}),$$
$$i = 1, \ldots, s, \tag{34}$$

where $\mathbf{0}$ denotes a zero matrix or vector of appropriate dimensions. It is worth noting that in obtaining the above form, extensive use is made of the formulas regarding the partial derivatives of the bilinear forms, quadratic forms and vectors with respect to algebraic vectors [9].

By means of the transformation

$$\begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{m}_e^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix}, \tag{35}$$

where $\mathbf{T} = \mathbf{I}_n$ and $\mathbf{y} = [y_1, \ldots, y_s]^{\mathrm{T}}$, the equations of motion in (33) can be written as

$$\begin{bmatrix} \ddot{\mathbf{p}} \\ \ddot{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega}^2 + \mathbf{e}\mathbf{k}_e\mathbf{e}^{\mathrm{T}} & -\mathbf{e}\mathbf{k}_e\mathbf{m}_e^{-1/2} \\ -\mathbf{k}_e\mathbf{m}_e^{-1/2}\mathbf{e}^{\mathrm{T}} & \boldsymbol{\Omega}_e^2 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix} = \mathbf{0}. \tag{36}$$

Here, the following abbreviations are introduced

$$\mathbf{e}_j = \mathbf{T}^{\mathrm{T}}\mathbf{I}_j = \mathbf{I}_j, \qquad \omega_{e_j}^2 = k_{e_j}/m_{e_j}, \qquad \boldsymbol{\Omega}_e^2 = \mathbf{diag}(\omega_{e_j}^2), \qquad \mathbf{e} = [\mathbf{e}_1, \ldots, \mathbf{e}_s],$$
$$j = 1, \ldots, s. \tag{37}$$

Harmonic solutions of the form

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{y}} \end{bmatrix} \mathrm{e}^{\mathrm{i}\omega t} \tag{38}$$

result in a set of homogeneous equations for the amplitude vectors $\bar{\mathbf{p}}$ and $\bar{\mathbf{y}}$ where $\bar{\mathbf{p}} = [\bar{p}_1, \ldots, \bar{p}_n]^{\mathrm{T}}$ and $\bar{\mathbf{y}} = [\bar{y}_1, \ldots, \bar{y}_s]^{\mathrm{T}}$. A non-trivial solution of this set is possible only if the determinant of the coefficient matrix vanishes. This condition leads to the following form of the frequency equation of the mechanical system.

$$\begin{vmatrix} \boldsymbol{\Omega}^2 + \mathbf{e}\mathbf{k}_e\mathbf{e}^{\mathrm{T}} - \boldsymbol{\omega}^2 & -\mathbf{e}\mathbf{k}_e\mathbf{m}_e^{-1/2} \\ -\mathbf{k}_e\mathbf{m}_e^{-1/2}\mathbf{e}^{\mathrm{T}} & \boldsymbol{\Omega}_e^2 - \boldsymbol{\omega}^2 \end{vmatrix} = 0, \tag{39}$$

where $\boldsymbol{\omega}^2 = \omega^2\mathbf{I}_n$ is introduced.

The above form is an alternative presentation of the frequency equation given in (25). Using non-dimensional quantities defined in (27), this equation can be brought into the following form

$$\begin{vmatrix} \boldsymbol{\Lambda} + \bar{\mathbf{e}}\boldsymbol{\alpha}_{k_e}\bar{\mathbf{e}}^{\mathrm{T}} - \boldsymbol{\omega}^{*2} & -\bar{\mathbf{e}}\boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{m_e}^{-1/2} \\ -\boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{\mathrm{m_e}}^{-1/2}\bar{\mathbf{e}}^{\mathrm{T}} & \boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{m_e}^{-1/2} - \boldsymbol{\omega}^{*2} \end{vmatrix} = 0, \tag{40}$$

where additional to those given in (27), the following abbreviations are introduced

$$\boldsymbol{\Lambda} = \mathbf{diag}(\lambda_i), \qquad \bar{\mathbf{e}}_i = [a_{1i}, \ldots, a_{ni}]^{\mathrm{T}}, \qquad i = 1, \ldots, n$$

$$\bar{\mathbf{e}} = [\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n], \qquad \boldsymbol{\alpha}_{k_e} = \mathbf{diag}(\alpha_{k_{e_j}}), \qquad \boldsymbol{\alpha}_{m_e} = \mathbf{diag}(\alpha_{m_{e_j}}), \qquad j = 1, \ldots, s. \tag{41}$$

$\boldsymbol{\omega}^{*2} = \omega^{*2}\mathbf{I}$, the unit matrix $\mathbf{I}$ having the dimensions $n \times n$ or $s \times s$.

The equation (40) represents now the alternative form of the frequency equation given in (26). This last form enables one to obtain ω^{*2}, i.e., the squares of the non-dimensional eigenfrequency parameters of the system, as the eigenvalues of the following matrix

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\Lambda} + \bar{\mathbf{e}}\boldsymbol{\alpha}_{k_e}\bar{\mathbf{e}}^{\mathrm{T}} & -\bar{\mathbf{e}}\boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{\mathrm{m_e}}^{-1/2} \\ -\boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{\mathrm{m_e}}^{-1/2}\bar{\mathbf{e}}^{\mathrm{T}} & \boldsymbol{\alpha}_{k_e}\boldsymbol{\alpha}_{\mathrm{m_e}}^{-1/2} \end{bmatrix}. \tag{42}$$

After having obtained also the second form of the frequency equation of the mechanical system, it is worth emphasizing the following point. The two forms (26) and (40) are obtained merely on the basis of physical considerations. Perhaps, mathematicians can prove mathematically as well that the determinant in (26) can actually be reformulated as in (40).

The suitability of the above mentioned alternatives of the frequency equation from the point of view of numerical calculations will be comparatively considered in the following section.

5. NUMERICAL RESULTS

This section is devoted to the numerical evaluation of the expressions established in the preceding sections. For the numerical applications, following values are chosen for the physical data of the mechanical system in Figure 1. Two secondary spring–mass systems are considered, i.e., $s = 2$ is chosen. $\eta_1 = 0{\cdot}5$, $\alpha_{k_{e_1}} = \alpha_{m_{e_1}} = \eta_2 = \alpha_{k_{e_2}} = \alpha_{m_{e_2}} = 1$. The number of the modes n in the expansion (3) is chosen as 10.

The 12 dimensionless eigenfrequency parameters $\omega^* = \omega/\omega_0$ are collected in Table 1. The first column contains those ω^* values obtained from numerical solution of equation (26) via MATLAB. The figures in the second column

TABLE 1

The dimensionless eigenfrequency parameters ω^* of the mechanical system in Figure 1, with $s = 2$.

Results from equation (26)	Results from Matrix **A** (42)
0·848489	0·848489
0·995872	0·995872
4·131800	4·131800
22·171642	22·171642
61·729698	61·729690
120·926747	120·926750
199·869531	199·869530
298·565601	298·565600
416·995570	416·995571
555·170662	555·170661
713·081751	713·081752
890·735150	890·735151

TABLE 2

The first five dimensionless eigenfrequency parameters ω^* *of the system in Figure* 1, *with* $s = 1$

Mode no.	Results from [4]	Results from matrix **A** (42)
1	2·71227	2·712284
2	5·01578	50·015806
3	22·03966	22·039659
4	61·73024	61·730238
5	120·92111	120·921117

represent simply the square roots of the eigenvalues of the matrix **A** in (42), obtained also with MATLAB. The comparison of the values from both columns indicates clearly that the results of both alternative forms of the frequency equation are identical, as expected. This is nothing else but the numerical justification of the fact that both alternative forms of the frequency equation are actually identical. Thus, the arguments proposed in the present study have been confirmed.

In the example system investigated in reference [4], $s = 1$, $\eta_1 = 0{\cdot}75$, $\alpha_{k_{e_1}} = 3$ and $\alpha_{m_{e_1}} = 0{\cdot}2$ are chosen. In Table 2, the first five dimensionless eigenfrequencies ω^* are given. The values in the first column are the exact values from [4], whereas those in the second column are the square roots of the eigenvalues of matrix **A** in (42), with $n = 10$. It can be seen that the values in two columns are very close to each other, indicating the effectiveness of the given method.

It is quite instructive to report also on the experience gained during the numerical solution of the equation (26) with MATLAB. After defining the determinant expression as a MATLAB-function, the first three roots were found without any problems, by using the MATLAB-function *fzero.m*. On the other hand, problems were encountered in finding the fourth and the following roots because of the ill-conditioned nature of the expression, in spite of using the eigenvalues of the **A** matrix as the starting values for the *fzero.m* function. The ill-conditionedness of the present problem gets worse with increasing frequencies, observed as the slope of the frequency curve getting smaller near the roots. As a result, the *fzero.m* function could not perform well with its internal defined step sizes, so the *zoom.m* function of MATLAB had to be used to find the roots manually.

Since numerical difficulties have been encountered with only two spring–masses and in spite of the availability of the eigenvalues of **A** as precise starting values, it seems natural that much great numerical problems will occur in case of several spring–mass systems. On the other hand, the solution of the eigenvalue problem of the matrix **A** gives all the dimensionless frequency squares ω^{*2} simultaneously and without any difficulties. As a result, it can be concluded that solving the eigenvalue problem of the matrix **A** is a much better means compared to the one by one numerical search of the roots of equation (26).

6. CONCLUSIONS

This note is concerned with the natural vibration problem of a mechanical system, consisting of a clamped–free Bernoulli–Euler beam to which several spring–mass systems are attached in-span. Two alternative forms for the frequency equation are derived where both formulations are based on the discretization of the elastic beam by its first n eigenfunctions, according to the assumed modes method. One of the alternatives enables one to determine the eigenfrequency parameters via the eigenvalues of a special matrix, whereas the second alternative yields the eigenfrequency parameters as the roots of a determinantal equation.

ACKNOWLEDGMENT

The author acknowledges gratefully the contribution of Mr. H. Batan in deriving the first alternative form of the frequency equation.

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