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EXACT CONTROLLABILITY OF A LINEAR EULER–BERNOULLI PANEL

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The problem of control of flexural vibrations of a flexible space structure (such as solar cell array) modelled by a thin uniform rectangular panel is considered here. The flexural vibrations of such a panel satisfies the one dimensional fourth order Petrowsky equation or Euler–Bernoulli equation. The panel is held at one end by a rigid hub and the other end is free. By attaching the hub to one side of the panel the dynamics creates a non-standard hybrid system of equations. It is shown that the vibrations of the overall system can be driven to rest by means of an active boundary control force applied on the rigid hub only. Also an estimate of the minimum time of control is obtained. A closed form approximate result is constructed by Galerkin's residual technique to support and implement the method.

1. INTRODUCTION

Recently, studies in vibration control of mechanical systems have developed significantly. The three most common classes of vibration control are of passive, active, and of the hybrid type. Passive vibration control uses resistive devices that disturb vibration or absorb vibration energy. Active vibration control is similar, but involves the use of force actuators linked with external energy. Hybrid vibration control is a combination of passive approach with an active control. The vibrations of flexible (space) structures is a problem of dynamical system theory governed by partial differential equations. The dynamical behavior of many practical systems consist of two parts: coupled elastic part and rigid part, constituting the class of hybrid systems, such as solar cell arrays, space craft with flexible attachments or robots with flexible links. For such systems generally, the situation arises where it is very difficult to apply the control force on the free end of the elastic part to obtain a good performance of the overall system where as, application is very easy on the rigid part. These problems are very significant mathematically. A common approach in engineering is to decompose the vibrations into normal modes and retain the first few modes to reduce the problem to a finite dimensional state space representation (cf. [1–3]). In the literature, exact controllability of a system is stated as follows: Let a system be disturbed from some initial state; find a suitable control function which drives the system to rest or to some desired state at finite time $T > 0$. For studying exact controllability, a rigorous method, the "Hilbert Uniqueness Method" (HUM), which avoids normal modes altogether, has been introduced by Lions [4] for distributed systems governed by second order wave equations and the fourth order Petrowsky equation with standard (Dirichlet or Neumann) boundary conditions. Considerable mathematical literature exists on the subject.

The question of exact controllability of the Euler–Bernoulli beam clamped at one end, with boundary control at the free end has been studied theoretically by Littman and Markus [5]. The idea was extended by Markus and You [6] to obtain an approximate control system. The problem of controllability and stability for serially connected beams with actuators and sensors co-located at nodal points has been discussed by Chen *et al*. [7]. Morgül [8] treated the case of controllability of Euler–Bernoulli beams using the energy functional of the system. Nagaya [9] studied the problems of vibration control of flexible beams to cancel resonances subject to forced vibrations by applying the inertia force cancellation method. Chen *et al*. [10] established the stabilisation property of a coupled vibrator. In order to make it stable, a point stabiliser is installed in the middle of the span. All of these investigations have shown the ability to control and stabilise the vibrations of an elastic beam whose behavior is modelled by the Euler–Bernoulli equation, clamped at one end and free at the other end, except for feedback damping and control forces or torques, applied on the free end.

Here, the exact controllability problem of transverse vibrations of a (large) simple flexible space structure is studied, mathematically modelled by a one-dimensional fourth order Petrowsky equation or in this case the Euler–Bernoulli equation. It may however be mentioned that varieties of structures exist in practice with mechanical damping and possibly non-linear characteristics. Flexible space structures are usually hoisted at one end by a rigid hub which is assumed here to be capable of motion in the transverse direction. Installation of the movable hub at one end of the panel leads to a non-standard hybrid system. By applying an active control force on the hub, the vibrations of the system can be suppressed exactly when the motion is set from a given initial displacement and velocity along the length of the panel. That this is possible at time $T > T_0 = (4l^2/\pi)\sqrt{m/D}$ where m, l, D are defined in section 2, is proved here theoretically by HUM.

The mathematical formulation for the active vibration control problem is described in section 2. Subsequently, in section 3, the systematic method for exactly controlling vibrations of the overall system is discussed. In section 4, an approximate closed form solution together with approximate boundary control is obtained by Galerkin's residual technique. Finally conclusions are drawn in section 5 which summarise practical aspects of the problem.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Referring to the schematic Figure 1, the simplest type of structure consisting of a uniform rectangular flexible panel of unit width and length l, with a rigid hub of mass m_h at one end, the other end being totally free is considered. Ones objective

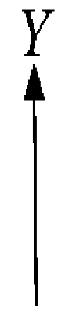

Figure 1. Schematic of the rigid hub and the panel.

is to control the vibrations of the panel exactly by applying suitable control force $Q(t)$ only on the rigid hub, in some finite time interval $[0, T]$, when it is initially set in motion. If $y_h(t)$ be the transverse displacement of the rigid hub and $y_p(x, t)$ that of the panel at the position x along the span of the panel relative to the hub at time t, then the total transverse deflection can be written as

$$y(x, t) = y_h(t) + y_p(x, t), \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T. \tag{1}$$

By assuming that the vibrations undergo only small deformations, $|y(x, t)| \ll l$ and $|(\partial y/\partial x)(x, t)| \ll 1$, and neglecting the gravitational effect and rotatory inertia of the panel cross-sections, $y(x, t)$ satisfies the fourth order Petrowsky equation

$$m(\partial^2 y/\partial t^2)(x, t) + D(\partial^4 y/\partial x^4)(x, t) = 0, \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T. \tag{2}$$

where $D = \frac{1}{12}Eh^3(1 - \nu^2)^{-1}$. The constants D, E, ν, m and h are the flexural rigidity, the Young's modulus, the Poisson's ratio, the mass per unit length and the thickness of the panel respectively.

At the hub end $x = 0$ where the control force $Q(t)$ is applied, the hub dynamics yields the differential equation

$$m_h(\partial^2 y_h/\partial t^2)(t) + D(\partial^3 y_p/\partial x^3)(0, t) + Q(t) = 0.$$

Since $y_p(0, t) = 0$, $y(0, t) = y_h(t)$ and hence the above equation becomes

$$(\partial^3 y/\partial x^3)(0, t) + \alpha(\partial^2 y/\partial t^2)(0, t) + \lambda Q(t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{3}$$

where $\alpha = m_h/D$ and $\lambda = 1/D$. Assuming at $x = 0$, there will be no rotational deflection of the panel relative to the hub (i.e., the hub is built into the panel at $x = 0$), $(\partial y_p/\partial x)(0, t) = 0$, implying

$$(\partial y/\partial x)(0, t) = 0, \qquad 0 \leqslant t \leqslant T. \tag{4}$$

Since the panel is assumed to be free at $x = l$, at this end

$$(\partial^2 y/\partial x^2)(l, t) = 0, \qquad (\partial^3 y/\partial x^3)(l, t) = 0, \qquad 0 \leqslant t \leqslant T. \tag{5}$$

Let the panel initially vibrate with arbitrary initial values

$$y(x, 0) = y_0(x), \qquad (\partial y/\partial t)(x, 0) = y_1(x), \qquad 0 \leqslant x \leqslant l, \tag{6}$$

satisfying the corresponding homogeneous boundary conditions (3–5). Therefore, to suppress vibrations of the panel as described above, a suitable control $Q(t)$ must be selected and the initial boundary value problem defined in equations (2–6) satisfied.

3. EXACT CONTROLLABILITY

To study the exact controllability at some finite time $T > 0$, the present aim is to find $Q(t)$ appropriately such that it drives the system (2–6) to rest at time $t = T$. Then the solution of system (2–6) must satisfy the desired final state

$$y(x, T) = 0, \qquad (\partial y/\partial t)(x, T) = 0. \tag{7}$$

Following exact controllability theory, $Q(t)$ is selected proportional to $\theta(0, t)$ say,

$$Q(t) = \beta_0 \theta(0, t) \tag{8}$$

where $\theta(x, t)$ is the solution of corresponding adjoint system of (2–6):

$$m(\partial^2\theta/\partial t^2)(x, t) + D(\partial^4\theta/\partial x^4)(x, t) = 0, \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T,$$

$$\theta(x, 0) = \theta_0(x), \qquad (\partial\theta/\partial t)(x, 0) = \theta_1(x), \qquad 0 \leqslant x \leqslant l, \tag{9}$$

$$(\partial^3\theta/\partial x^3)(0, t) + \alpha(\partial^2\theta/\partial t^2)(0, t) = 0, \qquad (\partial\theta/\partial x)(0, t) = 0, \qquad 0 \leqslant t \leqslant T,$$

$$(\partial^2\theta/\partial x^2)(l, t) = 0, \qquad (\partial^3\theta/\partial x^3)(l, t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{10}$$

under the assumptions $\theta_0(0) = 0$, $\theta_1(0) = 0$ and β_0 being an arbitrary positive constant independent of t. As the action of the control force $Q(t)$ in (8) depends on the solution of the adjoint system (9), the suppression of vibration at time T (exact control) entails coupling of these two systems. In this context, one should note that the adjoint system (8) is energy conserving (see Appendix), while the original system decays during the control process.

Now for given $\{\theta_0, \theta_1\}$ (though unknown till now), in a suitable Hilbert space (see Lions and Magenes [11]) the system (9) has a unique solution $\theta(x, t)$ for $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$. On the other hand, marching backward in time the system (2–5) and (7) must have a solution $y(x, t)$ depending on the initial values $\{\theta_0, \theta_1\}$ of (9), since $\theta(0, t)$ explicitly occurs in $Q(t)$ of the boundary condition (3). To obtain $Q(t)$, the major task is to find the initial states $\{\theta_0, \theta_1\}$, so that one can first acquire the solution $\theta(x, t)$ of (9). Supposing that one knows the values of $\{\theta_0, \theta_1\}$, the solution $y(x, t)$ can be obtained for the backward system (2–5) and (7). Hence one can then easily obtain $\{y_0, y_1\}$ as defined in (6). In other words, there is a mapping Λ from the set of values $\{\theta_0, \theta_1\}$ into $\{y_0, y_1\}$ which eventually can be written uniquely as

$$\Lambda\{\theta_0, \theta_1\} = \{y_1, -y_0\}. \tag{11}$$

It can be shown following Lions [4], that Λ is an isomorphism for $T > T_0$ (see Appendix), where $T_0 = (4l^2/\pi)\sqrt{m/D}$ being the estimated least time to control. Therefore one can uniquely invert Λ from (11) to find $\{\theta_0, \theta_1\}$ for given $\{y_0, y_1\}$ in some appropriate Hilbert space. Consequently the solution $\theta(x, t)$ of (9) can be found and hence the control $Q(t)$ by (8). Knowing $Q(t)$ one is then able to solve the backward system (2–5) and (7), and ultimately the exact controllability result immediately follows from (7).

An interpretation of T_0 is as follows. The frequency of vibrations of a uniform panel (or bar) fixed (clamped) at $x = 0$ and free at $x = l$ is $(1/2\pi l^2)\sqrt{(D/m)}p^2$, where p is a root of the equation (cf. [13])

$$\cos p \cosh p + 1 = 0. \tag{12}$$

The roots of (12) are approximately given by $p_1 = 1{\cdot}875$, $p_2 = 4{\cdot}694$ etc. If τ be the time period of the first (gravest) mode of vibration then $\tau = (2\pi l^2/p_1^2)\sqrt{m/D}$. Therefore $T_0/\tau = 2p_1^2/\pi^2 = 0{\cdot}71$ (approximately). Hence T_0 is somewhat less than τ. The deflation in time period may be ascribed to the compliant motion of the end $x = 0$ towards the equilibrium position $y = 0$.

4. SPACE-TIME GALERKIN APPROXIMATION

In this section a closed form approximation of the vibration control system (2–6) is constructed. One proceeds by constructing an admissible approximate displacement function as well as the approximate boundary control force that satisfy the final conditions (7) as closely as possible. The present approach is on the basis of Galerkin's weighted residual method [cf. 14]. For this it is convenient to treat the above boundary value problem in two steps.

In the first step, the approximate displacement for the system (2–6) is written as a superposition of polynomial shape functions of the following type:

$$y(x, t) = \sum_{i=1}^{n+1} f_i(x)\phi_i(t), \qquad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{13}$$

where

$$f_i = \sum_{j=1}^{p+1} a_j^i \left(\frac{x}{l}\right)^{j-1}$$

satisfy the homogeneous boundary conditions corresponding to equations (3–5) for $i = 1, 2, \ldots, n$, while f_{n+1} satisfies the non-homogeneous boundary conditions (3–5). The coefficient functions $\phi_i(t)$ for $i = 1, 2, \ldots, n$ are to be determined for finding the approximate solution of the system (2–6) by the Galerkin technique, while $\phi_{n+1}(t)$ on account of (3) is given by

$$\phi_{n+1}(t) = -l^3[\alpha(\partial^2 y/\partial t^2)(0, t) + \lambda Q(t)]. \tag{14}$$

In practical procedure, one may assume that $y_0(x)$ and $y_1(x)$ are approximated by suitable polynomials (by measurement at suitable discrete points along the length of the panel) satisfying the corresponding homogeneous boundary conditions (3–5). The functions thus become candidates for f_i and one can assume

$$f_1(x) = y_0(x), \qquad f_2(x) = y_1(x). \tag{15}$$

In addition to these, another function $f_3(x)$ can be taken which is a simple monotonic function, since with increasing time the equilibrium position $y = 0$ is approached. Thus

$$f_3(x) = x^2/l^2 - \tfrac{1}{2}x^4/l^4 + \tfrac{1}{5}x^5/l^5. \tag{16}$$

The last function $f_4(x)$ (with $n = 3$) is similarly taken as

$$f_4(x) = -\tfrac{1}{4}x^2/l^2 + \tfrac{1}{6}x^3/l^3 - \tfrac{1}{24}x^4/l^4. \tag{17}$$

With the above remarks, substituting (13) into the vibration equation (2), the integral of the weighted residue (with weight f_i) over $[0, l]$ set equal to zero:

$$\int_0^l f_i(x)(m\,\partial^2/\partial t^2 + D\,\partial^4/\partial x^4) \sum_{j=1}^{n+1} f_j(x)\phi_j(t)\,\mathrm{d}x = 0, \qquad (i = 1, 2, \ldots, n),$$

yields the matrix equation

$$\mathbf{A}\ddot{\mathbf{\Psi}} + \mathbf{B\Psi} + \mathbf{B}(\mathbf{\Phi}^0 + t\mathbf{\Phi}^1) + \mathbf{E}\ddddot{\mathbf{\Psi}} = \lambda[\mathbf{C}\ddot{Q}(t) + \mathbf{D}Q(t)]. \tag{18}$$

Similarly, the initial conditions (6) reduce to

$$\mathbf{\Psi}(0) = 0, \qquad \dot{\mathbf{\Psi}}(0) = 0, \tag{19}$$

where

$$\mathbf{\Phi}(t) = \mathbf{\Psi}(t) + \mathbf{\Phi}^0 + t\mathbf{\Phi}^1, \tag{20}$$

and $(\dot{\ })$ represents the time derivative. The square matrices **A**, **B**, **E** and the column vectors $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{\Phi}^0$, $\mathbf{\Phi}^1$, **C**, **D** are defined as

$$\mathbf{A} = \Bigg[ml \sum_{k=1}^{p+1}\sum_{s=1}^{p+1} \frac{a_k^i a_s^j}{s+k-1} - \alpha a_1^i D \sum_{k=1}^{p+1}\sum_{s=5}^{p+1} a_k^i a_s^{n+1} \frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5} \Bigg]_{n\times n},$$

$$\mathbf{B} = \left[\frac{D}{l^3} \sum_{k=1}^{p+1}\sum_{s=5}^{p+1} a_k^i a_s^j \frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5} \right]_{n\times n},$$

$$\mathbf{E} = \left[-ml^4 \alpha a_1^i \sum_{k=1}^{p+1}\sum_{s=1}^{p+1} \frac{a_k^i a_s^{n+1}}{s+k-1} \right]_{n\times n},$$

$$\mathbf{C} = \left[ml^4 \sum_{k=1}^{p+1}\sum_{s=1}^{p+1} \frac{a_k^i a_s^1}{s+k-1} \right]_{n\times 1},$$

$$\mathbf{D} = \left[D \sum_{k=1}^{p+1}\sum_{s=5}^{p+1} a_k^i a_s^1 \frac{(s-1)(s-2)(s-3)(s-4)}{s+k-5} \right]_{n\times 1},$$

$$\mathbf{\Phi} = \mathbf{\Phi}(t) = [\phi_i(t)]_{n\times 1}, \qquad \mathbf{\Psi} = \mathbf{\Psi}(t) = [\psi_i(t)]_{n\times 1},$$
$$\mathbf{\Phi}^0 = [\phi_i(0)]_{n\times 1}, \qquad \mathbf{\Phi}^1 = [\dot{\phi}_i(0)]_{n\times 1}. \tag{21}$$

In the next step, the Galerkin's weighted residual is repeated in the time domain for the system (18) with homogeneous initial conditions (19). As a tool, the approximation of $\mathbf{\Psi}(t)$ is written as

$$\mathbf{\Psi}(t) = \sum_{k=1}^{m^*} \mathbf{X}_k \left(\frac{t}{T}\right)^{k+1}, \tag{22}$$

where each $\mathbf{X}_k$ $(k = 1, 2, \ldots, m)$ is an $n \times 1$ column vector. The determination of $\mathbf{X}_k$ yields the approximate solution $\mathbf{\Psi}(t)$ of (22) and hence that of $\mathbf{\Phi}(t)$ from (20). Proceeding as in the previous step, one obtains

$$\sum_{k=1}^{m} \mathbf{M}_{lk} \mathbf{X}_k = -\xi_l \mathbf{B}\mathbf{\Phi}^0 - \eta_l \mathbf{B}\mathbf{\Phi}^1 + \mathbf{H}_l, \qquad l = 1, 2, \ldots, m^*, \tag{23}$$

where

$$\mathbf{M}_{lk} = \left[\frac{(k+1)k}{T(k+l+1)}\mathbf{A} + \frac{T}{k+l+3}\mathbf{B} + \frac{(k+1)k(k-1)(k-2)}{T^3(k+l-1)}\mathbf{E}\right],$$

$$\xi_l = \frac{T}{l+2}, \qquad \eta_l = \frac{T^2}{l+3}, \qquad \mathbf{H}_l = \lambda \int_0^T [\mathbf{C}\ddot{Q}(t) + \mathbf{D}Q(t)]\left(\frac{t}{T}\right)^{l+1} \mathrm{d}t. \tag{24}$$

To solve $\mathbf{X}_k$ from (23), the matrix $\mathbf{M} = [\mathbf{M}_{lk}]$ has to be inverted. By supposing that $\mathbf{M}$ is non-singular and $\mathbf{F} = [\mathbf{F}_{kl}]$ the inverse of $\mathbf{M}$, then from (23) one has the scheme

$$\mathbf{X}_k = -\sum_{l=1}^{m} \mathbf{F}_{kl}\xi_l \mathbf{B}\mathbf{\Phi}^0 - \sum_{l=1}^{m} \mathbf{F}_{kl}\eta_l \mathbf{B}\mathbf{\Phi}^1 + \sum_{l=1}^{m} \mathbf{F}_{kl}\mathbf{H}_l, \qquad k = 1, 2, \ldots, m^*. \tag{25}$$

To obtain control force $Q(t)$, one now solves the adjoint system (9) by a similar Galerkin residual technique. This leads to the scheme

$$\mathbf{Y}_k = -\sum_{l=1}^{m} \mathbf{F}_{kl}\xi_l \mathbf{B}\mathbf{\Theta}^0 - \sum_{l=1}^{m} \mathbf{F}_{kl}\eta_l \mathbf{B}\mathbf{\Theta}^1, \qquad k = 1, 2, \ldots, m^*, \tag{26}$$

where the vectors $\mathbf{Y}_k$ are corresponding to $\mathbf{X}_k$ $(k = 1, 2, \ldots, m^*)$ in the form (22) and the vectors $\mathbf{\Theta}^0 = [\theta_i(0)]_{n\times 1}$, $\mathbf{\Theta}^1 = [\dot{\theta}_i(0)]_{n\times 1}$ corresponding to $\mathbf{\Phi}^0$, $\mathbf{\Phi}^1$ in (20), for the adjoint system (9). Knowing $\mathbf{\Theta}^0$ and $\mathbf{\Theta}^1$ one can easily obtain $\mathbf{Y}_k$ from (26). But the control force $Q(t)$ following (13), (20) and (22), is taken as

$$Q(t) = \beta_0\theta(0, t) = \beta_0\mathbf{I}_1\mathbf{\Theta}(t) = \beta_0\mathbf{I}_1\left(\sum_{k=1}^{m} \mathbf{Y}_k\left(\frac{t}{T}\right)^{k+1} + \mathbf{\Theta}^0 + t\mathbf{\Theta}^1\right), \tag{27}$$

$\mathbf{\Theta}(t)$ being the corresponding term of $\mathbf{\Phi}(t)$ in the adjoint system and $\mathbf{I}_1 = [1, 0, 0, \ldots, 0]_{1\times n}$. Since the $\mathbf{Y}_k$'s are dependent on $\mathbf{\Theta}^0$, $\mathbf{\Theta}^1$ in (26), it follows that $\mathbf{\Theta}^0$, $\mathbf{\Theta}^1$ will explicitly occur in $Q(t)$. Now substituting $Q(t)$ from (27) into $\mathbf{H}_l$ in (24), one obtains $\mathbf{H}_l$ in terms of $\mathbf{\Theta}^0$ and $\mathbf{\Theta}^1$ which then helps to obtain $\mathbf{X}_k$ from (25), in terms of $\mathbf{\Theta}^0$, $\mathbf{\Theta}^1$, $\mathbf{\Phi}^0$, $\mathbf{\Phi}^1$. But by the Galerkin technique, the conditions $y(x, T) = 0$, $(\partial y/\partial t)(x, T) = 0$ yield

$$\mathbf{A}^*\mathbf{\Phi}(T) = \left[\alpha \sum_{i=1}^{n} a_1^i \ddot{\phi}_i(T) + \lambda Q(T)\right]\mathbf{C}^*,$$

$$\mathbf{A}^*\dot{\mathbf{\Phi}}(T) = \left[\alpha \sum_{i=1}^{n} a_1^i \dddot{\phi}_i(T) + \lambda \dot{Q}(T)\right]\mathbf{C}^*, \tag{28}$$

where $\mathbf{A}^*$ corresponds to $\mathbf{A}$ in (24) without the second term and the factor m. Similarly $\mathbf{C}^*$ corresponds to $\mathbf{C}$ without the factor m. With the help of the relations (25–27), (28) ultimately leads to the matrix equations of the form

$$\mathbf{P\Theta}^0 + \mathbf{Q\Theta}^1 = \mathbf{R\Phi}^0 + \mathbf{S\Phi}^1, \qquad \mathbf{U\Theta}^0 + \mathbf{V\Theta}^1 = \mathbf{W\Phi}^0 + \mathbf{Z\Phi}^1, \tag{29}$$

where the entries of all $n \times n$ square matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{Z}$ depend on the entries of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$. Solving $\mathbf{\Theta}^0$ and $\mathbf{\Theta}^1$ from (29), with the help of initial conditions of the control problem, the vectors $\mathbf{Y}_k$ can be obtained from the equation (26) and subsequently the approximate control force $Q(t)$ from (27) to compute $\mathbf{H}_l$ from (24) and then $\mathbf{X}_k$ from (25). Hence from (22), $\mathbf{\Psi}(t)$ can be obtained which finally gives the approximate shape function (13). Since the scheme is direct and low values of n and m^* are normally needed, computation proceeds very quickly.

The model parameters for numerical computation for the control problem are chosen as follows (in MKS units): length of the panel $l = 3{\cdot}6$ m, mass per unit length of the panel $m = 5{\cdot}9$ kg/m, Poisson ratio $\nu = 0{\cdot}33$, rigidity $D = 6{\cdot}9$ kg m^3/s^2, mass of the hub $m_h = 12{\cdot}2$ kg.

For this panel $T_0 = 15{\cdot}26$ s. Two examples of initial conditions are considered. In the first

$$y_0(x) = -\frac{1}{200} + \frac{x^2}{l^2} - 15\frac{x^4}{l^4} + \frac{67}{2}\frac{x^5}{l^5} - 27\frac{x^6}{l^6} + \frac{53}{7}\frac{x^7}{l^7}, \qquad y_1(x) = 0, \tag{30}$$

in which $y_0(x)$ has a wavy shape and in the second,

$$y_0(x) = \frac{1}{100}\left(1 + \frac{x^2}{l^2} + 2\frac{x^4}{l^4} - \frac{14}{5}\frac{x^5}{l^5} + \frac{x^6}{l^6}\right), \qquad y_1(x) = \frac{1}{5}\left(\frac{x^4}{l^4} - \frac{6}{5}\frac{x^5}{l^5} + \frac{2}{5}\frac{x^6}{l^6}\right), \tag{31}$$

where a monotonic velocity is imparted with a small monotonic displacement. Applying the above computational scheme with $n = 2$, $m^* = 4$ and $n = 3$, $m^* = 3$ respectively it is observed that the Galerkin approximation yields increasingly better results for T higher than 15·26. The above results for the dynamic deflection and the control force for the first example with $T = 20$ s are presented

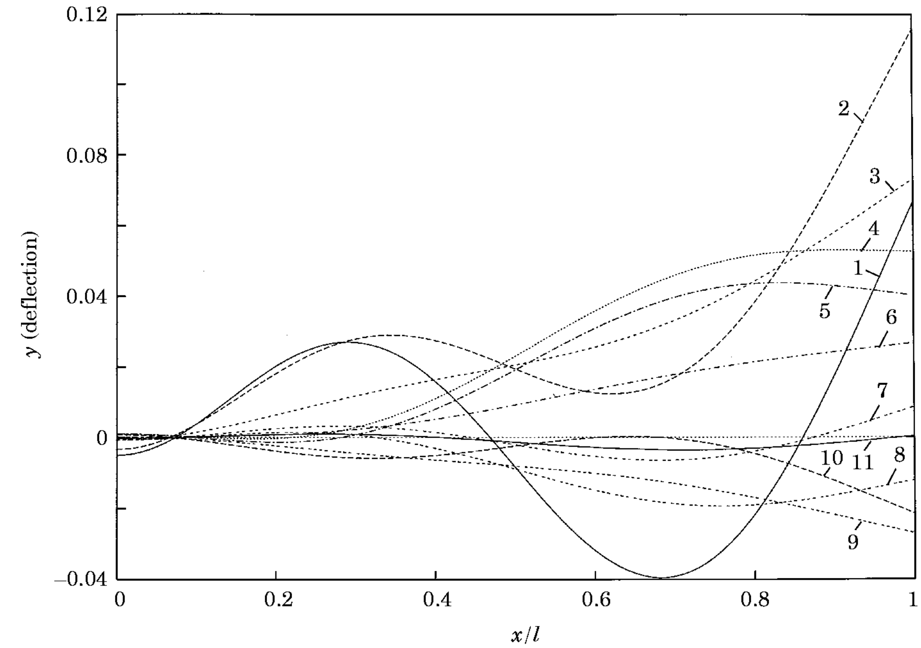

Figure 2. The approximate deflections of the panel along the length with different time for $T = 20$. Numbers 1–11 respectively are the approximate positions of the panel at the times $t = 0$, $t = 0{\cdot}1T$, $t = 0{\cdot}2T$, $t = 0{\cdot}3T$, $t = 0{\cdot}4T$, $t = 0{\cdot}5T$, $t = 0{\cdot}6T$, $t = 0{\cdot}7T$, $t = 0{\cdot}8T$, $t = 0{\cdot}9T$, $t = T$.

in Figures 2 and 3, while those for the second example with $T = 30$ s are presented in Figures 4 and 5 respectively. The velocities are lower by an order of magnitude and are not drawn. Finally, one observes that for very accurate results, even for low values of T (higher than T_0), one may need a full, space-time Galerkin finite element technique. Such a technique will however need greater computational time.

5. CONCLUSIONS

The exact controllability of transverse vibrations of a flexible panel attached to a rigid hub at one end and free at the other has been established. The authors have shown that by applying a suitable boundary control at the hub end only, one can exactly control the panel vibrations following prescribed initial displacement and velocity, without applying constraints at the free end. In this context, the minimum time T_0 of controllability for which the result is valid has also been estimated. The results are also valid for an Euler–Bernoulli beam held by a rigid hub at one end. In this case EI, the flexural rigidity of the beam replaces D of the panel. The analytical treatment of the problem is supported by closed form fast numerical result obtained by Galerkin's residual technique. The basic principle of exact controllability as treated herein is the Hilbert Uniqueness Method (HUM) due to Lions [4]. Explicit details of the method including various types of problems such as the wave equation, Petrowsky equation with standard Dirichlet or Neumann boundary conditions, have been discussed.

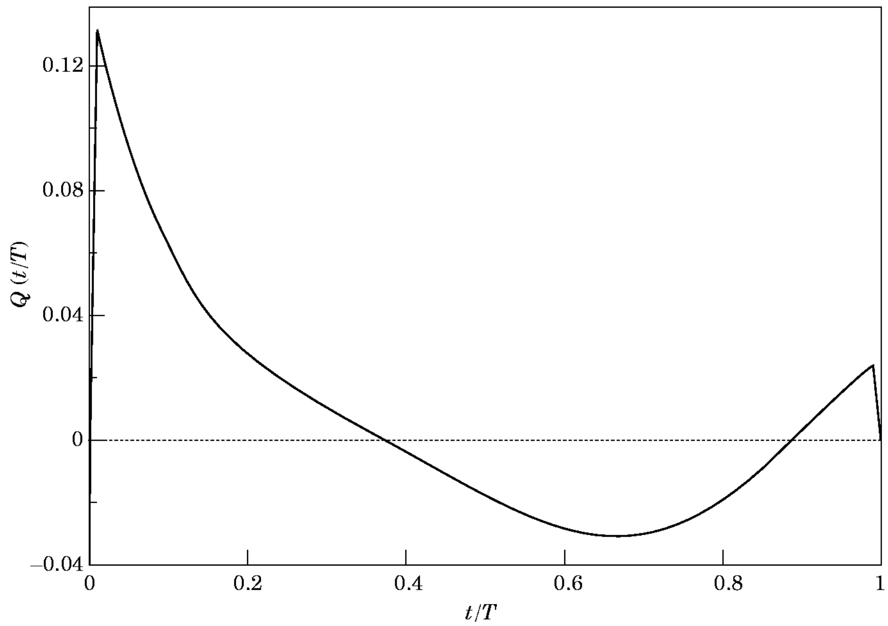

Figure 3. The approximate response of the control with time for $T = 20$. Note that application and removal of the control force at initial and final times is sudden.

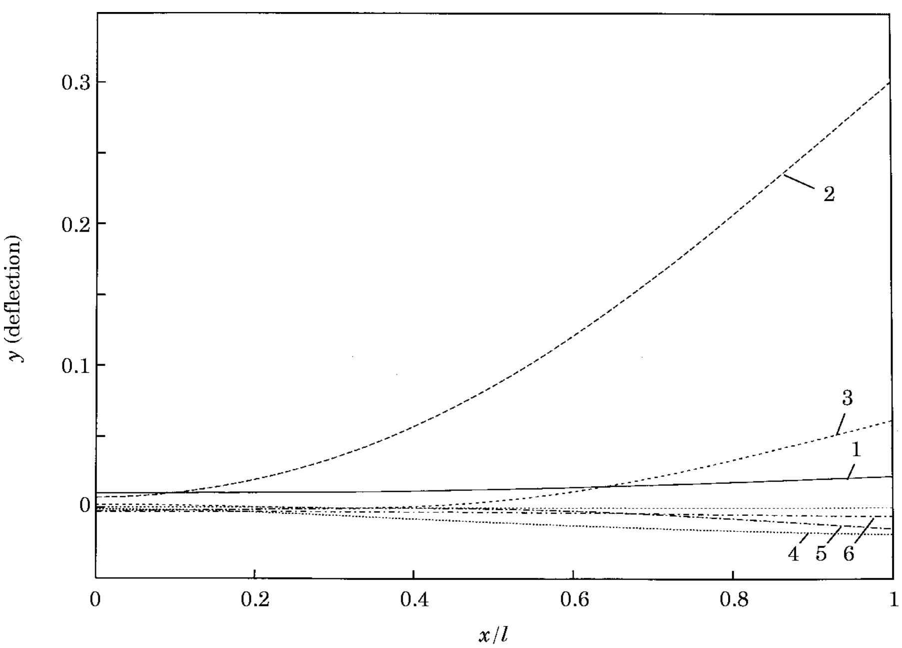

Figure 4. The approximate deflections of the panel along the length with different time for $T = 30$. Numbers 1–6 respectively are the approximate positions of the panel at the times $t = 0$, $t = 0{\cdot}2T$, $t = 0{\cdot}4T$, $t = 0{\cdot}6T$, $t = 0{\cdot}8T$, $t = T$.

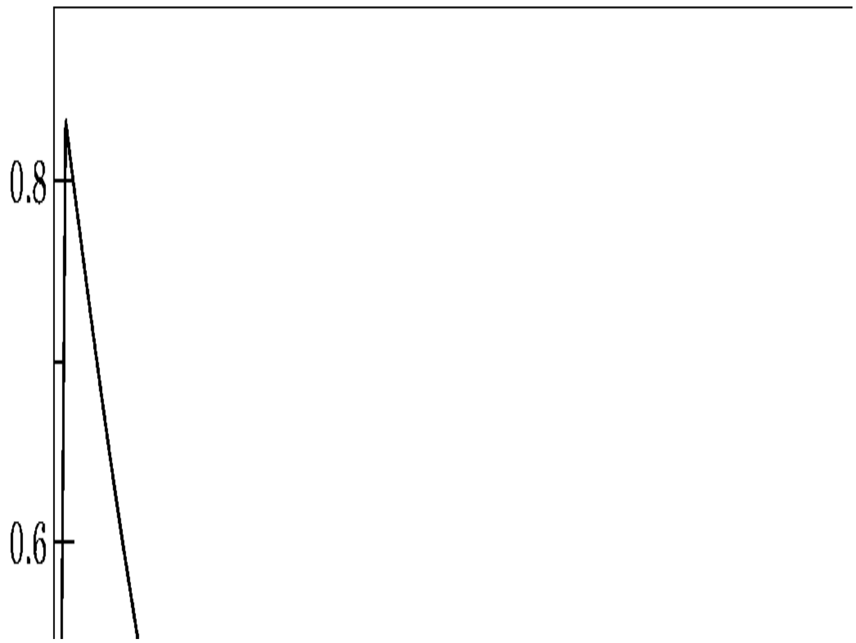

Figure 5. The approximate response of the control force with time for $T = 30$. Note that application and removal of the control force at initial and final times is sudden.

Since the theory is exact, the parameter involved should be known as accurately as possible. The initial displacement and velocity $y_0(x)$ and $y_1(x)$ when sufficiently smooth need be measured at a limited number of points along the length of the panel and approximated by polynomial functions.

Of the other parameters required in the theory D may be determined from some dynamical test, while m, m_h and l can be ascertained quite accurately. Nevertheless, approximations and uncertainties in measurements do pose the question of robustness of the exact theory and may need to be addressed theoretically. One may however note that dissipation of energy takes place in actual systems with significant material damping in the panel, and frictional and other losses in the hub rendering the system asymptotically stable. Preliminary investigations have shown that while T_0 increases slightly due to the former, damping proportional to velocity of the latter induces strong stability (cf. Gorain and Bose [15]). The level of performances should thus be good under these circumstances. In the earlier literature (cf. [1–3]) using modal decomposition followed by finite state representation, one notes that uncertainties are introduced as Gaussian white noise followed by Kalman filtering.

The dynamic behaviour of many other practical systems such as space craft with flexible attachments, robots with flexible links and certain parts of many mechanical systems obey more complicated hybrid systems of equations. For such systems the situation generally occurs when it is very difficult or undesirable to apply boundary control at the free end and where a good performance is needed from the whole system. The authors discussion in this paper has covered these types of active vibration control problems and produced an appropriate exact controllability of the overall system. For other such systems, one can produce similar exact controllability systems followed by corresponding control.

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APPENDIX

A1. CONSERVATION OF TOTAL ENERGY

In association with each solution of (9), the total energy $E(t)$ at time t is defined by

$$E(t) = \tfrac{1}{2}\int_0^l \left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 \right] \mathrm{d}x + \tfrac{1}{2}m_h\left[\frac{\partial\theta}{\partial t}(0, t)\right]^2. \tag{A.1}$$

Differentiating with respect to t and replacing $m\,\partial^2\theta/\partial t^2$ by $-D\,\partial^4\theta/\partial x^4$, (32) then leads to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = D\int_0^l \frac{\partial}{\partial x}\left(\frac{\partial^2\theta}{\partial t\,\partial x}\frac{\partial^2\theta}{\partial x^2} - \frac{\partial\theta}{\partial t}\frac{\partial^3\theta}{\partial x^3}\right)\mathrm{d}x + m_h\frac{\partial\theta}{\partial t}(0, t)\frac{\partial^2\theta}{\partial t^2}(0, t).$$

Integrating by parts and applying the boundary conditions of (9), one obtains $\mathrm{d}E/\mathrm{d}t = 0$ which implies

$$E(t) = \text{constant} = E(0) \qquad \text{for} \quad t \geqslant 0. \tag{A.2}$$

where

$$E(0) = \tfrac{1}{2}\int_0^l \left[m\left(\frac{\partial\theta}{\partial t}(x, 0)\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}(x, 0)\right)^2 \right]\mathrm{d}x, \tag{A.3}$$

by (10). Thus the adjoint system is energy conserving.

A2. ESTIMATE OF T_0

By multiplying the first equation of (9) by $(l - x)\,\partial\theta/\partial x$ and integrating by parts over $[0, l] \times [0, T]$, and using the boundary conditions of (9):

$$m\int_0^l \left[(l - x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}\right]_0^T \mathrm{d}x - \tfrac{1}{2}\int_0^l\int_0^T (l - x)\frac{\partial}{\partial x}\left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2\right]\mathrm{d}x\,\mathrm{d}t$$

$$+ D\int_0^l\int_0^T \frac{\partial\theta}{\partial x}\frac{\partial^3\theta}{\partial x^3}\,\mathrm{d}x\,\mathrm{d}t = 0$$

leads to

$$\frac{l}{2}\int_0^T \left[m\left(\frac{\partial\theta}{\partial t}(0, t)\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}(0, t)\right)^2 \right]\mathrm{d}t$$

$$\geqslant \tfrac{1}{2}\int_0^l\int_0^T \left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\left(\frac{\partial^2\theta}{\partial x^2}\right)^2 \right]\mathrm{d}t - m\int_0^l \left[(l - x)\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial t}\right]_0^T \mathrm{d}x. \tag{A.4}$$

Since $\partial\theta/\partial x(0, t) = 0$, from Wirtinger's inequality (cf. [16]), one can write

$$\int_0^l \left(\frac{\partial\theta}{\partial x}\right)^2 \mathrm{d}x \leqslant \frac{4l^2}{\pi^2} \int_0^l \left(\frac{\partial^2\theta}{\partial x^2}\right)^2 \mathrm{d}x. \tag{A.5}$$

On the other hand, if one sets $X = m \int_0^l (l - x)\, \partial\theta/\partial x\, \partial\theta/\partial t\, \mathrm{d}x$ then one notes that by inequality (A.5) and energy equation (A.1),

$$|X| \leqslant m \int_0^l |l - x| \left|\frac{\partial\theta}{\partial x}\right| \left|\frac{\partial\theta}{\partial t}\right| \mathrm{d}x$$

$$\leqslant \frac{l^2}{\pi} \sqrt{\frac{m}{D}} \int_0^l \left[m\left(\frac{\partial\theta}{\partial t}\right)^2 + D\,\frac{\pi^2}{4l^2}\left(\frac{\partial\theta}{\partial x}\right)^2 \right] \mathrm{d}x \leqslant \frac{2l^2}{\pi} \sqrt{\frac{m}{D}}\, E(t).$$

By conservation of energy (A.1)

$$|X|_0^T \leqslant \frac{2l^2}{\pi} \sqrt{\frac{m}{D}}\, [E(T) + E(0)] = \frac{4l^2}{\pi} \sqrt{\frac{m}{D}}\, E(0). \tag{A.6}$$

Introducing (A.1) and (A.6) into (A.4), one has therefore

$$\tfrac{1}{2}(ml + m_h) \int_0^T \left(\frac{\partial\theta}{\partial t}(0, t)\right)^2 \mathrm{d}t + \frac{Dl}{2} \int_0^T \left(\frac{\partial^2\theta}{\partial x^2}(0, t)\right)^2 \mathrm{d}t \geqslant \left(T - \frac{4l^2}{\pi}\sqrt{\frac{m}{D}}\right) E(0). \tag{A.7}$$

Again by Wirtinger's inequality, one can write

$$\int_0^T \theta^2(0, t)\, \mathrm{d}t \leqslant \frac{4T^2}{\pi^2} \int_0^T \left(\frac{\partial\theta}{\partial t}(0, t)\right)^2 \mathrm{d}t. \tag{A.8}$$

Therefore, if one defines a positive function $K(T)$ by

$$K(T) = \frac{\int_0^T \theta^2(0, t)\, \mathrm{d}t}{\tfrac{1}{2}(ml + m_h) \int_0^T \left(\frac{\partial\theta}{\partial t}(0, t)\right)^2 \mathrm{d}t + \frac{Dl}{2} \int_0^T \left(\frac{\partial^2\theta}{\partial x^2}(0, t)\right)^2 \mathrm{d}t}, \tag{A.9}$$

then by (A.9)

$$K(T) \leqslant \frac{\int_0^T \theta^2(0, t)\, \mathrm{d}t}{\tfrac{1}{2}(ml + m_h) \int_0^T \left(\frac{\partial\theta}{\partial t}(0, t)\right)^2 \mathrm{d}t} \leqslant \frac{8T^2}{\pi^2(ml + m_h)}, \tag{A.10}$$

which is bounded above for all finite T. Therefore, it follows from (A.7) and (A.9) that

$$\int_0^T \theta^2(0, t)\, \mathrm{d}t \geqslant K(T)(T - T_0)E(0), \tag{A.11}$$

where

$$T_0 = (4l^2/\pi)\sqrt{m/D}. \tag{A.12}$$

Again from energy equation (A.1), it follows easily with the help of (A.8), the reverse inequality

$$\int_0^T \theta^2(0, t)\, \mathrm{d}t \leqslant \frac{8T^3}{\pi^2 m_h} E(0). \tag{A.13}$$

Hence one has from (A.11) and (A.13),

$$\frac{8T^3}{\pi^2 m_h} E(0) \geqslant \int_0^T \theta^2(0, t)\, \mathrm{d}t \geqslant K(T)(T - T_0)E(0). \tag{A.14}$$

In mathematical literature, equation (A.11) provides an observability result for positivity of the right side of equation (A.14). Thus the adjoint system is observable for $T > T_0$. Hence the vibrations of the original problem can be exactly controlled for $T > T_0$ (cf. [12]), where T_0 is given by (43). Therefore the time T_0 can be described as the estimated least time for exact controllability of this system.

A3. INVERSION OF Λ

By multiplying the first equation of (9) by y and (2) by θ, integrating over $[0, l] \times [0, T]$ and then subtracting:

$$m \int_0^l \int_0^T \frac{\partial}{\partial t}\left(\frac{\partial \theta}{\partial t} y - \theta \frac{\partial y}{\partial t}\right) \mathrm{d}x\, \mathrm{d}t + D \int_0^l \int_0^T \frac{\partial}{\partial x}\left(\frac{\partial^3 \theta}{\partial x^3} y - \theta \frac{\partial^3 y}{\partial x^3}\right) \mathrm{d}x\, \mathrm{d}t$$

$$- D \int_0^l \int_0^T \frac{\partial}{\partial x}\left(\frac{\partial^2 \theta}{\partial x^2}\frac{\partial y}{\partial x} - \frac{\partial \theta}{\partial x}\frac{\partial^2 y}{\partial x^2}\right) \mathrm{d}x\, \mathrm{d}t = 0.$$

Using the boundary, initial and final conditions of the two systems, a straightforward calculation gives

$$m \int_0^l (\theta_0 y_1 - \theta_1 y_0)\, \mathrm{d}x = \beta_0 \int_0^T \theta^2(0, t)\, \mathrm{d}t - D\alpha \int_0^T \left[\frac{\partial^2 \theta}{\partial t^2}(0, t) y(0, t)\right.$$

$$\left. - \theta(0, t)\frac{\partial^2 y}{\partial t^2}(0, t)\right] \mathrm{d}t$$

which finally yields

$$\int_0^l (\theta_0 y_1 - \theta_1 y_0)\,\mathrm{d}x = C\int_0^T \theta^2(0, t)\,\mathrm{d}t \tag{A.15}$$

by (7) and (10), where $C = \beta_0/m$. Hence according to Lions [4], the functional

$$\{\theta_0, \theta_1\} \to \langle \Lambda\{\theta_0, \theta_1\}, \{\theta_0, \theta_1\}\rangle$$

is obtained as

$$\langle \Lambda\{\theta_0, \theta_1\}, \{\theta_0, \theta_1\}\rangle = \int_0^l (\theta_0 y_1 - \theta_1 y_0)\,\mathrm{d}x = C\int_0^T \theta^2(0, t)\,\mathrm{d}t. \tag{A.16}$$

By Poincaré inequality (cf. Aubin [17]), one knows that the norm

$$\left\| \frac{\partial^2\theta}{\partial x^2} \right\|^2_{L^2[0,l]} = \int_0^l \left(\frac{\partial^2\theta}{\partial x^2}\right)^2 \mathrm{d}x$$

is equivalent to the norm of θ in $H^2[0, l]$, where

$$H^2[0, l] = \left\{ v \mid v \in L^2[0, l], \frac{\partial v}{\partial x} \in L^2[0, l], \frac{\partial^2 v}{\partial x^2} \in L^2[0, l] \right\}.$$

Inequality (A.14) implies that (A.16) defines a norm of $\{\theta_0, \theta_1\}$ which is equivalent to the norm on the Hilbert space $F = H^2[0, l] \times L^2[0, l]$ for $T > T_0$. From the inequality (A.11), one can use the Lax–Milgram theorem (cf. Aubin [17]) by virtue of (A.16) to conclude that Λ is an isomorphism from F to F' for $T > T_0$, F' being the dual space of F. This proves the inversion of Λ.