Journal of Sound and Vibration (1998) **217**(4), 757–765
Article No. sv981805

AN APPLICATION OF CHEBYSHEV'S MIN-MAX CRITERION TO THE OPTIMAL DESIGN OF A DAMPED DYNAMIC VIBRATION ABSORBER

E. PENNESTRÌ

Università degli Studi di Roma Tor Vergata,
Dipartimento di Ingegneria Meccanica, via della Ricerca Scientifica, 00133 *Roma, Italy*

(*Received* 15 *January* 1997, *and in final form* 12 *June* 1998)

In this paper an application of the Chebyshev's criterion to the optimal design of the damped dynamic vibration absorber is presented. The results are summarized in ready-to-use computational graphs. The reliability of the proposed method is demonstrated through a comparison of the numerical results obtained by different authors.

1. INTRODUCTION

Watts [1] in 1883 and Frahm [2] in 1909 reported on the first use of a dynamic vibration absorber. The range of industrial applications of such device is wide and includes the minimization of the rolling of ships, the damping of torsional vibration of internal combustion engines and the smoothing of vibrations in a barber cutter. The minimization of oscillations of tall buildings by means of dynamic vibrations absorbers is considered in reference [3].

In particular, (see Figure 1) a secondary mass (the absorber) and a spring–dashpot system can be effectively designed to reduce the vibrations of the primary mass. The inertial force exherted on the secondary mass counteracts the disturbing force on the primary mass. Thus the vibratory motion of such mass is reduced. The main advantage of adopting a dynamic vibration absorber is the significant reduction of a primary mass vibration amplitude. The drawbacks are:

the sensitivity to forcing frequency change;

the high stresses in the elastic element connecting the secondary to the primary mass;

the introduction of two other resonant frequencies in the neighborhood of the suppressed frequency.

The first mathematical theory on the passive dynamic vibration absorber is described in a paper by Ormondroyd and Den Hartog [4].

Moreover, in the book authored by Den Hartog [5] a closed form optimal solution for the system with viscous damping only between the primary and secondary mass (i.e.; $b_1 = 0$) can be found. In the following treatment such a model

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will be referred to as the *classic* system. Because of its elegance and historical importance, the design procedure proposed by Den Hartog is reported by the vast majority of textbooks on mechanical vibrations.

The usual goals pursued in this type of design are the minimization of vibration amplitudes of both primary and secondary masses and a reduction in sensitivity to the variation of the forcing frequency. According to Den Hartog [5], the *most favorable response curve* of the main mass has the same maximum amplitudes. This makes the displacement of the main mass less sensitive to variations of the force frequency.

Randall *et al*. [6] considered the more realistic situation of viscous damping between the two masses. They have shown that the optimal parameters for the damped linear system differ significantly from those obtained for the classic system. Moreover, reference [6] reports computational graphs for the optimal design of a linear damped vibration absorber. The graphs were obtained through numerical minimization of the maximum value of the primary mass vibration amplitude. However, when the procedure described by Randall *et al*. is adopted, the frequency response curve of the main mass may not have two maxima with the same amplitude. Other criteria of optimality have been introduced. For example, Soom and M.-S. Lee [7] and Jordanov and Cheshankov [8] applied non-linear programming techniques to obtain the optimal tuning and damping parameters for dynamic absorbers with both linear and non-linear springs. The flexibility of the solving tool allowed them to seek for objective functions other than the maximum displacement of the primary mass.

The minimization of the maximum displacement of the primary mass is usually set as an objective by those authors (e.g. [9–12] who prefer an algebraic approach to the design problem taken into account. The optimality criterion originally proposed by den Hartog, widely accepted in many industrial applications, will be maintained in the present derivations. Thus, when there is viscous damping on both masses, the design problem can be formulated as follows: Given a primary mass m_1, connected to the ground with a spring–dashpot element and subjected to the force $F_1 \sin \omega t$, compute the values of secondary mass m_2, stiffness k_2 and

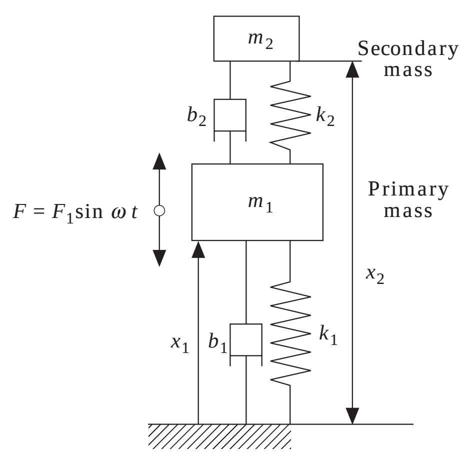

Figure 1. Damped vibration absorber system.

viscous damping c_2 such that the frequency response curve of the main mass has two equal maximum amplitudes.

Considering the requirements for the shape of such a response curve, it seems appropriate to solve the present design problem by making use of the mini-max Chebyshev's criterion. This will guarantee the uniqueness of the optimal solution (a question overlooked by many authors) and also provides the scheme for the analytical settings. The resulting system of non-linear equations will be solved numerically. However, for the ready use of the method herein presented, design charts are included at the end of this paper. These allow the user to bypass the numerical solution of non-linear equations. A comparison of the present method with those described in references [5–9] demonstrates the reliability of the proposed approach.

The nomenclature adopted in this paper is based on that of Randall *et al*. [6] and is listed in the Appendix.

2. THEORETICAL BASES: CHEBYSHEV'S THEOREM

The optimization tools used in this paper rest upon the following mathematical properties from the theory of best approximation of functions [13]. Let $f(x)$ be a continuous function in $[a, b]$ and $p(x)$ an approximant polynomial belonging to the class $\mathscr{P}_n$ of polynomials with degree less or equal to n. According to Chebyshev, the *best uniform approximation* is attained when the condition

$$\min_{p(x)\in\mathscr{P}_n} \max_{a \leqslant x \leqslant b} |f(x) - p(x)| \tag{1}$$

is satisfied. The solution to the minimization problem stated by (1) is unique and can be found considering the following theorem:

Let $f(x)$ be a continuous function in $[a, b]$ and $p(x)$ the best uniform approximant of degree n. Moreover, let

$$E_n = \max_{a \leqslant x \leqslant b} |f(x) - p(x)| \qquad \text{and} \qquad \epsilon(x) = f(x) - p(x).$$

There are at least $(n + 2)$ points $a \leqslant x_1 < x_2 \cdots < x_{n+2} \leqslant b$ where $\epsilon(x)$ assumes the values $\pm E_n$ and with alternating signs:

$$\epsilon(x_i) = \pm E_n, \qquad i = 1, 2, \ldots, n + 2;$$

$$\epsilon(x_i) = -\epsilon(x_{i+1}), \qquad i = 1, 2, \ldots, n + 1. \tag{2a, b}$$

Hence the best uniform approximant is completely characterized by the property of equioscillation at $(n + 2)$ points. This property is the basis of numerical schemes for computing the approximant.

3. DEDUCTION OF DESIGN EQUATIONS

For the two d.o.f. dynamic model shown in Figure 1, the normalized maximum vibration amplitude of the primary mass m_1 and of the maximum relative displacement between secondary and primary masses are, respectively,

$$\alpha = X_1/(F/k_1) = \sqrt{(1-\beta^2/T^2)^2 + 4(\zeta_2\beta/T)^2}/Z, \qquad \gamma_r = \beta^2/T^2 Z, \qquad (3, 4)$$

where

$$Z^2 = [\beta^4/T^2 - \beta^2/T^2 - \beta^2(1+\mu) - 4(\zeta_1\zeta_2\beta^2/T) + 1]^2 + 4[\zeta_1\beta^3/T^2 + \zeta_2\beta^3/T(1+\mu) - \zeta_2\beta/T - \zeta_1\beta]^2. \qquad (5)$$

As can be observed, the amplitude α depends on the five parameters β, ζ_1, ζ_2, μ and T. Following the same reasoning of Randall *et al*. [6], it has been assumed that ζ_1 and μ are independent parameters varying, respectively, in the range $0 \leqslant \zeta_1 \leqslant 0{\cdot}4$ and $0{\cdot}1 \leqslant \mu \leqslant 0{\cdot}4$. Thus, the remaining parameters to be optimized are ζ_2 and T.

3.1. THE CLASSIC SYSTEM

Since $\zeta_1 = 0$ a closed form analytical solution is possible. In reference [5] is proposed the following optimal choice of parameters:

$$\zeta_{2opt} = \sqrt{3\mu/8(1+\mu)}, \qquad T_{opt} = 1/(1+\mu). \qquad (6, 7)$$

3.2. THE LINEAR DAMPED SYSTEM

Ones goal is to determine the values of ζ_2 and T such that the curve α versus β has two equal peak values with minimum distance from a straight line. Hence, to make use of the results of the Chebyshev's equioscillation theorem, ones goal will be the *optimal* approximation of the curve $\alpha(\beta)$ with a straight line $\alpha = L$ (i.e., $n = 1$), where L is initially unknown.

Making use of the conditions required by the above mentioned theorem, the following system of non-linear algebraic equations can be written:

$$\mathrm{d}\alpha/\mathrm{d}\beta|_{\beta=\beta_1} = 0, \qquad \mathrm{d}\alpha/\mathrm{d}\beta|_{\beta=\beta_2} = 0, \qquad \mathrm{d}\alpha/\mathrm{d}\beta|_{\beta=\beta_3} = 0, \qquad (8a\text{–}c)$$

$$-\alpha(\beta_1) + L + \Delta = 0, \qquad -\alpha(\beta_2) + L - \Delta = 0, \qquad -\alpha(\beta_3) + L + \Delta = 0, \qquad (8d\text{–}f)$$

where Δ is the maximum deviation of the response curve from the value $\alpha = L$ and β_1, β_2 and β_3 are the frequency ratios where such a curve attains a maximum or a minimum. Therefore, system (8) is composed of six equations with seven unknown variables (i.e., ζ_2, T, L, Δ, β_1, β_2 and β_3).

In the present case the numerical values of the optimal parameters can be computed solving the system of non-linear equations for different prescribed values of ζ_2. The chosen solution set is the one with the minimum value of α_{max}. It is worth observing that, the optimal solution, when it exists, is unique. The graphs presented in Figures 2, 3 and 4 summarize the numerical results obtained from the solution of system of equations (8). After specifying the values of μ and

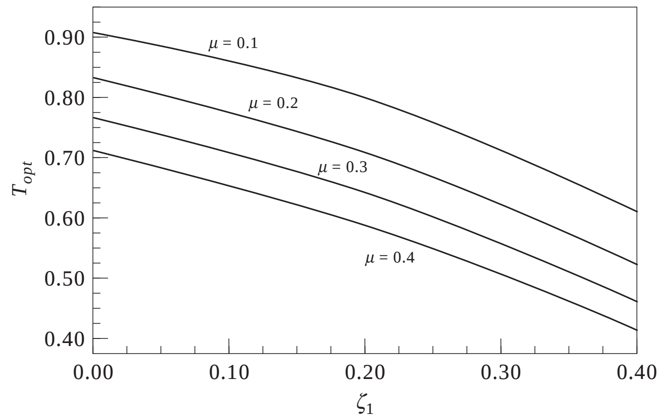

Figure 2. Optimal values of T for prescribed values of ζ_1 and μ.

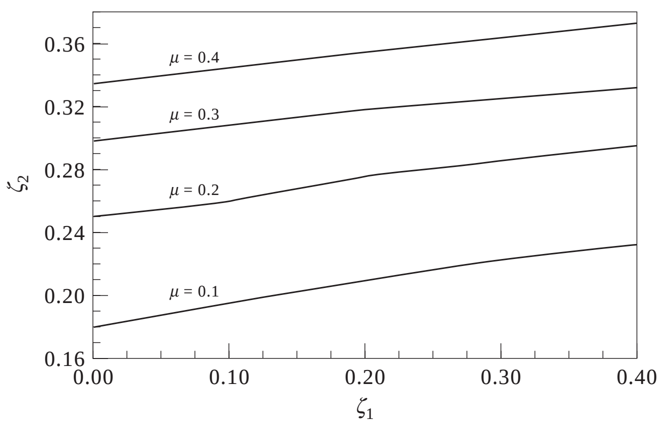

Figure 3. Optimal values of ζ_2 for prescribed values of ζ_1 and μ.

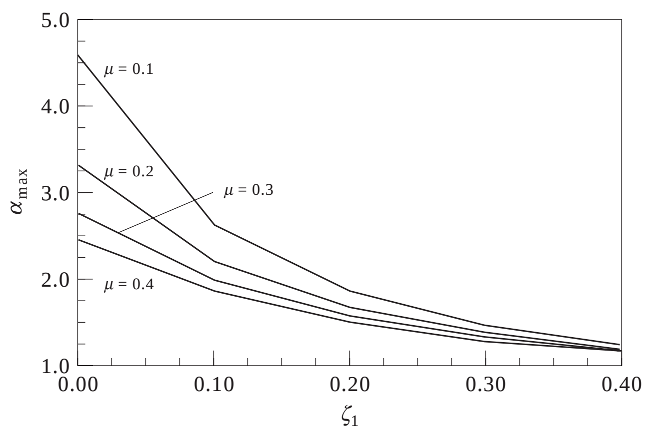

Figure 4. Optimal values of α for prescribed values of ζ_1 and μ.

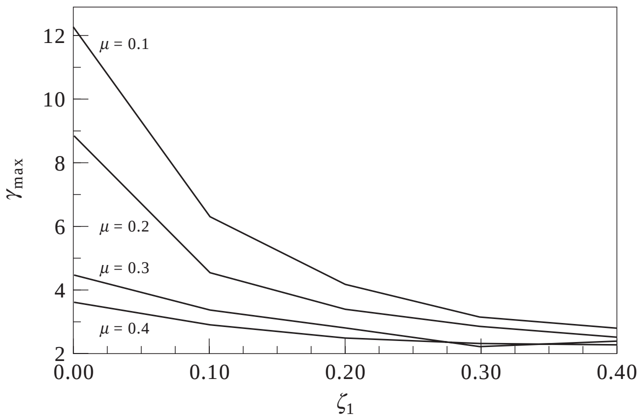

Figure 5. Maximum values of γ_r for prescribed values of ζ_1 and μ and optimal choice of parameters T and ζ_2.

ζ_1, the optimal parameters set ζ_2 and T can be easily deduced. The diagram shown in Figure 4 gives the normalized maximum vibration amplitude of the mass m_1. From the graph of Figure 5, the maximum vibration amplitude of the mass m_2 with respect to m_1 can be obtained. The knowledge of this value is important. It allows an estimate of the stress on the spring connecting the two masses. A copy of the FORTRAN program developed for the purpose of optimal design is available by contacting the author.

4. NUMERICAL RESULTS

This numerical example is taken from reference [6]. Consider a linear damped system with the following characteristics $m_1 = 100$ kg, $\zeta_1 = 0{\cdot}10$, $\omega_1 = 100$ rad/s. The design constraints are such that $\mu = 0{\cdot}10$. The optimal solution for the *classic system* (i.e., $\zeta_1 = 0$) is $T_{opt} = 0{\cdot}909$ and $\zeta_{2opt} = 0{\cdot}185$ and follows from formulas (6) and (7), respectively. A comparison of the results obtained with the proposed method and those from other authors is reported in Table 1. Figures 6, 7 show,

TABLE 1

Comparison of results given by different methods

Parameters	ζ_1	μ	T_{opt}	ζ_{2opt}	α_{max}	γ_{max}
Den Hartog	0·00	0·10	0·909	0·185	4·59	11·14
Randall†	0·10	0·10	0·861	0·204	2·63	6·48
Soom‡	0·10	0·10	0·86	0·26	2·72	5·64
Jordanov§	0·123	0·10	0·868	0·084	2·79	9·13
Thompson‖	0·10	0·10	0·862	0·192	2·62	6·55
This investigation	0·10	0·10	0·861	0·202	2·62	6·49

† See Table 1 of reference [6].
‡ See graphs reported in Figures 2, 4 and 5 of reference [7].
§ See graph reported in Figure 9 of reference [8].
‖ Numerically computed. See also graphs 5, 6 and 7 of reference [9].

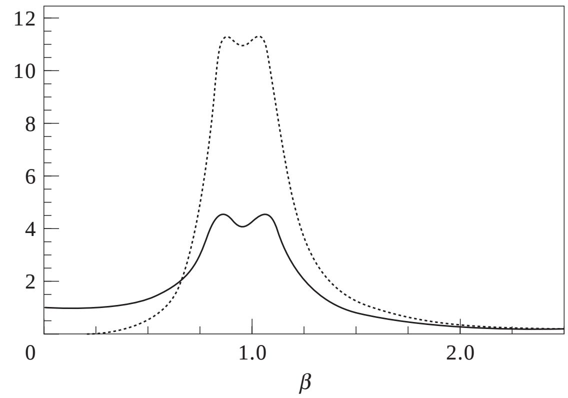

Figure 6. Vibration amplitude α and γ versus β for the method of Den Hartog. Key: ——, α; · · · ·, γ.

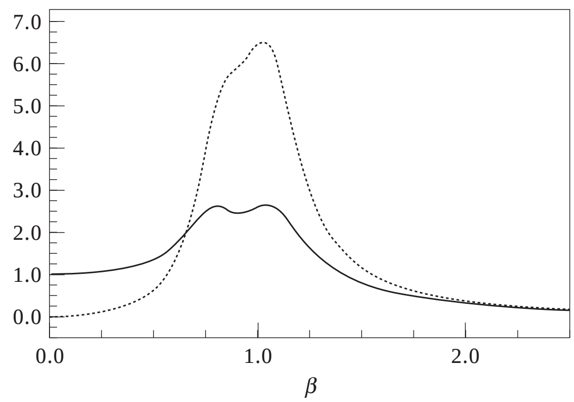

Figure 7. Vibration amplitude α and γ versus β for this investigation. Key: ——, α; · · · ·, γ.

respectively, the vibration amplitudes of the primary and secondary masses for two different solutions. In particular, Figure 6 refers to the classic system with the optimal parameters computed according to Den Hartog's approach. Figure 7 refers to a linearly damped vibration absorber with the computed optimal parameters.

5. CONCLUSIONS

The design method presented allows the optimal choice of parameters of the damped dynamic vibration absorber. Observing Figure 4 one concludes that the case $\zeta_1 = 0$ gives an upper bound for the primary mass vibration amplitude. Moreover, by considering the mathematical bases of the method herein discussed and the optimality criterion chosen, one can state that, for a given set of parameters ζ_1 and μ, the optimal solution is unique.

For the case $\zeta_1 = 0$ there is a negligible difference between the results given by this method and those obtained with formulas (6) and (7). Such a difference can be explained considering that the points P and Q (see Figure 3.13 on p. 100 of reference [5]), through which all amplitudes curves pass, are not the peak points. Regarding this matter Den Hartog [5] stated "*. . . practically no error is made by taking the amplitude of either point as the maximum amplitude of the curve*".

It was found that the value of ζ_2 has more influence on the relative displacement γ than on displacement α of the primary mass. The comparison with others methods show good correlation between the results obtained by different authors. Such results are almost coincident for those authors who preferred the algebraic approach to the use of non-linear programming codes. The graphs presented in this paper show the influence of the design parameters on the performances of the damped dynamic vibration absorber. In particular, the normalized relative displacement of the secondary mass, not reported in the cited references, is useful during the design phase.

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APPENDIX: NOMENCLATURE

For the purpose of direct comparison, most of the nomenclature introduced by Randall *et al*. [6] is adopted.

b	damping coefficient
F_1	amplitude of forcing function
k	spring constant
m	mass
ω	frequency of forcing function
ω_i	natural frequency of uncoupled system [$\sqrt{k_i/m_i}$ $(i = 1, 2)$]
X	frequency response amplitude
X_r	frequency response amplitude of relative displacement of mass m_2 w.r.t. m_1
T	ω_2/ω_1
β	ω/ω_1
ζ_i	damping ratio [$b_i/2\sqrt{k_i m_i}$ $(i = 1, 2)$]
μ	mass ratio (m_2/m_1)
α	primary mass displacement amplification $X_1/(F_1/k_1)$
γ_r	relative displacement amplification $X_r/(F_1/k_1)$
α_{max}	peak value of α
α_{opt}	optimum amplitude α for a fixed set of ζ_1 and μ
γ_{r-opt}	optimum amplitude γ_r for a fixed set of ζ_1 and μ
ζ_{2opt} and T_{opt}	optimum set of design parameters for fixed values of μ and ζ_1

Subscripts 1 and 2 refer to the primary and absorber system, respectively.