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LETTERS TO THE EDITOR

RANDOM VIBRATIONS OF A DUFFING OSCILLATOR USING THE VOLTERRA SERIES

K. Worden and G. Manson

Department of Mechanical and Process Engineering, *University of Sheffield*, *Mappin Street*, *Sheffield S*1 3*JD*, *England*

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1. INTRODUCTION

The important problem of computing the response spectrum of a randomly excited non-linear oscillator has been approached by various methods including: statistical linearization [1], closure schemes [2] and the Fokker–Planck–Kolmogorov equation [3]. The comparatively recent approaches in references [4] and [5] also show considerable merit. One approach which has been pursued sporadically is based on the Volterra series [2, 6]. However, the results produced have been limited by the fact that a low-order truncation of the series was used. The object of the current paper is to show that interesting results become available if slightly higher-order calculations are pursued. The calculations presented here are arguably routine, but the results are of value.

It is well-known that many non-linear systems or input–output processes $x(t) \rightarrow y(t)$ can be realized as a mapping,

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \cdots + y_n(t) + \cdots, \tag{1}$$

where

$$y_n(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n h_n(\tau_1, \ldots, \tau_n) x(t-\tau_1) \cdots x(t-\tau_n). \tag{2}$$

This is the *Volterra series* and the functions h_n are the *Volterra kernels*. The dual frequency–domain representation is based on the *higher-order FRFs* (HFRFs) or *Volterra kernel transforms*, $H_n(\omega_1, \ldots, \omega_n)$, $n = 1, \ldots, \infty$, which are defined as the multi-dimensional Fourier transforms of the kernels.

The definition of the FRF of a linear system based on the input/output cross-spectrum, $S_{yx}(\omega)$, and input autospectrum, $S_{xx}(\omega)$, is also well-known,

$$H(\omega) = H_1(\omega) = S_{yx}(\omega)/S_{xx}(\omega). \tag{3}$$

The *composite FRF* $\Lambda_r(\omega)$, of a non-linear system under random excitation, is defined similarly,

$$\Lambda_r(\omega) = S_{yx}(\omega)/S_{xx}(\omega). \tag{4}$$

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Using the Volterra series representation in (1) results in the expression

$$\Lambda_r(\omega) = \frac{S_{y_1x}(\omega) + S_{y_2x}(\omega) + \cdots + S_{y_nx}(\omega) + \cdots}{S_{xx}(\omega)}. \quad (5)$$

$\Lambda_r(\omega)$ will be approximated here by obtaining expressions for the various cross-spectra between the input and the individual output components. For example, it is shown in references [7] and [8] that

$$S_{y_3x}(\omega) = \frac{3S_{xx}(\omega)}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 H_3(\omega_1, -\omega_1, \omega) S_{xx}(\omega_1). \quad (6)$$

The general term is obtained straightforwardly as

$$S_{y_{2n-1}x}(\omega) = \frac{(2n)!S_{xx}(\omega)}{n!2^n(2\pi)^{n-1}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1}$$
$$\times H_{2n-1}(\omega_1, -\omega_1, \ldots, \omega_{n-1}, -\omega_{n-1}, \omega) S_{xx}(\omega_1)\cdots S_{xx}(\omega_{n-1}). \quad (7)$$

Now, given that the input autospectrum is constant over all frequencies for a Gaussian white noise input (i.e., $S_{xx}(\omega) = P$), the composite FRF for random excitation follows. Substituting equation (7) into equation (5) gives

$$\Lambda_r(\omega) = \sum_{n=1}^{n=\infty} \frac{(2n)!P^{n-1}}{n!2^n(2\pi)^{n-1}}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\cdots\mathrm{d}\omega_{n-1}$$
$$\times H_{2n-1}(\omega_1, -\omega_1, \ldots, \omega_{n-1}, -\omega_{n-1}, \omega). \quad (8)$$

2. RANDOM EXCITATION OF A DUFFING OSCILLATOR

The system studied here is the ubiquitous Duffing oscillator with equation of motion

$$m\ddot{y}(t) + c\dot{y}(t) + k_1 y(t) + k_3 y(t)^3 = x(t), \quad (9)$$

where m represents mass, c viscous damping, k_1 linear spring stiffness and k_3 cubic spring stiffness. By using the theory developed in the last section, an expression for $\Lambda_r(\omega)$ up to $O(P^2)$ will be calculated for this system. From equation (8) the first three terms are given by

$$\frac{S_{y_1x}(\omega)}{S_{xx}(\omega)} = H_1(\omega), \qquad \frac{S_{y_3x}(\omega)}{S_{xx}(x)} = \frac{3P}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 H_3(\omega_1, -\omega_1, \omega),$$

$$\frac{S_{y_5x}(\omega)}{S_{xx}(\omega)} = \frac{15P^2}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\,\mathrm{d}\omega_2 H_5(\omega_1, -\omega_1, \omega_2, -\omega_2, \omega). \quad (10)$$

The first term of equation (10) needs no further work but the others require expressions for the HFRF terms as functions of the H_1s and k_3. This may be achieved by harmonic probing of the system equation of motion [9]. The results are

$$H_3(\omega_1, -\omega_1, \omega) = -k_3 H_1(\omega)^2 |H_1(\omega_1)|^2 \tag{11}$$

and

$$\begin{aligned} H_5(\omega_1, -\omega_1, \omega_2, -\omega_2, \omega) = (3k_3^2/10) H_1(\omega)^2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \{ & 2H_1(\omega) + H_1(\omega_1) \\ & + H_1(-\omega_1) + H_1(\omega_2) + H_1(-\omega_2) \\ & + H_1(\omega_1 + \omega_2 + \omega) + H_1(\omega_1 - \omega_2 + \omega) \\ & + H_1(-\omega_1 + \omega_2 + \omega) + H_1(-\omega_1 - \omega_2 + \omega)\}. \end{aligned} \tag{12}$$

Substituting equation (11) into the $S_{y_3x}(\omega)/S_{xx}(\omega)$ term of equation (10) gives

$$\frac{S_{y_3x}(\omega)}{S_{xx}(\omega)} = -\frac{3Pk_3 H_1(\omega)^2}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 |H_1(\omega_1)|^2. \tag{13}$$

The integral may be found in standard tables of integrals used for the calculation of mean square response, e.g. reference [10]. The result is

$$S_{y_3x}(\omega)/S_{xx}(\omega) = -3Pk_3 H_1(\omega)^2 / 2ck_1 \tag{14}$$

for this system. The third-order component of the response does not change the position of the poles of $\Lambda_r(\omega)$ from those of the linear FRF. It actually converts them to double poles. This is as far as previous calculations have gone. The effect of the next non-zero term will now be analyzed.

Substituting equation (12) into the $S_{y_5x}(\omega)/S_{xx}(\omega)$ term of equation (10) gives, after a little effort, the integrals

$$\begin{aligned} \frac{S_{y_5x}(\omega)}{S_{xx}(\omega)} = & \frac{9P^2k_3^2 H_1(\omega)^3}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \\ & + \frac{9P^2k_3^2 H_1(\omega)^2}{2\pi^2} \left\{ \mathrm{Re} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2 H_1(\omega_1) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \right. \\ & \left. + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2 H_1(\omega_1 + \omega_2 + \omega) |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 \right\}. \end{aligned} \tag{15}$$

The first integral may be evaluated straightforwardly as

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2 |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 = \left\{ \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 |H_1(\omega_1)|^2 \right\}^2 = \frac{\pi^2}{c^2 k_1^2}, \tag{16}$$

on making use of the tabulated results in reference [10].

The second integral can also be factorized,

$$\mathrm{Re}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 H_1(\omega_1)|H_1(\omega_1)|^2 \int_{-\infty}^{+\infty} \mathrm{d}\omega_2 |H_1(\omega_2)|^2, \tag{17}$$

and the second integral is as before. Contour integration can be used to evaluate the first part and gives

$$\mathrm{Re}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\,\mathrm{d}\omega_2 H_1(\omega_1)|H_1(\omega_1)|^2|H_1(\omega_2)|^2 = \frac{\pi^2}{2c^2k_1^3}. \tag{18}$$

The third and final integral of the $S_{y_5x}(\omega)/S_{xx}(\omega)$ expression is a little more complicated as the integrand does not factorize. However, contour integration again suffices to give

$$I(\omega) = \frac{-\pi^2(\omega^2 - 3\omega_d^2 - 10\mathrm{i}\zeta\omega_n\omega - 27\zeta^2\omega_n^2)}{mc^2k_1^2(\omega - \omega_d - 3\mathrm{i}\zeta\omega_n)(\omega + \omega_d - 3\mathrm{i}\zeta\omega_n)(\omega - 3\omega_d - 3\mathrm{i}\zeta\omega_n)(\omega + 3\omega_d - 3\mathrm{i}\zeta\omega_n)}, \tag{19}$$

and the overall expression for $S_{y_5x}(\omega)/S_{xx}(\omega)$ is

$$\frac{S_{y_5x}(\omega)}{S_{xx}(\omega)} = \frac{9P^2k_3^2H_1(\omega)^3}{4c^2k_1^2} + \frac{9P^2k_3^2H_1(\omega)^2}{4c^2k_1^3} + \frac{9P^2k_3^2H_1(\omega)^2}{2\pi^2} I(\omega) \tag{20}$$

for the classical Duffing oscillator. The above equation shows that the first two terms do not affect the position of the poles of the linear system, but convert them to triple poles. The term of greatest interest is the final one which has introduced four new poles at

$$\omega_d + 3\mathrm{i}\zeta\omega_n; \qquad -\omega_d + 3\mathrm{i}\zeta\omega_n; \qquad 3\omega_d + 3\mathrm{i}\zeta\omega_n; \qquad -3\omega_d + 3\mathrm{i}\zeta\omega_n. \tag{21}$$

The pole structure to this order is shown in Figure 1. The poles at $\pm 3\omega_d + 3\mathrm{i}\zeta\omega_n$ explain the secondary observed peak at three times the resonance frequency in the output spectra of non-linear oscillators [3].

Combining equations (14), (18) and (19) into equation (5) yields an expression for the composite FRF $\Lambda_r(\omega)$ up to $O(P^2)$. The magnitude of the composite FRF

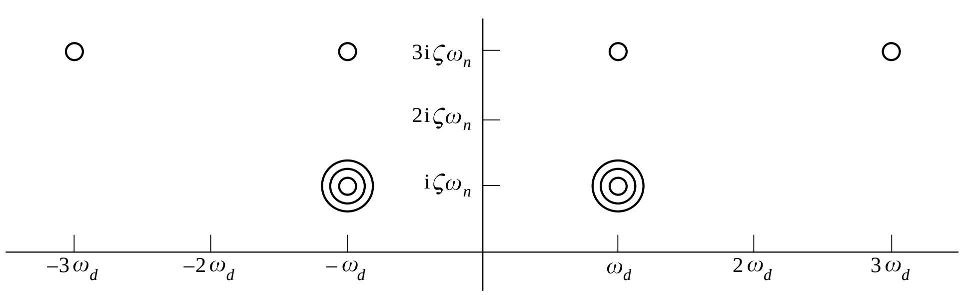

Figure 1. Pole structure of the first three terms of $\Lambda_r(\omega)$ for the classical Duffing oscillator system.

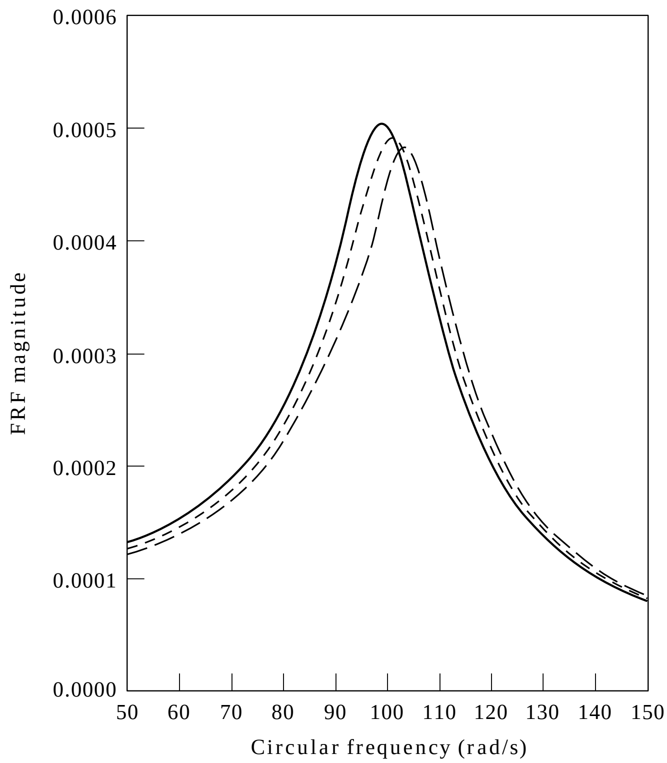

Figure 2. Composite FRF $\Lambda_r(\omega)$ for the Duffing oscillator to order P^2. FRF distortion ($k_2 = 0$, $k_3 = 5 \times 10^9$). P values: ——, 0·0 (linear system); ---, 0·01; – – –, 0·02.

is plotted in Figure 2 for values of P equal to 0 (linear system), 0·01 and 0·02. The Duffing oscillator parameters are $m = 1$, $c = 20$, $k_1 = 10^4$ and $k_3 = 5 \times 10^9$.

In order to determine whether or not the inclusion of further terms in the $\Lambda_r(\omega)$ approximation results in further poles arising at new locations, the fourth non-zero term (i.e., $S_{y_7x}(\omega)/S_{xx}(\omega)$) for this system was considered. The expression consists of 280 integrals when the problem is expressed in H_1 terms. However repeating the procedure of combining terms which yield identical integrals results in 13 integrals. After evaluating the integrals, again by contour integration, it was found that no new poles arose.

Due to the rapidly increasing level of difficulty associated with the addition of further terms to $\Lambda_r(\omega)$ it was not possible to completely examine the $S_{y_9x}(\omega)/S_{xx}(\omega)$ term. However, one integral was considered which would be included in the overall expression: i.e.,

$$\frac{27P^4k_3^4H_1(\omega)^2}{128\pi^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,\mathrm{d}\omega_4 H_1(\omega+\omega_1+\omega_2+\omega_3+\omega_4)$$
$$\times\, H_1(\omega+\omega_1+\omega_2)H_1(-\omega_1-\omega_2-\omega_3)|H_1(\omega_1)|^2|H_1(\omega_2)|^2|H_1(\omega_3)|^2|H_1(\omega_4)|^2. \tag{22}$$

This integral was evaluated as before and was found to have triple poles at the locations given in equation (21) and simple poles at the locations

$$\omega_d + 5i\zeta\omega_n, \quad -\omega_d + 5i\zeta\omega_n, \quad 3\omega_d + 5i\zeta\omega_n, \quad -3\omega_d + 5i\zeta\omega_n,$$
$$5\omega_d + 5i\zeta\omega_n, \quad -5\omega_d + 5i\zeta\omega_n,$$
$$\omega_d + 7i\zeta\omega_n, \quad -\omega_d + 7i\zeta\omega_n, \quad 3\omega_d + 7i\zeta\omega_n, \quad -3\omega_d + 7i\zeta\omega_n,$$
$$5\omega_d + 7i\zeta\omega_n, \quad -5\omega_d + 7i\zeta\omega_n, \quad 7\omega_d + 7i\zeta\omega_n, \quad -7\omega_d + 7i\zeta\omega_n. \quad (23)$$

Although it is possible that these contributions cancel when combined with other integrals from $S_{y_9x}(\omega)/S_{xx}(\omega)$, it can be conjectured that including all further terms would result in FRF poles for this system being witnessed at all locations $a\omega_d + bi\zeta\omega_n$ where $a \leqslant b$ are both odd integers.

If a quadratic stiffness term is added to equation (9),

$$m\ddot{y}(t) + c\dot{y}(t) + k_1 y(t) + k_2 y(t)^2 + k_3 y(t)^3 = x(t), \quad (24)$$

the calculation up to $O(P^2)$ gives poles at

$$2\omega_d + 2i\zeta\omega_n, \quad -2\omega_d + 2i\zeta\omega_n, \quad 2i\zeta\omega_n,$$
$$\omega_d + 3i\zeta\omega_n, \quad -\omega_d + 3i\zeta\omega_n, \quad 3\omega_d + 3i\zeta\omega_n, \quad -3\omega_d + 3i\zeta\omega_n. \quad (25)$$

Note that these poles arise not only due to the k_3 term but also in integrals which depend only upon k_2. This suggests that even non-linear terms result in poles in the composite FRF at all locations $a\omega_d + bi\zeta\omega_n$ where $a \leqslant b$ are both odd integers or both even. It might be expected that the inclusion of all terms in the $\Lambda(\omega)$ expansion will result in an infinite array of poles, positioned at $a\omega_d + bi\zeta\omega_n$ where $a \leqslant b$ are both odd integers or both even.

An interesting feature of this analysis is that the mutiplicity of the poles increases with order P. The implication is that the poles will become isolated essential singularities in the limit.

It is significant that all the poles are located in the upper half of the ω-plane. It is known (e.g. see reference [11]), that applying the Hilbert transform test to a system with all its poles in the upper half of the complex plane results in the system being labelled linear. If this behaviour continues for higher terms this then shows agreement with apparent linearization of FRFs obtained under random excitation.

3. VALIDITY OF THE VOLTERRA SERIES

The convergence of the Volterra series is examined here. Results will be obtained for the classical Duffing oscillator by using the criteria developed by Barrett [12]. The first step is to convert the equation of motion (9) to the normalized form,

$$\ddot{y}' + 2\zeta\dot{y}' + y' + \varepsilon y'^3 = x'(t'). \quad (26)$$

This is accomplished by the transformation,

$$y' = \omega_n^2, \qquad x' = x/m, \qquad t' = \omega_n t, \quad (27)$$

so that

$$\varepsilon = k_3/m\omega_n^6 \tag{28}$$

and ζ has the usual definition. Once in this co-ordinate system, convergence of the Volterra series is assured as long as

$$\|y'\| < y'_b = 1/\sqrt{3\varepsilon H}, \tag{29}$$

where

$$H = \coth\,(\pi\zeta/\sqrt{1-\zeta^2}). \tag{30}$$

The norm $\|y'\|$ on an interval of time is simply the maximum value of y' over that interval. Upon using the values of section 4, $m = 1$, $c = 20$ and $k_1 = 10^4$, the value of y'_b obtained is 4·514. This translates into a physical bound $y_b = 4{\cdot}514 \times 10^{-4}$.

Now, the mean-square response σ_{yl}, of the underlying linear system ($k_3 = 0$) is given by the standard formula

$$\sigma_{yl}^2 = \pi P/ck_1, \tag{31}$$

and in this case, $\sigma_{yl} = 3{\cdot}96332 \times 10^{-4}$ if $P = 0{\cdot}01$ and $\sigma_{yl} = 5{\cdot}60499 \times 10^{-4}$ if $P = 0{\cdot}02$. However, these results will be conservative if a non-zero k_3 is assumed. In fact for the non-linear system [1],

$$\sigma_{y_{nl}}^2 = \sigma_{y_l}^2 - 3\alpha\sigma_{y_l}^2, \tag{32}$$

to first order in $\alpha = k_3/k_1$. If this results is assumed valid (α is by no means small), the mean-square response of the cubic system can be found. It is $\sigma_{y_{nl}} = 3{\cdot}465 \times 10^{-4}$ when $P = 0{\cdot}01$ and $\sigma_{y_{nl}} = 4{\cdot}075 \times 10^{-4}$ when $P = 0{\cdot}02$. In the first case, the Barrett bound is 1·3 standard deviations and in the second case it is 1·11 standard deviations. Thus, by using standard tables for Gaussian statistics [13], it is found that the Volterra series is valid with 80·6% confidence if $P = 0{\cdot}01$ and with 73·3% confidence if $P = 0{\cdot}02$. As the Barrett bound is known to be conservative [14], these results were considered to lend support to the assumption of validity for the Volterra series.

4. Hilbert transforms

The Hilbert transform has attracted attention recently as a means of diagnosing structural non-linearity on the basis of measured Frequency Response Function data [15]. It is essentially a mapping on the FRF $G(\omega)$,

$$\mathscr{H}[G(\omega)] = \tilde{G}(\omega) = -\frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty} \mathrm{d}\Omega\, \frac{G(\Omega)}{\Omega - \omega}\,, \tag{33}$$

where the integral is to be understood as a principal value. This mapping reduces to the identity on those functions corresponding to linear systems. For non-linear systems, the Hilbert transform results in a distorted version $\tilde{G}$, of the original FRF, with the form of the distortion yielding some indication of the type of non-linearity. Unfortunately, no analytical expressions are available for the

Hilbert transform of even the simplest non-linear system FRF, although an attempt at approximating $\tilde{G}(\omega)$ for the Duffing oscillator was given in reference [16]. The approximation, based on the Volterra series, reproduced well the qualitative features of $\tilde{G}(\omega)$.

The origin of the distortion is well-known [11]; suppose $G(\omega)$ is decomposed as

$$G(\omega) = G^{+}(\omega) + G^{-}(\omega), \tag{34}$$

where $G^{+}(\omega)$ (resp. $G^{-}(\omega)$) has poles only in the upper (resp. lower) half of the complex ω-plane. It can be shown that

$$\mathscr{H}[G^{\pm}(\omega)] = \pm G^{\pm}(\omega). \tag{35}$$

It is clear from this that the distortion suffered in passing from the FRF to the Hilbert transform will be given by the simple relation,

$$\Delta G(\omega) = \mathscr{H}[G(\omega)] - G(\omega) = -2G^{-}(\omega). \tag{36}$$

In the case of a non-linear system excited by a stepped-sinusoid $X\sin(\omega t)$, the resulting FRF $\Lambda_s(\omega)$ can be obtained by a straightforward calculation [17],

$$\begin{aligned} \Lambda_s(\omega) = H_1(\omega) &+ \tfrac{3}{4}k_3X^2H_1(\omega)^3H_1^*(\omega) \\ &+ \tfrac{3}{16}k_3^2X^4[3H_1(\omega)^4H_1^*(\omega)^3 + 6H_1(\omega)^5H_1^*(\omega)^2 \\ &+ H_1(\omega)^4H_1^*(\omega)^2H_1(3\omega)] + O(X^6). \end{aligned} \tag{37}$$

The critical fact is that the FRF contains terms of the form H_1^*. This means that it will always have poles in the lower half-plane. As a result, according to equation (36), the FRF will distort under the action of the Hilbert transform.

In contrast to the situation for a stepped-sine FRF, the random excitation FRF appears to have all poles in the upper half-plane. The Hilbert transform result (36) is independent of the multiplicity of the poles, so the FRF $\Lambda_r(\omega)$ will *not* distort under the action of the Hilbert transform. These remarks are completely consistent with experimental findings.

Note that a generic non-linearity will almost always have a second or third order term and the FRF will therefore almost always have the complex conjugate poles described above. It follows that general non-linear system FRFs will distort under the Hilbert transform if stepped-sine excitation is used.

5. CONCLUSIONS

An expansion is obtained for the FRF of a Duffing oscillator subjected to white Gaussian excitation. The expansion reproduces the observed variations in the FRF when subjected to varying input powers. The expansion also indicates why the Hilbert transform test fails to signal non-linearity on FRFs from random testing, a question which has stood since the introduction of the test. It is also possible on the basis of the results above, to make a tentative conjecture on the pole structure of the Duffing oscillator and in fact of non-linear systems in general. It remains to be seen if there are any discernible consequences for the stability analysis of non-linear systems for example.

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