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LIÉNARD SYSTEMS, LIMIT CYCLES, MELNIKOV THEORY, AND THE METHOD OF SLOWLY VARYING AMPLITUDE AND PHASE

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An important class of second-order, ordinary differential equations that play an important role in the modelling of dynamical systems in the natural and engineering sciences is the Liénard equation [1–4]. This equation takes the form

$$\ddot{x} + \varepsilon f(x,\mu)\dot{x} + g(x,\mu) = 0, \tag{1}$$

where, in general, $f(x,\mu)$ and $g(x,\mu)$ are, respectively, even and odd degree polynomials; ε is a small, positive parameter; and μ is a set of n-parameters which characterize the dynamical system. For the purposes of this paper, we select $g(x) = x$. In summary,

$$0 < \varepsilon \ll 1, \quad \mu \in R^n, \qquad g(x) = x. \tag{2}$$

The Liénard systems have the property that isolated periodic orbits may exist in their 2-dim phase space. These special orbits are called limit cycles [1–4] and play an essential role in both the understanding and analysis of the corresponding dynamical systems. For a given set of parameter values, two important questions to answer are: (i) How many limit cycles does the Liénard system, equations (1) and (2), possess? (ii) Where are these limit cycles located in the 2-dim phase plane, i.e., their radii? A large number of investigators have considered these questions. In particular, Burnette and Mickens [5] studied the number of limit cycles for the generalized Rayleigh–Liénard oscillator

$$\ddot{x} + x = \varepsilon[bx^3 + (c_1 + c_2x^2 + c_3\dot{x}^2 + c_4x^4)\dot{x}], \tag{3}$$

where (b, c_1, c_2, c_3, c_4) are constants. More recently, Giacomini and Neukirch [6] investigated the properties of Liénard systems by constructing a sequence of polynomials whose roots provide information on both the number and location of the limit cycles. It was then shown by Sanjuán [7] that the use of Melnikov theory [8, 9] gave the same results. The purpose of this paper is to show that the Melnikov function for the Liénard system of equations (1) and (2) is given exactly by the well known perturbation procedure called the method of slowly varying amplitude and phase [1, 10], except for an unessential known factor.

The main concepts underlying Melnikov theory are presented in Sanjuán [7]. For a more detailed discussion see Guckenheimer and Holmes [8], and Perko [9]. Equations (1) and (2) may be rewritten to the 2-dim system form

$$\dot{x} = y, \qquad \dot{y} = -x - \varepsilon f(x, \mu)y. \tag{4}$$

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For $\varepsilon = 0$, equations (4) have the following one-parameter family of periodic solutions having period $T_\alpha = 2\pi$:

$$x_\alpha(t) = \alpha \cos t, \qquad y_\alpha(t) = -\alpha \sin t. \tag{5}$$

The corresponding Melnikov function is

$$M(\alpha, \mu) = -\alpha^2 \int_0^{2\pi} f(\alpha \cos t, \mu) \sin^2 t \, \mathrm{d}t. \tag{6}$$

The paper by Sanjuán states the conditions under which a knowledge of $M(\alpha, \mu)$ allows the determination of the number and location of the limit cycles for equation (4).

The method of slowly varying amplitude and phase is an averaging method [1, 10] which was justified in a mathematically rigorous manner by Bogoliubov and Mitropolsky [11]. Details of the method's derivation, as well as numerous applications, appear in Mickens [12]. The basic idea is to assume that the solutions to equations (1) and (2) and their derivatives take the form

$$x(t) = A(t) \cos[t + \phi(t)], \qquad \dot{x}(t) = -A(t) \sin[t + \phi(t)], \tag{7}$$

where the (ε, μ) dependencies of the amplitude $A(t)$ and phase $\phi(t)$ have been suppressed. The first approximation to $A(t)$ and $\phi(t)$ are given by solutions to the following first-order differential equations [10, 12]:

$$\begin{aligned}
\frac{\mathrm{d}\bar{A}}{\mathrm{d}t} &= \left(\frac{\varepsilon}{2\pi}\right) \int_0^{2\pi} f(\bar{A} \cos\psi, \mu)(-\bar{A} \sin\psi) \sin\psi \, \mathrm{d}\psi \\
&\equiv \varepsilon G(\bar{A}, \mu), \\
\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}t} &= \left(\frac{\varepsilon}{2\pi\bar{A}}\right) \int_0^{2\pi} f(\bar{A} \cos\psi, \mu)(-\bar{A} \sin\psi) \cos\psi \, \mathrm{d}\psi \\
&\equiv \varepsilon H(\bar{A}, \mu).
\end{aligned} \tag{8}$$

Note that the right sides of equations (8) depend on the parameters (ε, μ) and the averaged amplitude $\bar{A}(t)$, but not the averaged phase function $\bar{\phi}(t)$. The number of limit cycles and their location (radii) are obtained from the real solutions of the equation [12, 13]

$$G(\bar{A}, \mu) = 0. \tag{9}$$

In the first of equations (8), make the substitutions $\bar{A} \to \alpha$ and $\psi \to t$. Comparing the resulting expression with the Melnikov function of equation (6) gives the relation

$$M(\alpha, \mu) = 2\pi\alpha G(\alpha, \mu). \tag{10}$$

This gives a relation linking the Melnikov function, $M(\alpha, \mu)$, to the amplitude function, $G(\alpha, \mu)$, appearing in the method of slowly varying amplitude and phase

(MSVAP). Consequently, all results obtained for Liénard systems using Melnikov theory also hold for MSVAP. In fact, it was our original derivation [14] of the results of Giacomini and Neukirch [6] and the subsequent appearance of the paper by Sanjuán [7] that led us to investigate the connection between the Melnikov function and $G(\alpha, \mu)$.

A major advantage of MSVAP over the use of Melnikov theory, as presented by Sanjuán, is that the stability of the various limit cycles may be easily determined [12, 13, 15]. Let α_i be a real root of

$$G(\alpha, \mu) = 0. \tag{11}$$

(Note that $+|\alpha_i|$ and $-|\alpha_i|$ correspond to the same limit cycle; they differ only in the value of their phases.) The limit cycle having radius, $\alpha = \alpha_i$, is stable if

$$\frac{\mathrm{d}G(\alpha_i, \mu)}{\mathrm{d}\alpha_i} < 0. \tag{12}$$

The opposite sign implies that the limit cycle is unstable.

A Liénard system having no limit of linear oscillations when $\varepsilon = 0$ is the equation

$$\ddot{x} + x^3 = \varepsilon f(x)\dot{x}, \tag{13}$$

where $f(x)$ is a polynomial of even degree. The 2-dim phase space equations are

$$\dot{x} = y, \qquad \dot{y} = -x^3 + \varepsilon f(x)y. \tag{14}$$

A number of papers have appeared recently on this equation; see Mickens [15] for a listing of these references. We are currently investigating the application of Melnikov theory to this class of Liénard equations.

In summary, Giacomini and Neukirch [6] introduced a sequence of polynomial functions whose roots allow the determination of both the number and location of the limit cycles for Liénard systems. Soon after, Sanjuán [7] showed that the same information is provided by a certain polynomial which Melnikov theory associates with a given Liénard system. This paper demonstrated that the method of slowly varying amplitude and phase not only reproduces the results from Melnikov theory, but also allows the stability of the limit cycles to be calculated. Finally, it should be indicated that the method studied by Giacomini and Neukirch [6] is a non-perturbative procedure which works for arbitrary values of the parameter ε in equation (1). Consequently, their method is not equivalent to either the Melnikov method or the method of slowly varying amplitude and phase except for small values of ε.

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