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HIGHER ORDER NORMAL FORM AND PERIOD AVERAGING

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Calculation of the higher order averaging equations of a non-linear oscillator is very tedious using the classical averaging method. This is also true for higher order normal forms. This paper presents an alternative method, which is a combination of the method of normal form and the classical averaging method. A simple and efficient program is given to calculate the higher order averaging equations by using the symbolic computer algebra system Mathematica. Furthermore, the program can be used to calculate the higher order coefficients of normal form. Four examples are given and compare well with the existing results

1. INTRODUCTION

The normal form of vector fields (or Poincaré normal form, standard form) is a simplified analytical expression for a non-linear oscillator [1]. Such a simplified expression is obtained by using nearly identical non-linear transformations of variables. By introducing non-linear transformations of variables to simplify the analytical expression, the qualitative behaviour of the oscillator can be efficiently described. The qualitative geometric bifurcation structure can be obtained for flows through the analysis of dynamical character of normal forms.

The averaging method, originally due to Krylov and Bogoliubov, is particularly useful for a weakly non-linear oscillator by utilizing small perturbations of the corresponding linear oscillator. This method is one of the most important methods of studying the bifurcation problems at present. It has been proved [2, 3] that the averaging method is equivalent to the normal form method. So the problems of calculating higher order averaging equation are equivalent to the problems of calculating higher order coefficients of normal form.

There are three basic methods [1] to calculate the coefficients of normal form: matrix representation method, adjoint operator method and a method based on representation theory of sl(2, **R**) [4]. Though the first method makes the calculation

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of the coefficients of normal form, or higher order averaging equations, in a routine manner, it is still very complicated to program. In the last two methods, one needs a higher level of mathematical skills. There are classical methods [5, 6] which obtain the normal form by elimination of the secular term one step at a time and which are not suitable for automatic computation.

The tedious computations required here are implemented by using the computer algebra system—Mathematica [7]. A short and efficient program is given to calculate the higher order averaging equations. It takes two minutes to calculate the second order averaging equation for a general single degree freedom non-linear oscillator and four minutes to calculate the fourth order by this method in a Pentium 133 Personal Computer with 16M memory. The user is required to type in the equations only.

2. NORMAL FORM

Consider the non-linear ordinary differential equations [5, 8],

$$\dot{\mathbf{u}} = \mathbf{g}(\mathbf{u}), \qquad \mathbf{g}\in\mathbf{C}^r(\mathbf{R}^n), \qquad \mathbf{u}\in\mathbf{R}^n, \tag{1}$$

where $\mathbf{g}$ is an n-vector function of $\mathbf{u}$ which is an n-vector function of time t. Function $\mathbf{g}$ is differentiable up to an integer order r. A dot denotes differentiation with respect to time t. When equation (1) has a fixed point at $\mathbf{u} = \mathbf{u}_0$ i.e., $\mathbf{g}(\mathbf{u}_0) = 0$, a few linear transformations are performed to simplify equation (1). By the variable change $\mathbf{v} = (\mathbf{u} - \mathbf{u}_0)$ one can eliminate the constant terms and shift the fixed point to the origin under which equation (1) becomes

$$\dot{\mathbf{v}} = \mathbf{g}(\mathbf{v} + \mathbf{u}_0) = \mathbf{H}(\mathbf{v}) = \mathbf{H}_1\mathbf{v} + \mathbf{H}_2(\mathbf{v}),$$

where $\mathbf{H}(\mathbf{v})$ is at least linear in $\mathbf{v}$ which is readily split into the linear part $\mathbf{H}_1$ and the non-linear part $\mathbf{H}_2$. The linear part $\mathbf{H}_1 = D_v\mathbf{H}(\mathbf{v} = 0)$, which is the Jacobian of $\mathbf{H}(\mathbf{v})$ evaluated at $\mathbf{v} = 0$, where D_v is a differential operator with respect to $\mathbf{v}$, and the non-linear part $\mathbf{H}_2 = (\mathbf{H}(\mathbf{v}) - \mathbf{H}_1\mathbf{v})$ is at least quadratic in $\mathbf{v}$. By transforming $\mathbf{H}_1$ into Jordan canonical form by the canonical matrix $\mathbf{Q}$, i.e., $\mathbf{v} = \mathbf{Q}\mathbf{x}$, where $\mathbf{Q}$ is the matrix consisting of all eigenvectors of $\mathbf{H}_1$ if $\mathbf{H}_1$ is non-defective and of all generalised (principal) vectors if $\mathbf{H}_1$ is defective, one obtains

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{J}\mathbf{x} + \mathbf{f}_2(\mathbf{x}) + \cdots + \mathbf{f}_r(\mathbf{x}) + \mathscr{O}(|\mathbf{x}|^{r+1}), \qquad \mathbf{x}\in\mathbf{R}^n, \tag{2}$$

where $\mathbf{J}$ is the Jordan canonical form $\mathbf{J} = \mathbf{Q}^{-1}\ \mathbf{H}_1\mathbf{T}\mathbf{v}$, $\mathbf{f}(\mathbf{x}) = \mathbf{Q}^{-1}\ \mathbf{H}_2(\mathbf{Q}\mathbf{x})$ and $\mathbf{f}_k(\mathbf{x})\in H_n^k$, the linear space of all n vector-valued homogenous polynomials in n variables $\mathbf{x}$, $k = 2, \ldots, r$, or just simply $\mathbf{f}_k(\mathbf{x})$ is the kth order homogeneous polynomial of $\mathbf{x}$.

To transform equation (2) into its normal form, a series of nearly identity nonlinear co-ordinate transformations of the form [9–11] is introduced,

$$\mathbf{x} = \mathbf{y} + \mathbf{P}_k(y), \qquad \mathbf{P}_k(\mathbf{y})\in H_n^k, \quad k \leqslant 2, \tag{3}$$

where $\mathbf{P}_k(\mathbf{x})$ is an unknown kth order homogeneous polynomial of $\mathbf{x}$ to be determined. The system of equation (2) after the change of variables is

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{I} + D_y\mathbf{P}_k(\mathbf{y}))\dot{\mathbf{y}} \\ &= \mathbf{J}(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) + \mathbf{f}_2(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) + \cdots + \mathbf{f}_r(\mathbf{y} + \mathbf{P}_k(\mathbf{y})) + \mathcal{O}(|\mathbf{y}|^{k+1}), \quad y \in \mathbf{R}^n,\end{aligned}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix and the term $(\mathbf{I} + D_y\mathbf{P}_k(\mathbf{y}))$ is invertible for small $\mathbf{y}$ so that $(\mathbf{I} + D_y\mathbf{P}_k(\mathbf{y}))^{-1} = \mathbf{I} - D_y\mathbf{P}_k(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^2)$. Define the adjoint operator, which is a kth order homogeneous polynomial

$$\mathrm{ad}_J^k\mathbf{P}_k(\mathbf{y}) = D_y\mathbf{P}_k(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{P}_k(\mathbf{y}), \tag{4}$$

where $D_y\mathbf{P}_k(\mathbf{y})$ is the Jacobian matrix of $\mathbf{P}_k(\mathbf{y})$. Equation (2) becomes

$$\begin{aligned}\dot{\mathbf{y}} &= \mathbf{J}\mathbf{y} + \mathbf{f}_2(\mathbf{y}) + \cdots \mathbf{f}_{k-1}(\mathbf{y}) + \mathbf{f}_k(\mathbf{y}) - (D_y\mathbf{P}_k(\mathbf{y})\mathbf{J}\mathbf{y} - \mathbf{J}\mathbf{P}_k(\mathbf{y})) + \mathcal{O}(|\mathbf{y}|^{k+1}) \\ &= \mathbf{J}\mathbf{y} + \mathbf{f}_2(\mathbf{y}) + \cdots \mathbf{f}_{k-1}(\mathbf{y}) + \mathbf{f}_k(\mathbf{y}) - \mathrm{ad}_J^k\mathbf{P}_k(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^{k+1}) \end{aligned}\tag{5}$$

One notes that $\mathbf{f}_j (2 \leqslant j \leqslant k)$ are unchanged under transformation (3). If one can find a polynomial $\mathbf{P}_k(\mathbf{y})$ so that $\mathbf{f}_k(\mathbf{y}) - \mathrm{ad}_J^k\mathbf{P}_k(\mathbf{y}) = 0$, then the kth order homogeneous polynomial terms in $\dot{\mathbf{y}}$ are completely eliminated. It is not always the case, in particular, when the kth order homogeneous polynomial contains resonance terms, i.e., the matrix representation of ad_J^k is rank deficient. The latter is dealt with below.

Let $\mathrm{Im}\,\mathrm{ad}_J^k$ be the image of ad_J^k which is rank deficient and G^k be any complement to $\mathrm{Im}\,\mathrm{ad}_J^k$ to span the space of H_n^k: $H_n^k = \mathrm{Im}\,\mathrm{ad}_A^k \oplus G^k$. Assume that $\mathbf{f}_k(\mathbf{y}) = \mathbf{v}_k(\mathbf{y}) + \mathbf{f}_k^0(\mathbf{y})$, where $\mathbf{v}_k(\mathbf{y}) \in \mathrm{Im}\,\mathrm{ad}_A^k$, $\mathbf{f}_k^0(\mathbf{y}) \in \mathbf{G}^k$. If one chooses $\mathbf{P}_k(\mathbf{y})$ such that $\mathrm{ad}_J^k\mathbf{P}_k(\mathbf{y}) = \mathbf{v}_k(\mathbf{y})$, then equation (5) becomes

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{f}_2(\mathbf{y}) + \cdots \mathbf{f}_{k-1}(\mathbf{y}) + \mathbf{f}_k^0(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^{k+1}). \tag{6}$$

Thus by induction one has the following Theorem [12, 13].

Theorem 1. Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a $\mathbf{C}^r$ system of differential equations with $\mathbf{f}(\mathbf{0}) = 0$ and $D\mathbf{f}(0) = \mathbf{J}$. Let G^k be complementary subspaces to $\mathrm{Im}\,\mathrm{ad}_J^k$ in H_n^k for $k = 2, \ldots, r$, then $H_n^k = \mathrm{Im}\,\mathrm{ad}_J^k + G^k$. There exists a series of near identity transformations

$$\mathbf{x} = \mathbf{y} + \mathbf{P}_k(\mathbf{y}), \qquad \mathbf{P}_k(\mathbf{y}) \in H_n^k, \qquad k = 2, \ldots r,$$

such that the system of equation (2) becomes

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{f}_2^0(\mathbf{y}) + \cdots + \mathbf{f}_r^0(\mathbf{y}) + \mathcal{O}(|\mathbf{y}|^{r+1}), \tag{7}$$

where $\mathbf{f}_k^0(\mathbf{y}) \in G^k$, $k = 2, \ldots, r$.

It is clear that equation (7) is the simplest form up to order r among all equivalent systems of equation (2) with respect to smooth transformations. If ad_J^k has full rank for $k = 2, 3, \ldots, r$, then all $\mathbf{f}_k^0$ are identically zero. $\mathbf{f}_k^0$ is non-zero only when ad_J^k is rank deficient.

3. LIE GROUP DEFINITION OF NORMAL FORM

In general, the normal form of an ordinary differential equation satisfies the following homology equation:

$$\mathrm{ad}_J^k \mathbf{f}_k^0(\mathbf{y}) = 0, \qquad k = 2, \ldots, r. \tag{8}$$

References [3, 13] have given another one parameter Lie group definition of normal form which is equivalent to equations (8). Choose the one parameter Lie group

$$\boldsymbol{\Gamma} = \{\mathrm{e}^{tJ}, t \in \mathbf{R}\}, \tag{9}$$

Then the normal form $\mathbf{f}_k^0(\mathbf{y})$ has the symmetry about $\boldsymbol{\Gamma}$, that is

$$\mathbf{f}_k^0(\mathrm{e}^{tJ}y) = \mathrm{e}^{tJ}\mathbf{f}_k^0(\mathbf{y}), \qquad \forall t \in \mathbf{R} \quad \text{and} \quad \mathrm{e}^{TJ} = \mathbf{I}, \tag{10}$$

where $\mathbf{I}$ is an identity matrix and T is a primitive period and is proven below.

$$(\partial/\partial t)[\mathrm{e}^{-tJ}\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x})] = -J\,\mathrm{e}^{-tJ}\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x}) + \mathrm{e}^{-tJ}D\mathbf{f}^0(\mathbf{e}^{tJ}\mathbf{x})\mathbf{J}^{tJ}\mathbf{x})$$

$$= \mathrm{e}^{-tJ}[J\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x}) - D\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x})J^{tJ}\mathbf{x})] = -\mathrm{e}^{-tJ}\,\mathrm{ad}_J^k\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x}) = 0,$$

where equation (8) has been employed with $\mathbf{y} = \mathrm{e}^{tJ}\mathbf{x}$. Obviously, $\mathrm{e}^{-tJ}\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x}) = c$ which is independent of t. Let $t = 0$, $\mathbf{c} = \mathbf{f}^0(\mathbf{x})$ and $\mathrm{e}^{-tJ}\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x}) = \mathbf{f}^0(x)$. Therefore, equation (10) is proved.

Equation (8) is an automonous definition of normal form $\mathbf{f}^0(y)$ and equation (10) is a non-automonous definition of normal form $\mathbf{f}^0(\mathrm{e}^{tJ}\mathbf{x})$ where time t is explicitly involved.

4. AVERAGING METHOD

As with normal form transformations, averaging uses a near identity co-ordinate transformation to simplify a given system of ordinary differential equations. Applying the classical normal form transformation to autonomous systems but applying averaging to non-autonomous systems, the co-ordinate transformation is chosen to transform the non-autonomous system into an autonomous one called the averaged system through integrating the time variable.

With the method of averaging, non-autonomous differential equations of the type

$$\dot{\mathbf{x}} = \epsilon\mathbf{h}(\mathbf{x}, t, \epsilon), \qquad \mathbf{x} \in \mathbf{R}^p, \qquad |\epsilon| \ll 1$$

are studied and analyzed, where $\mathbf{h}(\mathbf{x}, t, \epsilon)$ is a T-periodic vector field. First a non-autonomous T-periodic transformation of the form

$$\mathbf{x} = \boldsymbol{\xi} + \epsilon\mathbf{u}_1(\boldsymbol{\xi}, t) + \epsilon^2\mathbf{u}_2(\boldsymbol{\xi}, t) + \cdots + \epsilon^k\mathbf{u}_k(\boldsymbol{\xi}, t)$$

is applied resulting in an averaged equation of which all terms up to $\mathcal{O}(\epsilon^k)$ are autonomous

$$\dot{\boldsymbol{\xi}} = \epsilon\tilde{\mathbf{h}}_1(\boldsymbol{\xi}) + \epsilon^2\tilde{\mathbf{h}}_2(\boldsymbol{\xi}) + \cdots + \epsilon^k\tilde{\mathbf{h}}_k(\xi) + \epsilon^{k+1}\mathbf{R}(\boldsymbol{\xi}, t, \epsilon).$$

Next, the non-autonomous part is truncated and the remaining equation is analyzed. By using an asymptotic theory, one may show that the solution of the final equation can be used as approximation for the original equation.

5. PERIOD AVERAGED NORMAL FORMS

Suppose $\mathbf{J}\in\mathbf{R}^{n\times n}$, $\mathbf{\Gamma} = \{e^{tJ}, t\in\mathbf{R}\}$, is defined by equation (9), and T is defined by equation (10). Consider the following non-linear ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \epsilon\mathbf{f}(\mathbf{x}, \epsilon), \qquad \mathbf{x}\in\mathbf{\Omega}\subset\mathbf{R}^n, \tag{11}$$

where $0 < |\epsilon| \ll 1$, $\mathbf{f}\in C^{r+1}$ and $\mathbf{f}(0, \epsilon) = 0$. Ω is a domain which contains the origin and invariant under $\mathbf{\Gamma}$, and $\mathbf{\Gamma}x\in\Omega$ for any $x\in\Omega$. By making the transformation

$$\mathbf{x} = e^{tJ}\mathbf{y}, \qquad \dot{\mathbf{x}} = \mathbf{J}\, e^{tJ}\mathbf{y} + e^{tJ}\dot{\mathbf{y}} \tag{12}$$

and substituting into equation (11), gives

$$e^{tJ}\dot{\mathbf{y}} = \epsilon\mathbf{f}(e^{tJ}\mathbf{y}, \epsilon).$$

Let

$$\mathbf{g}(\mathbf{y}, t, \epsilon) = e^{-tJ}\mathbf{f}(e^{tJ}\mathbf{y}, \epsilon), \tag{13}$$

then equation (11) becomes

$$\dot{\mathbf{y}} = \epsilon\, e^{-tJ}\mathbf{f}(e^{tJ}\mathbf{y}, \epsilon) = \epsilon\mathbf{g}(\mathbf{y}, t, \epsilon) \tag{14}$$

Equation (14) is explicitly dependent on time while the original equation (11) is not. Equation (14) is usually solved by the averaging method and equation (11) by normal form method. The period averaged normal forms of equation (14) are constructed by means of Theorem 2.

Theorem 2. The change of variable

$$\mathbf{y} = \boldsymbol{\zeta} + \sum_{l=1}^{m} \epsilon^l \mathbf{h}_l(\boldsymbol{\zeta}, t) \tag{15}$$

transforms equation (14) by the averaging method to the following normal form up to order m

$$\dot{\boldsymbol{\zeta}} = \sum_{k=1}^{m} \epsilon^k \mathbf{f}_k^0(\boldsymbol{\zeta}) + \mathcal{O}(\epsilon^{m+1}) \tag{16}$$

where the transformations $\mathbf{h}_k(\boldsymbol{\zeta}, t)$ are determined from

$$\mathbf{h}_k(\boldsymbol{\zeta}, t) = \frac{1}{T}\int_0^T \tau[\mathbf{g}_k(\boldsymbol{\zeta}, \tau + t) - \mathbf{f}_k^0]\, d\tau \tag{17}$$

and the normal forms $\mathbf{f}_k^0(\boldsymbol{\zeta})$ are given by

$$\mathbf{f}_k^0(\boldsymbol{\zeta}) = \frac{1}{T}\int_0^T \mathbf{g}_k(\boldsymbol{\zeta}, \tau)\,\mathrm{d}\tau \tag{18}$$

and the expansion

$$\mathbf{g}_k(\boldsymbol{\zeta}, t) = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\epsilon^{k-1}}\mathbf{g}\left(\boldsymbol{\zeta} + \sum_{l=1}^{k-1}\epsilon^l\mathbf{h}_l(\boldsymbol{\zeta}, t), t, \epsilon\right)\Bigg|_{\epsilon=0} - \sum_{l=1}^{k-1}\mathbf{h}'_{k-l}(\boldsymbol{\zeta}, t)\mathbf{f}_l^0(\boldsymbol{\zeta}), \tag{19}$$

in which a prime denotes differentiation with respect to $\boldsymbol{\zeta}$, and

$$\mathbf{f}_k^0 \in C^{m+1-k}(\Omega, \mathbf{R}^m), \qquad \mathbf{h}_k \in C^{m+1-k}(\Omega \times \mathbf{R}, \mathbf{R}^m), \qquad \mathbf{f}_k^0(0) = 0,$$

$$\mathbf{h}_k(0, t) = 0, \qquad \mathbf{h}_k(\mathbf{x}, t) = \mathbf{h}_k(\mathbf{x}, t + T), \qquad \forall \mathbf{x} \in \Omega, \quad t \in \mathbf{R} \tag{20}$$

here $\mathbf{g}_k(\boldsymbol{\zeta}, t)$, $\mathbf{h}_k(\boldsymbol{\zeta}, t)$ and $\mathbf{f}_k^0(\boldsymbol{\zeta})$ satisfy the following time symmetries

$$\mathbf{h}_k(\mathrm{e}^{tJ}\mathbf{x}, t) = \mathrm{e}^{sJ}\mathbf{h}_k(\mathbf{x}, s + t), \forall s, t \in \mathbf{R}, \tag{21}$$

$$\mathbf{f}_k^0(\mathrm{e}^{sJ}\mathbf{x}) = \mathrm{e}^{sJ}\mathbf{f}_k^0(\mathbf{x}), \qquad \forall s \in \mathbf{R}, \qquad \mathrm{e}^{-\tau J}\mathbf{g}_k(\mathrm{e}^{\tau J}\boldsymbol{\zeta}, t) = \mathbf{g}_k(\boldsymbol{\zeta}, t + \tau) \tag{22, 23}$$

A proof of Theorem 2 is given in Appendix A. The time variable disappears from equations (15) and (16) as a result of the averaging process. Theorem 2 suggests a convenient way to find the normal form. It eliminates the determination of the complement to $\mathrm{Im}\,\mathrm{ad}_J^k$ in the matrix representation and the adjoint operator methods [10, 11]. The theorem unifies the formulation of averaging and normal form for periodic oscillation.

6. THE MATHEMATICA PROGRAM

A concise program in Mathematica language is given in Appendix B for a set of two first order ordinary differential equations whose Jordan matrix is

$$\omega\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The first three lines are the parameters and functions input by the user. The first line k = the highest order of ϵ required. The second line w = ω which is the linear frequency. The third line $\mathbf{f} = \{\mathrm{f1}, \mathrm{f2}\}$ where f1 and f2 are the two expressions of the two dimensional vector function $\mathbf{f}$ in equation (11). After running, the result is saved in the last line of the programme to a file with the filename "olf" in the current directory. The first part of olf is $\dot{r}$, and the second part of olf is $\dot{\theta}$. Solutions of four examples are produced in the following section by this programme.

7. EXAMPLES

Four examples whose higher normal form or averaged solutions are obtained by the above programme are considered. The initial conditions are of no concern because interest is in the limit cycles obtained from the steady state $\dot{r} = 0$. The actual value of the small parameter ϵ is not needed before the analysis.

Example 1. Consider the following ordinary differential equations as the first example:

$$\ddot{x} + x = \epsilon[x^2 + \epsilon\dot{x}(1 - \dot{x}^2/3)],$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon\begin{bmatrix} 0 \\ x_1^2 + \epsilon x_2(1 - x_2^2/3) \end{bmatrix}, \qquad \epsilon > 0 \text{ is small.} \tag{24}$$

To calculate the fourth order averaging equations of equation (24), take k = 4, w = 1 and $f = x_1^2 + \epsilon x_2(1 - x_2^2/3)$ (i.e., $\mathbf{f} = \{0, \text{x1\^{}2+ep x2}(1 - \text{x2\^{}2}/3)\}$) in the programme. The fourth order averaging equations are

$$\dot{\boldsymbol{\zeta}} = \epsilon^2\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) + \mathcal{O}(\epsilon^6), \tag{25}$$

where

$$\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) = \begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & -\zeta_1 \end{bmatrix}\begin{bmatrix} (4 - r^2)/8 - (8 - 5r^2)r^2\epsilon^2/144 \\ -5r^2/12 - (864 - 432r^2 + 3221r^4)\epsilon^2/6912 \end{bmatrix},$$

$$r^2 = \zeta_1^2 + \zeta_2^2.$$

In order to translate equation (25) into polar co-ordinate form, let $\zeta_1 = r\cos\theta$, $\zeta_2 = r\sin\theta$, then

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{bmatrix}\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix}$$

$$= \epsilon^2\begin{bmatrix} (4 - r^2)r/8 - (8 - 5r^2)r^3\epsilon^2/144 \\ 5r^2/12 + (864 - 432r^2 + 3221r^4)\epsilon^2/6912 \end{bmatrix} + \mathcal{O}(\epsilon^6)$$

The result obtained by this program is the same as the results in reference [2].

Example 2. Consider the Duffing oscillator with no damping term:

$$\ddot{x} + \omega_1^2 x = cx^3\epsilon^2. \tag{26}$$

Let $x_1 = x$, $\dot{x}_1 = -\omega_1 x_2$, and substitute these transformations into equation (26),

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon\begin{bmatrix} 0 \\ -(c/\omega_1)x_1^3\epsilon \end{bmatrix}.$$

By taking k = 6, w = $w1$ and $f = -(c/\omega_1)x_1^3\epsilon$ (i.e., $\mathbf{f} = \{0, -c/w1 \text{ x1\^{}3 ep}\}$) in the program, the sixth order averaging equations are

$$\dot{\boldsymbol{\zeta}} = \epsilon^2\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) + \mathcal{O}(\epsilon^8), \tag{27}$$

where

$$\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) = \begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & -\zeta_1 \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{3cr^2}{8\omega_1} - \dfrac{51c^2r^4}{256\omega_1^3}\epsilon^2 + \dfrac{1419c^3r^6}{8192\omega_1^5}\epsilon^4 \end{bmatrix},$$

$$r^2 = \zeta_1^2 + \zeta_2^2$$

In order to translate the equation into polar co-ordinate form, let $\zeta_1 = r\cos\theta$, $\zeta_2 = n\sin\theta$, then

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{bmatrix} \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix}$$

$$= \epsilon^2 \begin{bmatrix} 0 \\ -\dfrac{3cr^2}{8\omega_1} + \dfrac{51c^2r^4}{256\omega_1^3}\epsilon^2 - \dfrac{1419c^3r^6}{8192\omega_1^5}\epsilon^4 \end{bmatrix} + \mathcal{O}(\epsilon^8).$$

The fourth order result obtained by this program is the same as the result in reference [8].

Example 3. Consider the Van der Pol oscillator with no damping term:

$$\ddot{x} + \epsilon(\epsilon x^2 - c)\dot{x} + x = 0. \tag{28}$$

Let $x_1 = x$, $\dot{x}_1 = x_2$, and substitute these transformations into equation (28),

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ -(\epsilon x_1^2 - c)x_2 \end{bmatrix}. \tag{29}$$

By taking k = 4, w = 1 and $f = -(\epsilon x_1^2 - c)x_2$ (i.e., **f** = {0, −(ep x1^2−c)x2}) in the program, the fourth order averaging equations are

$$\dot{\boldsymbol{\zeta}} = \epsilon\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) + \mathcal{O}(\epsilon^5)$$

where

$$\mathbf{f}^0(\boldsymbol{\zeta}, \epsilon) = \begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_2 & -\zeta_1 \end{bmatrix} \begin{bmatrix} c/2 - r^2\epsilon/8 + c^2(4c - 3r^2)\epsilon^3/128 \\ -c^2/8 + 3cr^2\epsilon^2/16 - (2c^4 + 11r^4)\epsilon^3/256 \end{bmatrix},$$

$$r^2 = \zeta_1^2 + \zeta_2^2.$$

In order to translate the equation into polar co-ordinate form, let $\zeta_1 = r\cos\theta$, $\zeta_2 = r\sin\theta$, then

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{bmatrix} \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix}$$

$$= \epsilon \begin{bmatrix} cr/2 - r^3\epsilon/8 + c^2r(4c - 3r^2)\epsilon^3/128 \\ c^2/8 - 3cr^2\epsilon^2/16 + (2c^4 + 11r^4)\epsilon^3/256 \end{bmatrix} + \mathcal{O}(\epsilon^5)$$

For steady state, $\dot{r} = \epsilon r[c/2 - r^2\epsilon/8 + c^2(4c - 3r^2)\epsilon^3/128] = 0$, one can always find a relation between c and ϵ to determine the non-trivial solution for r for limit cycles. For example, up to $\mathcal{O}(\epsilon^4)$, $4c - r^2\epsilon = 0$, or $r = 2\sqrt{c/\epsilon}$, which is possible when both c and ϵ are greater than zero. Up to $\mathcal{O}(\epsilon^5)$ $c/2 - r^2\epsilon/8 + c^2(4c - 3r^2)\epsilon^3/128 = 0$, or $r^2 = (64c + 4c^3\epsilon^3)/(16\epsilon - 3c^2\epsilon^3)$, so long as $16 > 3c^2\epsilon^2$, the limit cycle exists.

Example 4. Consider the following equation which was requested by one of the reviewers.

$$\ddot{x} + \epsilon a\dot{x} + x = \epsilon b x^2 + \epsilon c\dot{x}^3 \tag{30}$$

where a, b, c are parameters and the linear natural frequency has been normalized to unity. Equation (30) is equivalent to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ -ax_2 + bx_1^2 + cx_2^3 \end{bmatrix}. \tag{31}$$

Let k = 4, for the fourth order solution, and $f = \epsilon(-ax_2 + bx_1^2 + cx_2^3)$, then one has the fourth order averaged equation

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{bmatrix} \begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon^2 r(3cr^2/8 - a/2) \\ \epsilon^4(-a^2/8 + r^2(3ac/16 - 5b^2/12) - 27c^2r^4/256) \end{bmatrix} + \mathcal{O}(\epsilon^5).$$

For steady state, $\dot{r} = \epsilon^2 r(3cr^2/8 - a/2) = 0$, one has the limit cycle $r^2 = 4a/3c$. It is interesting to note that the parameter b has no contribution to the limit cycle.

8. CONCLUSION

A combined method of normal form and averaging which takes the advantages of the both methods has been considered. The simplicity of the method enables one to develop a short program to find the higher order averaged equations or normal forms. Four examples have been given for comparison with existing results.

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APPENDIX A

Proof of Theorem 2. Theorem 2 is proved in three steps. In the first step, it is proved that equations (20) are indeed the conditions for the transformations $\mathbf{h}_k$ and the normal forms $\mathbf{f}_k^0$. In the second step, the relation between $\mathbf{g}$ and $\mathbf{g}_k$ is given so that $\dot{\zeta} = \Sigma_{k=1}^{m}\epsilon^k[-\dot{\mathbf{h}}_k + \mathbf{g}_k] + \mathcal{O}(\epsilon^{m+1})$. The notion $\mathcal{O}(\epsilon^{m+1})$ will be omitted below. Finally, it is proved that $\mathbf{f}_k^0 = -\dot{\mathbf{h}}_k + \mathbf{g}_k$ and $\mathbf{f}_k^0$, $\mathbf{h}_k$ and $\mathbf{g}_k$ satisfy the symmetry relations (21)–(23).

Step 1. Putting $\mathbf{y} = 0$ in equation (10) gives $\mathbf{f}_k^0(0) = \mathrm{e}^{tJ}\mathbf{f}_k^0(0)$. Since e^{tJ} is not always zero, $\mathbf{f}_k^0(0) = 0$. Putting $\mathbf{y} = 0$ in equation (15) gives $0 = 0 + \Sigma_{\ell=1}^{m}\epsilon^{\ell}\mathbf{h}_{\ell}(0, t)$, therefore $\mathbf{h}_{\ell}(0, t) = 0$. Because $\mathbf{h}_k$ is T-periodic, equations (20) are indeed the conditions for the transformations $\mathbf{h}_k$ and the normal forms $\mathbf{f}_k^0$.

Step 2. From equation (15), $\mathbf{y} = \boldsymbol{\zeta} + \Sigma_{l=1}^{m}\epsilon^l\mathbf{h}_l(\boldsymbol{\zeta}, t)$. After differentiation with respect to time, one has from equation (14), $\dot{\mathbf{y}} = \dot{\boldsymbol{\zeta}} + \Sigma_{l=1}^{m}\epsilon^l[\mathbf{h}_l'(\boldsymbol{\zeta}, t)\dot{\boldsymbol{\zeta}} + \dot{\mathbf{h}}_l] = \epsilon\mathbf{g}(y, t, \epsilon)$. Putting $\dot{\boldsymbol{\zeta}} = \epsilon\mathbf{g}(\mathbf{y}, t, \epsilon) - \Sigma_{l=1}^{m}\epsilon^l[\mathbf{h}_l'\dot{\zeta} + \dot{\mathbf{h}}_l] = \Sigma\epsilon^k[\mathbf{g}_k - \dot{\mathbf{h}}_k]$, then, one obtains the relation (19) between $\mathbf{g}$ and $\mathbf{g}_k$.

Step 3. The time symmetry conditions (21) to (23) are proved by mathematical induction. When $k = 1$, from definitions (13) and (19), $\mathbf{g}_1(\boldsymbol{\zeta}, t) = \mathbf{g}(\boldsymbol{\zeta}, t, 0) = \mathrm{e}^{-tJ}\mathbf{f}(\mathrm{e}^{tJ}\boldsymbol{\zeta}, 0)$, one has $\mathbf{g}_1(\boldsymbol{\zeta}, t + s) = \mathrm{e}^{-sJ}[\mathrm{e}^{-tJ}\mathbf{f}(\mathrm{e}^{tJ}\,\mathrm{e}^{sJ}\boldsymbol{\zeta}, 0)] = \mathrm{e}^{-sJ}\mathbf{g}_1(\mathrm{e}^{sJ}\boldsymbol{\zeta}, t)$, for equation (23). Also, $\mathrm{e}^{-tJ}\mathbf{f}_1^0(\mathrm{e}^{tJ}\boldsymbol{\zeta}) = (1/T)\int_0^T\mathbf{g}_1(\boldsymbol{\zeta}, t + \tau)\,\mathrm{d}\tau = (1/T)\int_t^{t+T}\mathbf{g}_1(\boldsymbol{\zeta}, s)\,\mathrm{d}s = \mathbf{f}_1^0(\boldsymbol{\zeta})$, for equation (22). Therefore,

$$\dot{\mathbf{h}}_1(\zeta, t) = \frac{1}{T}\int_0^T \tau\,\frac{\partial}{\partial t}\,[\mathbf{g}_1(\boldsymbol{\zeta}, \tau + t) - \mathbf{f}_1^0(\boldsymbol{\zeta})]\,\mathrm{d}\tau$$

$$= \frac{1}{T}\int_0^T \tau\,\frac{\partial}{\partial \tau}\,[\mathbf{g}_1(\boldsymbol{\zeta}, \tau + t) - \mathbf{f}_1^0(\boldsymbol{\zeta})]\,\mathrm{d}\tau$$

$$= \mathbf{g}_1(\zeta, t) - \mathbf{f}_1^0(\boldsymbol{\zeta})$$

$$e^{-tJ}\dot{\mathbf{h}}_1(e^{tJ}\boldsymbol{\zeta}, s) = e^{-tJ}\mathbf{g}_1(e^{tJ}\boldsymbol{\zeta}, s) - e^{-tJ}\mathbf{f}_1^0(e^{tJ}\boldsymbol{\zeta})$$

$$= \mathbf{g}_1(\boldsymbol{\zeta}, t+s) - \mathbf{f}_1^0(\boldsymbol{\zeta}) = \dot{\mathbf{h}}_1(\boldsymbol{\zeta}, s+t)$$

and equation (21) for $k = 1$ is correct.

Assume equations (21)–(23) are correct for $l < k$. Then replace $\boldsymbol{\zeta}$ by $e^{\tau J}\boldsymbol{\zeta}$ in equation (19) and premultiply the equation by $e^{-\tau J}$, it can be proven that equation (23) is valid for $l = k$. Hence, one can prove that

$$e^{-tJ}\mathbf{f}_k^0(e^{tJ}\boldsymbol{\zeta}) = \frac{1}{T}\int_0^T \mathbf{g}_k(\boldsymbol{\zeta}, t+\tau)\,d\tau = \frac{1}{T}\int_t^{t+T} \mathbf{g}_k(\boldsymbol{\zeta}, s)\,ds = \mathbf{f}_k^0(\boldsymbol{\zeta}),$$

$$\dot{\mathbf{h}}_k(\boldsymbol{\zeta}, t) = \mathbf{g}_k(\boldsymbol{\zeta}, t) - \mathbf{f}_k^0(\boldsymbol{\zeta}) \qquad \text{and} \qquad e^{-tJ}\dot{\mathbf{h}}_k(e^{tJ}\boldsymbol{\zeta}, s) = \dot{\mathbf{h}}_k(\boldsymbol{\zeta}, s+t)$$

similarly as for the case $l = 1$. □

APPENDIX B: CALCULATING PROGRAM IN MATHEMATICA LANGUAGE

k = the order of the high order averaging equation, w = natural frequency of the non-linear oscillator, f = {0, the nonlinear function}.

```
          y = {y1,y2}
        eta = {{Cos[w t],Sin[w t]},{-Sin[w t],Cos[w t]}}
          x = eta.y
         x1 = x[[1]]
         x2 = x[[2]]
       enta = Simplify[Inverse[eta]]
          g = enta.f>>og
         ff = Table[0,{k},{2}]
         gg = Table[0,{k},{2}]
         hh = Table[0,{k},{2}]
         dh = Table[0,{k},{2},{2}]
        dhf = {0,0}
         lf = {0,0}
     gg[[1]] = g/.ep->0>>og1
         tt = 2 Pi/w
        fit = {{Cos[ct],Sin[ct]},{-Sin[ct]/r,Cos[ct]/r}}
     ff[[1]] = 1/tt Integrate[gg[[1]],{t, 0, tt}];
         lf = Simplify[Expand[fit.ff[[1]]/.{y1->r Cos[ct],y2->r Sin[ct]}]];
        Do[
        yy1 = y1;
        yy2 = y2;
hh[[i-1]] = 1/tt Integrate[tao(gg[[i-1]] - ff[[i-1]])/.t->(t + tao),{tao,0,tt}];
dh[[i-1]] = Outer[D,hh[[i-1]],{y1,y2}];
   Do[yy1 = yy1 + ep^j hh[[j,1]],{j,1,i-1}];
   Do[yy2 = yy2 + ep^j hh[[j,2]],{j,1,i-1}];
```

```
Do[dhf = dhf + dh[[i − j]].ff[[j]],{j,1,i − 1}];
  gg[[i]] = (1/(i − 1)!D[g/.{y1 − > yy1,y2 − > yy2},{ep,i − 1}]/.ep − > 0)
            − dhf;
   ff[[i]] = 1/tt Integrate[gg[[i]],{t,0,tt}];
       lf = lf + ep^(i−1) Simplify[Expand[fit.ff[[i]]/
                                      .{y1 − > r  Cos[ct], y2 − > r Sin[ct]}]],
            {i,2,k}]
            ep lf≫olf
```