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AN APPROXIMATE ANALYTICAL SOLUTION FOR PLANE SOUND WAVE TRANSMISSION IN INHOMOGENEOUS DUCTS

E. Dokumaci

Department of Mechanical Engineering, Dokuz Eylul University, Bornova, Izmir, Turkey

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A general approximate analytical solution is presented for transmission of plane sound waves in subsonic low Mach number ducts with axial ambient gradients and cross-sectional area variations. This is essentially a high frequency approximation and explicit criteria are presented for the estimation of the lower limiting frequency in practical applications. For uniform ducts, the proposed solution is almost exact for the temperature gradients and frequencies that are likely to be encountered in practice. In the case of non-uniform ducts, the lower limiting frequency increases with the cross-sectional area gradient and in applications where low frequencies are important, the utility of the approximate solution will be limited to ducts with a relatively small area ratio.

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1. INTRODUCTION

The object of this paper is to present a simple general approximate analytical solution for transmission of plane sound waves in subsonic low Mach number ducts with axial ambient gradients and cross-sectional area variations. Approximate analytical solutions for uniform ducts with axial temperature gradients have been presented previously by Cummings [1, 2] using the WKB method. Non-uniform ducts carrying an incompressible mean flow with Mach number squared much less than unity [3], and the classical horn equation, admit exact analytical solutions for certain duct shapes; however, a general analytical solution applicable for ducts of any shape, subsonic low mean flow Mach numbers and axial ambient gradients is not available.

As is well known, the plane sound wave field in a uniform homogeneous duct is given by the superposition of two uncoupled plane waves travelling in opposite directions. With ambient gradients and cross-sectional area variations present, the two travelling waves become coupled; however, in many practical cases, the axial non-uniformities are slowly varying functions of the duct axis and the coupling between the travelling waves is weak. The coupling terms in the state–space equations for the travelling wave components may then be neglected to obtain the simple general approximate analytical solution proposed in this paper. This approach has been used previously in reference [4] for the analysis of sound

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propagation in uniform pipes with axial temperature gradients. In the present paper it is extended to the general case of plane sound wave propagation in ducts with axial ambient gradients and cross-sectional area variations. In the former case [4], the approximate solutions are almost exact for temperature gradients and frequencies that are likely to be encountered in practice. With cross-sectional area variations present, however, the accuracy of the proposed approximate approach becomes more critical. Therefore, in the present paper, further considerations are given to the question of accuracy. In particular, it is shown that the proposed analytical solution is essentially a high frequency approximation and explicit criteria are presented for the estimation of the lower limiting frequency in practical applications. Approximate solutions are compared with the corresponding exact ones, which are computed by using a numerical matrizant method which is described briefly in the Appendix.

2. PROBLEM FORMULATION AND AN APPROXIMATE SOLUTION

2.1. BASIC ACOUSTIC EQUATIONS

The equations of plane wave duct acoustics come from the one-dimensional gas dynamic equations. Upon assuming $e^{-i\omega t}$ time dependence, where i denotes the unit imaginary number, ω is the radian frequency and t is the time, the basic acoustic equations are as follows.

The continuity equation is

$$-\mathrm{i}\omega\rho + v_o\rho' + v\rho_o' + \rho_o v' + \rho v_o' + (\rho_o v + \rho v_o)(\ln S)' = 0. \tag{1}$$

The momentum equation is

$$\rho_o(-\mathrm{i}\omega v + v_o v' + v v_o') + v_o v_o'\rho + p' = 0. \tag{2}$$

For a perfect gas, the energy equation, which is tantamount to the statement that the sound propagation is isentropic, can be expressed as [5]

$$-\mathrm{i}\omega p + v_o p' + v p_o' + \gamma_o p_o v' + \gamma_o p v_o' + \gamma_o(p_o v + p v_o)(\ln S)' = 0. \tag{3}$$

Here, a prime (′) denotes differentiation with respect to x, the duct axis, S denotes the duct cross-sectional area and γ_o is the ratio of specific heat coefficients of the ambient gas. ρ, p and v, the acoustic density, pressure and particle velocity, respectively, are first order perturbations, with zero averages, superimposed on the corresponding steady mean flow values ρ_o, p_o and v_o, which are to be understood as cross-sectional averages and are assumed to be known as functions of x, with the continuity equation $(\rho_o v_o S)' = 0$ and the state equation $p_o = \rho_o R T_o$ satisfied along the duct. Here, R denotes the gas constant and T_o is the ambient temperature.

2.2. TRANSFORMATION TO PRESSURE WAVE COMPONENTS

The following transformation is applied to equations (1)–(3) in order to recast them in a form that displays the coupling between the travelling wave components, namely,

$$p = p^+ + p^-, \qquad \rho_o c_o v = p^+ - p^-, \qquad c_o^2 \rho = p + \varepsilon, \tag{4a, b, c}$$

where $c_o = \sqrt{\gamma_o R T_o}$ is the local speed of sound. By using equations (4) in the basic acoustic equations, it can be shown, after some algebra, that the pressure wave components, p^+ and p^-, are given by

$$\begin{bmatrix} p^{+\prime} \\ p^{-\prime} \end{bmatrix} = \begin{bmatrix} A[M_o] & B[M_o] \\ B[-M_o] & A^*[-\mathrm{M}_o] \end{bmatrix} \begin{bmatrix} p^+ \\ p^- \end{bmatrix} - \frac{M_o^2 (\ln v_o)'}{2(1 - M_o^2)} \begin{bmatrix} 1 - M_o \\ 1 + \mathrm{M}_o \end{bmatrix} \varepsilon, \tag{5}$$

where $M_o = v_o/c_o$ denotes the local mean flow Mach number, an asterisk (*) denotes complex conjugate and $A[M_o]$, $B[M_o]$ and ε are given by

$$A[M_o] = (\mathrm{i}2k_o + (1 + M_o)(\ln \rho_o c_o)' - M_o(1 + \gamma_o + M_o)(\ln v_o)'$$
$$- (\ln p_o)'/\gamma_o - (1 + \gamma_o M_o)(\ln S)')/2(1 + M_o), \tag{6}$$

$$B[M_o] = (- (1 + M_o)(\ln \rho_o c_o)' - M_o(-1 + \gamma_o + M_o)(\ln v_o)'$$
$$+ (\ln p_o)'/\gamma_o + (1 - \gamma_o M_o)(\ln S)')/2(1 + M_o), \tag{7}$$

$$\varepsilon' - (\mathrm{i}k_o/M_o + (\ln \gamma_o p_o)')\varepsilon = (\ln \gamma_o)' p, \tag{8}$$

where $k_o = \omega/c_o$ denotes the local wavenumber. It should be noted that $A[M_o]$ and $B[M_o]$ are functions of x: that is, $A[M_o] = A(x)$ and $B[M_o] = B(x)$. The symbolic notation $A[M_o]$ and $B[M_o]$ is used here to indiacate the relationship between the diagonal and the off-diagonal elements of the state–space matrix in equation (5). In the case of a compressible subsonic mean flow, the following isentropic relationships apply for the gradients in equations (6)–(8), namely, $(\ln v_o)' = -(\ln S)'/(1 - M_o^2)$, $(\ln \rho_o)' + M_o^2(\ln v_o)' = 0$ and $(\ln p_o)' = \gamma_o(\ln \rho_o)'$. Equations (5) and (8) can also be solved for prescribed ambient temperature and pressure distributions.

2.3. A LOW MACH NUMBER APPROXIMATION

Equations (5) and (8) have been solved numerically, by using a numerical matrizant method which is similar to that described in the Appendix, as a coupled system of three first order differential equations in p^+, p^- and ε. This part of the analysis, which has been presented elsewhere [5], has shown that, for the subsonic low Mach numbers that are of interest here, say $M_o < 0{\cdot}3$, the second term on the right of equation (5) is negligibly small. Thus, the following simplified form of equation (6) is assumed to be valid throughout the present analysis, namely,

$$\begin{bmatrix} p^{+\prime} \\ p^{-\prime} \end{bmatrix} = \begin{bmatrix} A[M_o] & B[M_o] \\ B[-M_o] & A^*[-M_o] \end{bmatrix} \begin{bmatrix} p^+ \\ p^- \end{bmatrix}. \tag{9}$$

The more usual approach to reduce equations (1)–(3) into a system of differential equations with two variables is to neglect the product $v_o' v_o \rho$ as a small term of the second order in the momentum equation, equation (2). This omission decouples equation (1) from equations (2) and (3), and p and v can then be determined by solving only the latter two equations. If this approach were adopted in the present analysis, equation (4c) would not be needed, and in equations (6) and (7) the term M_o in the coefficients of $(\ln v_o)'$ in brackets would be absent. The present approach, which is tantamount to assuming that $v_o' v_o \varepsilon / c_o^2$ is small to second order, is more accurate because, for subsonic low Mach number ducts, the error in the isentropic relationship $p = c_0^2 \rho$ is negligibly small and, therefore, the term $v_o' v_o \varepsilon / c_o^2$ is some orders of magnitude smaller than the term $v_o' v_o \rho$. Indeed, equation (9) can be derived directly by replacing the energy equation by $p = c_o^2 \rho$.

2.4. GENERAL FORM OF THE PROPOSED APPROXIMATE SOLUTION

For a duct of length L, the general solution of equation (9) can be expressed as,

$$\begin{bmatrix} p^+(L) \\ p^-(L) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} p^+(0) \\ p^-(0) \end{bmatrix}, \tag{10}$$

where the square 2×2 matrix is called the scattering matrix of the duct. The exact duct scattering matrix can be evaluated accurately by using the numerical matrizant method described in the Appendix.

An important feature of equation (9) is that the off-diagonal elements of the state–space matrix, $B[M_o]$ and $B[-M_o]$, are independent of frequency and their magnitudes are determined only by the spatial gradients. Then, if the latter are small enough, equation (9) may be decoupled by neglecting the off-diagonal terms of the state–space matrix, in which case, the solution of equation (9) is given by

$$p^+(x) = p^+(0) \exp\left(\int_0^x A[M_o] \, \mathrm{d}x\right), \qquad p^-(x) = p^-(0) \exp\left(\int_0^x A^*[-M_o] \, \mathrm{d}x\right), \tag{11a, b}$$

Hence, for a duct of length L, the elements of the scattering matrix can be expressed as, approximately,

$$T_{11} = \exp\left(\int_0^L A[M_o] \, \mathrm{d}x\right), \qquad T_{12} = 0,$$

$$T_{21} = 0, \qquad T_{22} = \exp\left(\int_0^L A^*[-M_o] \, \mathrm{d}x. \tag{12a, b}$$

This is the general approximate analytical solution that is proposed in this paper. Thus, insofar as the deletion of the off-diagonal terms of the state–space matrix is valid, the solution of equation (9) consists of two slightly attenuated uncoupled waves travelling in $+x$ and $-x$ directions with phase velocities $c_o(1 + M_o)$ and $c_o(1 - M_o)$, and attenuation determined by the real parts of $A[M_o]$ and $A^*[-M_o]$, respectively. The ellimination of the coupling between the pressure wave components may cause significant errors if the spatial gradients are not small enough. The next section is, therefore, devoted to the conditions under which the proposed approximate solution will give satisfactory results.

3. A CRITICAL STUDY OF THE PROPOSED APPROXIMATE SOLUTION

This section will present criteria which can be used to estimate the accuracy to expect when using equations (12) in a given problem. The analysis is based on ducts with no mean flow, but the results derived on this basis are applicable also when there exists a subsonic low Mach number mean flow. Indeed, uniform mean flow in a uniform duct causes no coupling between the pressure wave components. The coupling of the plane wave components is caused primarily by ambient gradients and cross-sectional area variations, the non-uniformity which the mean flow acquires by the presence of these primary effects having only a secondary contribution. Some numerical results which confirm this are presented later in the section.

3.1. THE GENERAL FORM OF THE LOWER LIMITING FREQUENCY CRITERION

For a duct in which the mean flow is negligible but there exists arbitrary ambient gradients, equations (6) and (7) can be expressed as

$$A(x) = \mathrm{i}k - \Psi/2, \qquad B(x) = \Psi/2, \tag{13a, b}$$

where

$$\Psi(x) = (\ln S/\rho_o c_o)' + (\ln p_o)'/\gamma_o. \tag{14}$$

Substituting equations (13a) and (13b) into equation (9) and backtransforming to the variables p and v, one obtains, after some algebra,

$$p'' + (\Psi + (\ln c_o)')p' + k^2 p = 0, \tag{15}$$

$$V'' + (\Psi + (\ln c_o)')V' + (k^2 + \Psi(\ln c_o)' + \Psi')V = 0, \tag{16}$$

where $V = \rho_o c_o v$. A similar backtransformation can also be applied to equation (9) where $A(x)$ is given by equation (13a) and $B(x) = 0$. Obviously, this is equivalent to deletion of the off-diagonal terms in equation (9), as was assumed in the proposed approximate analytical solution, equations (11). Thus, it follows that the latter is in fact the exact solution of the following set of equations:

$$p'' + (\Psi + (\ln c_o)')p' + (k^2 + (\ln c_o)'\Psi/2 + \Psi'/2 + \Psi^2/4)p = 0, \tag{17}$$

$$V'' + (\Psi + (\ln c_o)')V' + (k^2 + (\ln c_o)'\Psi/2 + \Psi'/2 + \Psi^2/4)V = 0. \tag{18}$$

Equations (15) and (17) will be approximately identical if

$$4k^2 \gg |2\Psi(\ln c_o)' + 2\Psi' + \Psi^2|. \tag{19}$$

If this condition is satisfied, then equation (18) will reduce to

$$V'' + (\Psi + (\ln c_o)')V' + k^2 V = 0, \tag{20}$$

which will coincide with equation (16) if

$$k^2 \gg |\Psi(\ln c_o)' + \Psi'|. \tag{21}$$

The more critical of the conditions (19) and (21) then determines the condition for the proposed approximate analytical solution to be close to the corresponding exact solution. These conditions will obviously be satisfied if the frequency is high enough. This proves that the proposed approximate analytical solution is essentially a high frequency approximation.

With mean flow, the elements of the state–space matrix in equation (9) are considerably more complicated and do not lend themselves to a similar explicit analysis, however, as has been pointed out at the beginning of this section, the foregoing results are applicable also for ducts carrying a subsonic low Mach number mean flow. Numerical results that confirm this are presented in what follows together with the implementations of conditions (19) and (21) for some practical duct acoustics problems. The main question here is how to replace a much greater than "$\gg$" criterion by greater than "$>$" criterion. Here, this question is dealt with heuristically for each case considered, by comparing the predictions of the approximate theory to the exact results computed by using a numerical matrizant method. A brief description of this method is given in the Appendix. The trapezoid formula is used for the evaluation of the integrals in equations (12)

3.2. UNIFORM DUCTS WITH TEMPERATURE GRADIENTS

Consider a uniform duct with $p_o' = 0$, $v_o = 0$ and $\gamma_o' = 0$. Previously, this problem has been considered by Cummings [1] who presented an approximate analytical solution by using the WKB method. For this duct, $\Psi = (\ln c_o)' = -(\ln \rho_o)'/2$ and, therefore, conditions (19) and (21) can be expressed as,

$$k \gg \sqrt{|7\delta^2 - \delta(\ln \rho_o')'|}, \qquad k \gg \sqrt{2|6\delta^2 - \delta(\ln \rho_o')'|}, \tag{22a, b}$$

respectively, where $\delta = -\Psi/2$. Conditions (22) are applicable for any form of temperature distribution. Here, it is expedient to consider the case of linear temperature distribution, that is, $T_o(x) = (1 + \tau x)T_o(0)$, where $|\tau x|$ is not necessarily small. For this case $(\ln \rho_o')' = 8\delta$ and, therefore, the second condition turns out to be the more critical one. With the appropriate substitutions made for δ, this condition can be expressed as, for a duct of length L,

$$kL \gg |\Delta T_o|/2T_o, \tag{23a}$$

or, more explicitly,

$$f \gg f^* = \frac{|\Delta T_o|}{4\pi L}\sqrt{\frac{\gamma_o R}{T_{oc}}}, \tag{23b}$$

where f is the frequency, Δ denotes the difference between the terminal values of its argument, in this case T_o, and T_{oc} is the temperature, in K, of the colder end of the duct.

The elements of the exact and approximate solutions for the duct scattering matrix were computed for inlet mean flow Mach numbers, $M_o(0)$, up to 0·3 and temperature gradients as high as 300°C/m. No results are presented here as no deviation worthy of reporting really exists between the exact and approximate solutions, and condition (23b) appears to be applicable as $f > f^*$, approximately. This applies for both the diagonal and the off-diagonal elements of the duct scattering matrix. Thus, for example, if the lower limiting frequency is set at 10 Hz, then, using $\gamma_o = 1{\cdot}4$ and $R = 287$, condition (23b) reduces to $\Delta T_o < 6{\cdot}27\, L\sqrt{T_{oc}}$: that is, if the cold end temperature of a 1-m long duct is, say, 600°C, the approximate analytical solution will be accurate for high end temperatures up to 780°C.

It should be noted that, while the present results are complementary to those presented in reference [4], there is a slight difference between the governing equations used in the two analyses, which is due to the adoption in reference [4] of the simplification based on the omission of the term $v_o' v_o \rho$, as described in section 2.3. Also, it may be of interest to note that, condition (22a) is a required condition in the WKB solution of reference [1], too.

3.3. NON-UNIFORM DUCTS WITH NO TEMPERATURE GRADIENT

Consider a non-uniform duct with $p_o' = 0$, $\rho_o' = 0$, $v_o = 0$ and $c_o' = 0$. For this duct $\Psi = (\ln S)'$ and conditions (19) and (21) become, respectively,

$$2k \gg \sqrt{|(\ln S)'\{2(\ln S')' - (\ln S)'\}|}, \qquad k \gg \sqrt{|(\ln S)'\{(\ln S')' - (\ln S)'\}|}. \quad \text{(24a, b)}$$

These will be applied here to exponential and conical ducts. Explicit criteria for ducts with other shapes can be derived similarly.

For an exponential duct, $S(x) = S(0)\,\mathrm{e}^{2mx}$, that is, $(\ln S')' = (\ln S)'$ and, therefore, conditions (19) and (21) reduce to $k \gg |m|$ and $k \gg 0$, respectively. Hence, in this case condition (24a) is the more critical of the two conditions and, for a duct of length L, gives the following criterion, namely

$$kL \ll |\Delta \ln S|/2, \quad \text{(25a)}$$

or,

$$f \gg f^* = (c_o/4\pi L)|\ln\{S(L)/S(0)\}|. \quad \text{(25b)}$$

This result is valid for both convergent and divergent exponential ducts.

Shown in Figure 1, for the case of no mean flow, are the exact and approximate solutions, as computed by using the numerical matrizant method and equations (12), respectively, for the elements of the scattering matrix of a 1-m long exponential duct with $S(0) = 0{\cdot}01$ m^2, $S(L) = 0{\cdot}02$ m^2 and $T_o = 25$°C. Figure 2 shows the elements of the scattering matrix of the same duct but with an incompressible mean flow of Mach number $M_o(0) = 0{\cdot}3$. For this duct $f^* = 19$ Hz and, as can be seen from Figures 1 and 2, condition (25b) can be implemented

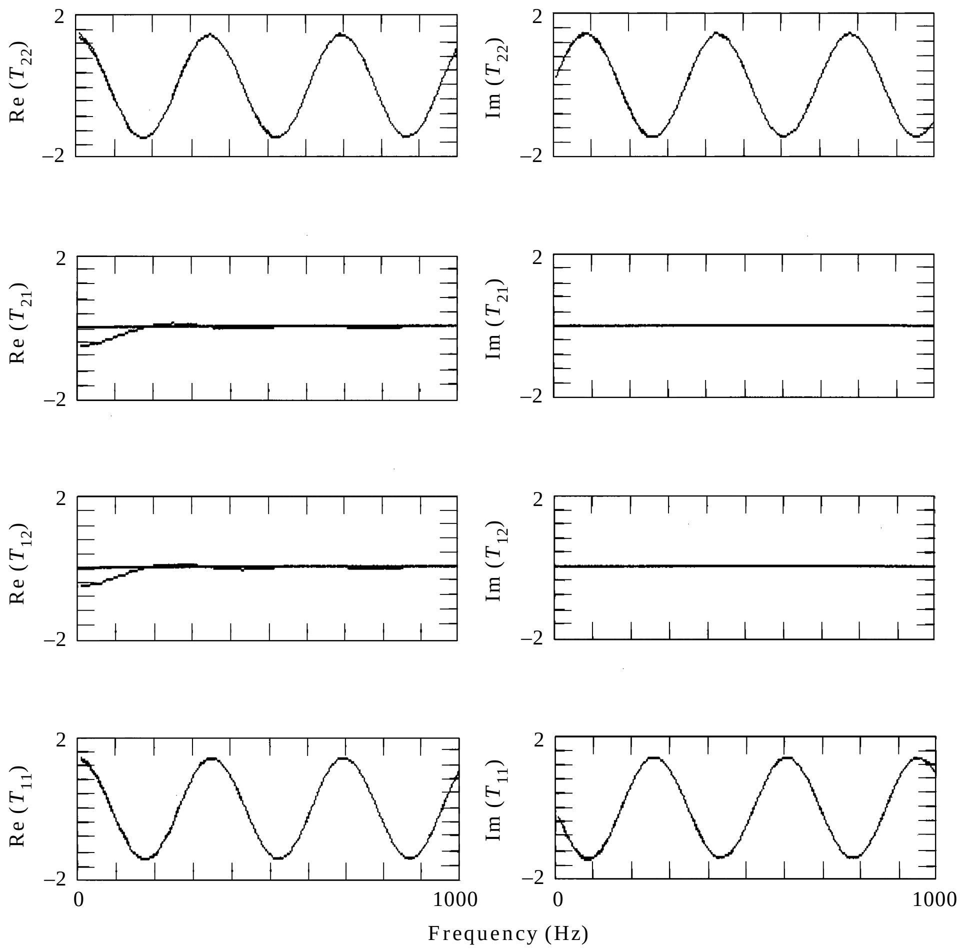

Figure 1. The elements of the scattering matrix of a 1-m long exponential diverging duct with inlet cross-sectional area of 0·01 m^2 and outlet cross-sectional area of 0·02 m^2. $M_o(0) = 0$, $T_o = 25$°C. Curves with the larger amplitudes are the numerical matrizant solutions.

as approximately $f > 190$ Hz. This condition is needed for the exact off-diagonal elements to be considered negligibly small. The diagonal elements given by the approximate solution are almost exact for all frequencies. That the lower limiting frequency criterion is not appreciably affected by the presence of mean flow is clearly observable from Figures 1 and 2.

For a conical duct with a truncated cone length a, $S(x) = S(0)(1 + x/a)^2$, that is, $2(\ln S')' = (\ln S)'$ and, therefore, conditions (24a) and (24b) reduce to $k \gg 0$ and $k \gg |(\ln S)'|/\sqrt{2}$, respectively. Hence, in this case, condition (24b) is the more critical of the above two conditions and, for a duct of length L, leads to the following criterion:

$$kL \gg |(\Delta\sqrt{S})/\sqrt{S}|\sqrt{2}, \tag{26a}$$

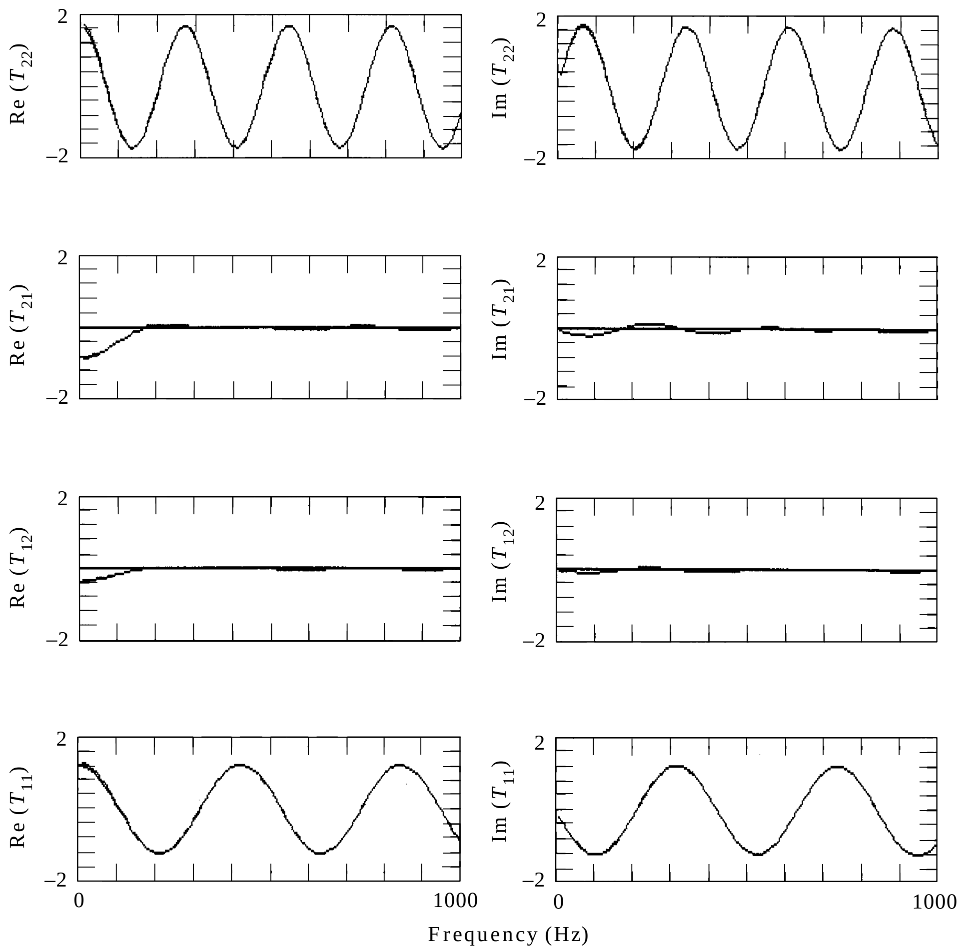

Figure 2. The elements of the scattering matrix of a 1-m long exponential diverging duct with inlet cross-sectional area of 0·01 m^2 and outlet cross-sectional area of 0·02 m^2. $M_o(0) = 0{\cdot}3$, $T_o = 25°$C. Curves with the larger amplitudes are the numerical matrizant solutions.

or

$$f \gg f^* = (c_o/\pi L\sqrt{2})|\sqrt{S(L)} - \sqrt{S(0)}|\sqrt{S_{min}}, \qquad (26b)$$

where S_{min} denotes the cross-sectional area of the end with the smaller diameter. This criterion holds for both divergent and convergent conical ducts. Figure 3 shows, for the case of no mean flow, the elements of the scattering matrix of a 0·444-m long diverging conical duct with inlet diameter of 0·0246 m and truncated cone length of 0·141 m and $T_o = 25°$C. One set of the results given in Figure 3 is computed by using the numerical matrizant solution, and the other set by using equations (12). Figure 4 shows the diagonal elements of the scattering matrix of the same duct but with an incompressible mean flow with $M_0(0) = 0{\cdot}3$. For this conical duct $f^* = 550$ Hz and, as can be deduced from Figures 3 and 4, for practical purposes, condition (26b) can be implemented as approximately $f > 1100$ Hz. The area ratio of this duct is 17·2, compared to 2 of the exponential duct considered above and, as can be expected, the lower limiting frequency is considerably higher. Thus, the applicability of the proposed approximate solution to non-uniform ducts will be limited in most practical cases to relatively small area ratios. That the lower limiting frequency criterion is not appreciably affected by the presence of mean flow is also confirmed by Figures 3 and 4.

The duct examples considered above are taken from reference [3] where exact analytical solutions are presented for exponential and conical ducts carrying an incompressible mean flow the Mach number of which is assumed to satisfy the condition $M_o^2 \ll 1$. This assumption is not required in the present analysis.

3.4. NON-UNIFORM DUCTS WITH TEMPERATURE GRADIENTS

Consider a non-uniform duct with $p_o' = 0$, $v_o = 0$ and $\gamma_o' = 0$. For this duct, equation (14) becomes $\Psi = (\ln S)' + (\ln c_o)'$ and conditions (19) and (21) can be

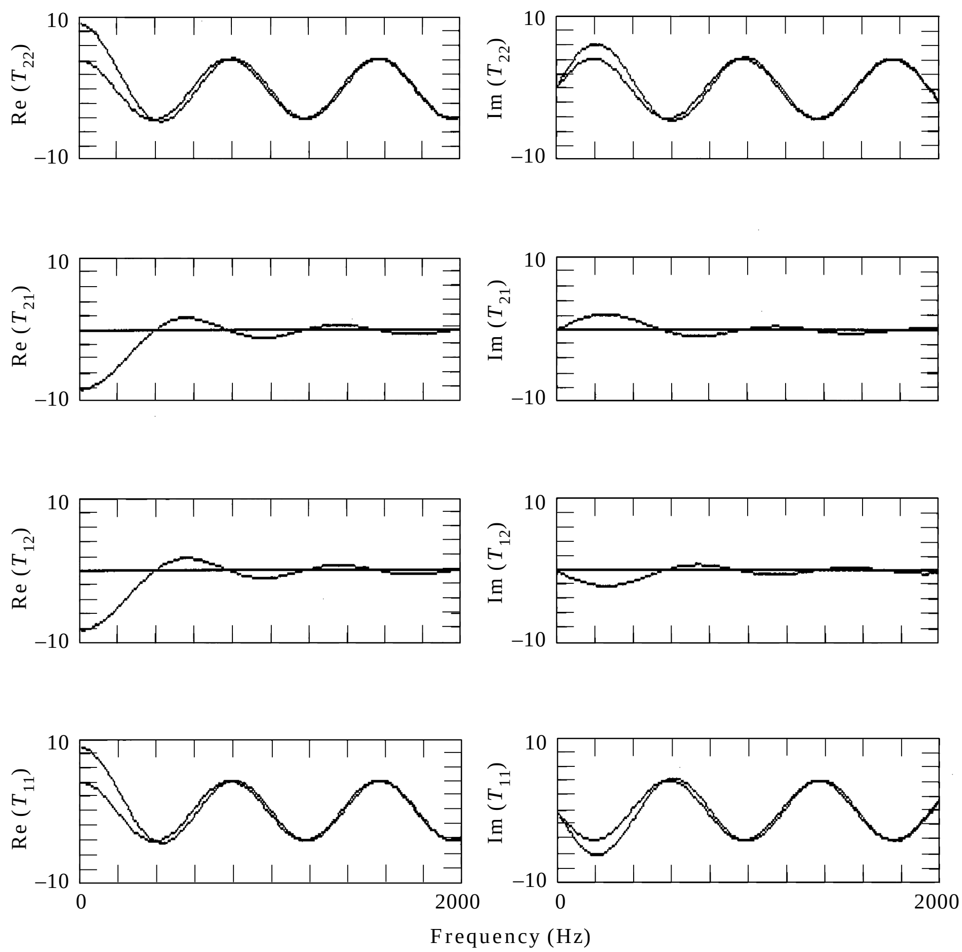

Figure 3. The elements of the scattering matrix of a 0·444-m long diverging conical duct with inlet diameter 0·0246 m and truncated cone length 0·141 m. $M_o(0) = 0$, $T_o = 25°C$. Curves with the larger amplitudes are the numerical matrizant solutions.

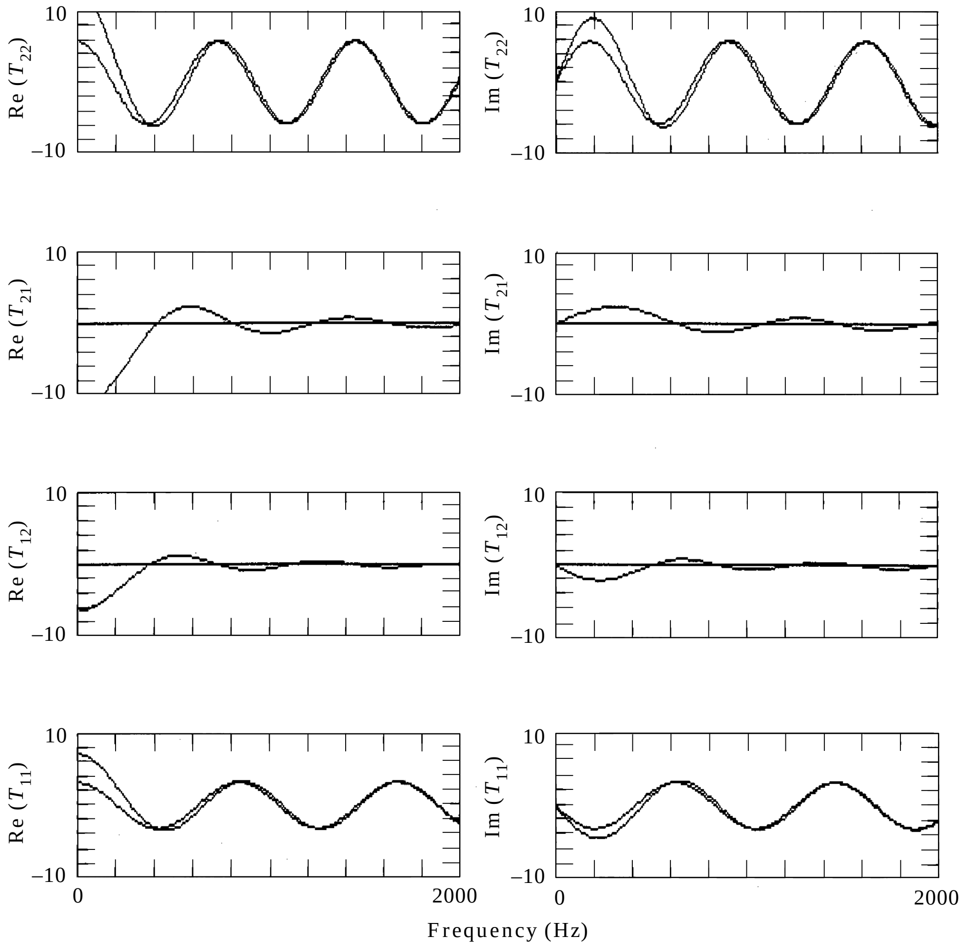

Figure 4. The elements of the scattering matrix of a 0·444-m long diverging conical duct with inlet diameter 0·0246 m and truncated cone length 0·141 m. $M_o(0) = 0{\cdot}3$, $T_o = 25°C$. Curves with the larger amplitudes are the numerical matrizant solutions.

implemented similarly for a given duct. The criteria related to exponential and conical ducts with a linear axial ambient temperature distribution are summarized in the following.

For an exponential duct of length L, conditions (19) and (21) yield, respectively,

$$4kL \gg \sqrt{|(2\Delta \ln S)^2 - (\Delta T_o/T_o)^2 - 8(\Delta \ln S)(\Delta T_o/T_o)|}, \tag{27a}$$

$$2kL \gg \sqrt{|(\Delta T_o/T_o)^2 + 2(\Delta \ln S)(\Delta T_o/T_o)|}. \tag{27b}$$

The more critical of these depends on the relative magnitudes of the temperature and cross-sectional area gradients. For example, if the end temperatures of the exponential duct considered in Figures 1 and 2 are $T_o(0) = 800°C$ and $T_o(L) = 600°C$, then conditions (27a) and (27b) reduce to $kL \gg 0{\cdot}2128$ and $kL \gg 0{\cdot}185$, respectively. Hence, condition (27a) is the more critical one in this case and gives the criterion $f \gg 25$ Hz. The elements of the scattering matrix of this duct, as calculated by the numerical matrizant method and the proposed approximate solution, equations (12), are shown in Figure 5 for $M_o(0) = 0{\cdot}3$. It is seen that, the lower limiting frequency for this case is approximately $f > 250$ Hz. Actually, this is required for the exact off-diagonal elements to become small enough so that they can be neglected, as the diagonal elements are almost exact for all frequencies. This lower limiting frequency is also valid if there is no mean flow.

For a conical duct having a linear distribution of ambient temperature over its length, L, conditions (19) and (21) can be expressed as, respectively,

$$4kL \gg \sqrt{|(\Delta T_o/T_o)^2 + 16(\Delta\sqrt{S}/\sqrt{S})(\Delta T_o/T_o)|}, \tag{28a}$$

$$2kL \gg \sqrt{|8(\Delta\sqrt{S}/\sqrt{S})^2 + (\Delta T_o/T_o)^2 + 4(\Delta\sqrt{S}/\sqrt{S})(\Delta T_o/T_o)|}. \tag{28b}$$

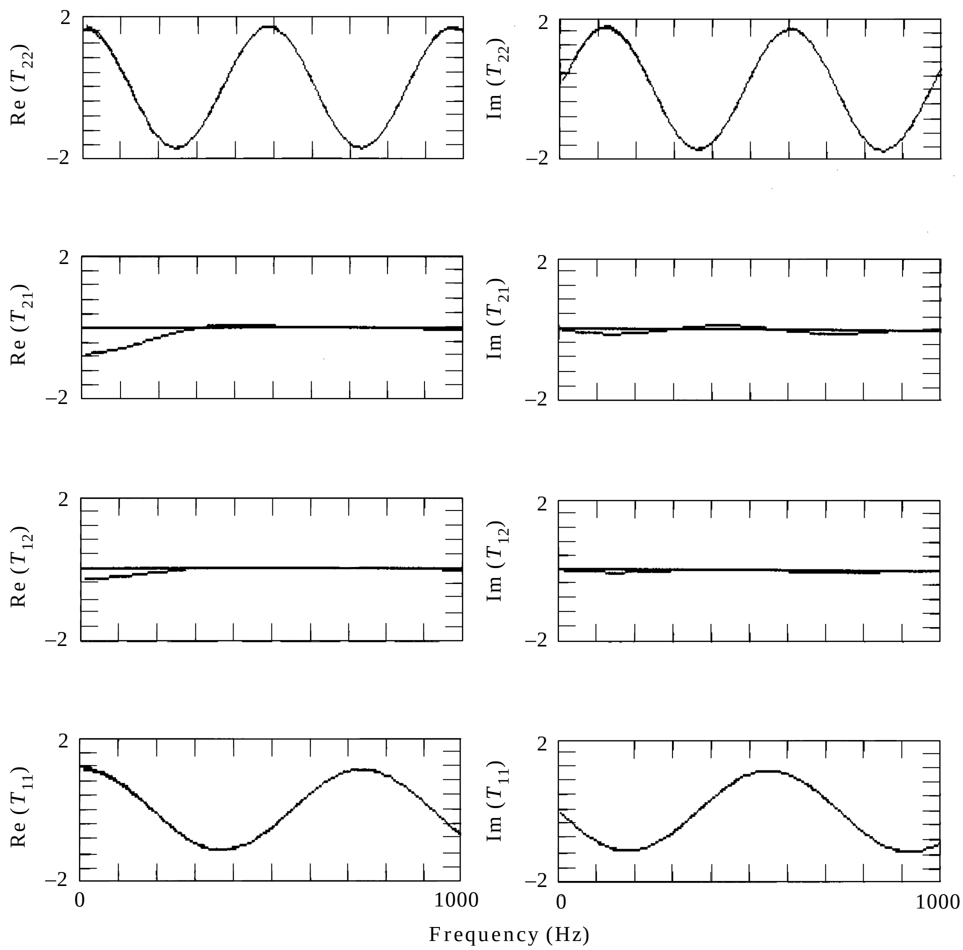

Figure 5. The elements of the scattering matrix of a 1-m long exponential diverging conical duct with inlet cross-sectional area of 0·01 m^2 and outlet cross-sectional area of 0·02 m^2. $M_o(0) = 0{\cdot}3$, $T_o(0) = 800°C$, $T_o(L) = 600°C$. Curves with the larger amplitudes are the numerical matrizant solutions.

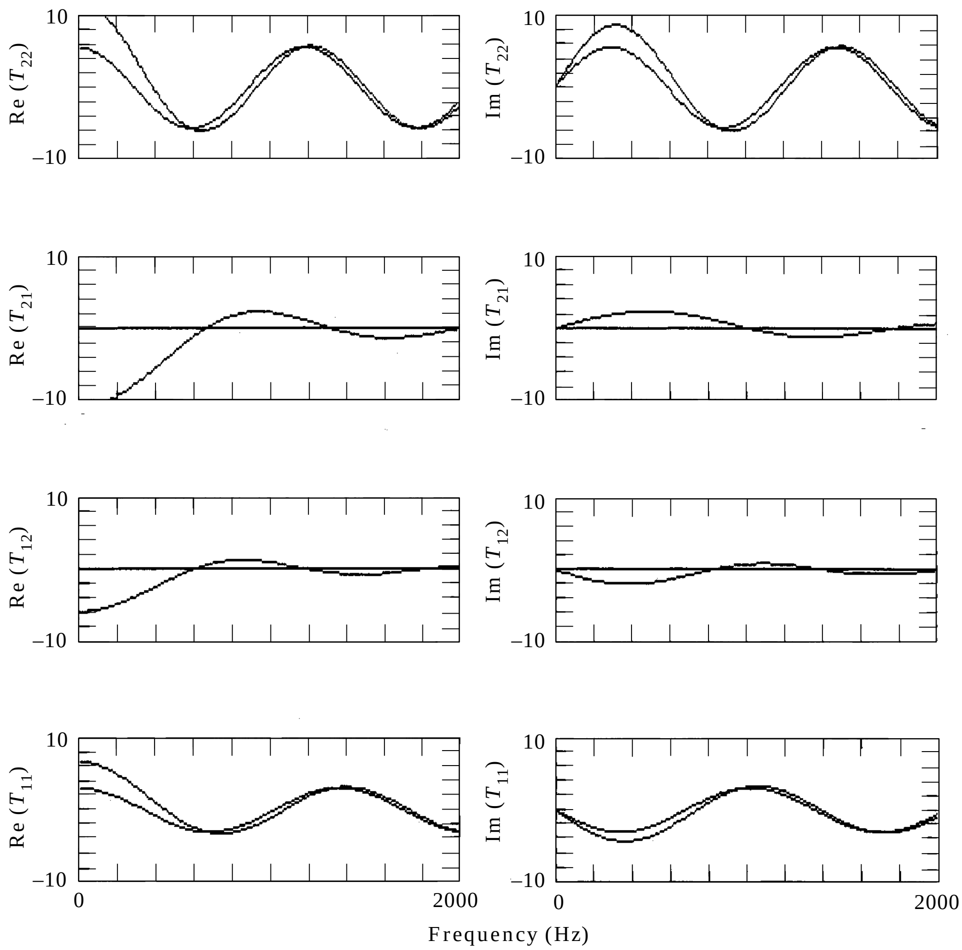

Figure 6. The elements of the scattering matrix of a 0·444-m long diverging conical duct with inlet diameter 0·0246 m and truncated cone length 0·141 m. $M_o(0) = 0{\cdot}3$, $T_o(0) = 600°C$, $T_o(0) = 500°C$. Curves with the larger amplitudes are the numerical matrizant solutions.

For example, if the end temperatures of the conical duct considered in Figures 3 and 4 are $T_o(0) = 600°C$ and $T_o(L) = 500°C$, then conditions (28a) and (28b) reduce to $kL \gg 0{\cdot}639$ and $kL \gg 4{\cdot}5$, respectively. Hence, condition (28b) is the more critical one in this case and leads to the criterion $f \gg 950$ Hz. The elements of the scattering matrix of this duct, as calculated by the numerical matrizant method and also by equations (12), are shown in Figure 6 for $M_o(0) = 0{\cdot}3$. It is seen that, although in this case a lower limiting frequency of $f > 2000$ Hz or more is required, $f > 1500$ Hz may still be acceptable for practical purposes.

5. CONCLUSION

The solution of equation (9) can be expressed in the scattering matrix form defined by equation (10) where the elements of the scattering matrix are given by equations (12) if the frequency is high enough. In the case of uniform ducts, equations (12) are almost exact for the temperature gradients and frequencies that are likely to be encountered in practice. In the case of non-uniform ducts, the lower limiting frequency increases with the cross-sectional area gradient and in applications where low frequencies are important, the utility of equations (12) will be limited to ducts with a relatively small area ratio. Conditions (19) and (21) can be used to estimate the lower limiting frequency with no mean flow and also in the presence of a subsonic mean flow. The present formulation allows one to set a prescribed ambient pressure gradient along the duct, however, for the usual pressure losses that are encountered in practice, the effect of the pressure gradient on the elements of duct scattering matrices is small.

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APPENDIX: THE NUMERICAL MATRIZANT SOLUTION

Equation (9) can be expressed briefly as

$$\mathbf{P}'(x) = \mathbf{H}(x)\mathbf{P}(x) \tag{A1}$$

where $\mathbf{P}(x) = \{p^+(x)\ p^-(x)\}$ and the elements of matrix $\mathbf{H}(x)$ are given by equations (6) and (7). The general solution of equation (A1) can be expressed as,

$$\mathbf{P}(x) = [\mathbf{T}]_0^x\mathbf{P}(0) \tag{A2}$$

where $[\mathbf{T}]_0^x$ denotes the matrizant [6] for the interval $(0, x)$, which is a function of $\mathbf{H}(x)$. The scattering matrix of a duct of length L is given by the matrizant $[\mathbf{T}]_0^L$. This can be evaluated numerically by dividing the interval $0 \leqslant x \leqslant L$ into N parts by introducing intermediate points $x_1, x_2, \ldots, x_{N-1}$. For simplicitly, the lengths of the parts are assumed to be all equal, i.e., $l = x_k - x_{k-1}$, $k = 1, 2, \ldots, N$ and $x_N = L = Nl$. Then, using the well known properties of the matrizant, it can be shown that

$$[\mathbf{T}]_0^L = [\mathbf{T}]_{x_{N-1}}^{x_N} \cdots [\mathbf{T}]_{x_1}^{x_2} [\mathbf{T}]_0^{x_1}, \tag{A3}$$

where $[\mathbf{T}]_{x_{k-1}}^{x_k}$ denotes the matrizant for the interval (x_{k-1}, x_k), $x_{k-1} \leqslant x \leqslant x_k$. Now, let ξ_k be a point, always the mid-point in this paper, in the interval (x_{k-1}, x_k), $k = 1, 2, \ldots, N$. By regarding l as a sufficiently small quantity, one may assume, for the approximate evaluation of the matrizant, $\mathbf{H}(x) \cong \mathbf{H}(\xi_k) = \mathbf{H}_k$, say. Then, $\mathbf{H}_k$ is independent of x and the matrizant for the interval (x_{k-1}, x_k) can be shown to be

$$[\mathbf{T}]_{x_{k-1}}^{x_k} = \exp(\mathbf{H}_k l) = \Phi_k^{-1} \exp(\underset{\sim}{\Lambda}_k l)\underset{\sim}{\Phi}_k \tag{A4}$$

where $\underset{\sim}{\Lambda}_k$ and $\underset{\sim}{\Phi}_k$ are the eigenvalue and eigenvector matrices of the matrix $\mathbf{H}_k$. As the number of parts is increased, equation (A3) will converge to the exact solution. Convergence is in general very fast and only few number of divisions are required to obtain accurate results.

For a duct with a given axial temperature distribution, the local mean flow Mach number can be computed from,

$$M_o = M(x)/C(x) \tag{A5}$$

where

$$M(x) = v_o(0)p_o(0)T_o(x)S(0)/c_o(0)p_o(x)T_o(0)S(x), \tag{A6}$$

$$C(x) = \sqrt{T_o(x)\gamma_o(x)/T_o(0)\gamma_o(0)}. \tag{A7}$$

Then, the mean flow velocity and the speed of sound are given by $v_o(x) = M(x)c_o(0)$ and $c_o(x) = C(x)c_o(0)$, respectively.

Hence, given $T_0 = T_o(x)$, $\gamma_o = \gamma_o(T_o)$, $p_o = p_o(x)$, $S = S(x)$ and $v_o(0)$, the matrix $\mathbf{H}(x)$ can be evaluated as a function of x.