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AN EXACT TRANSFER MATRIX FORMULATION OF PLANE SOUND WAVE TRANSMISSION IN INHOMOGENEOUS DUCTS

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The impedance, or the reflection coefficient, of plane sound waves in inhomogeneous ducts satisfies a Riccati equation. The present paper shows that the duct impedance matrix, or the scattering matrix, can be related explicitly to the solutions of the associated linear equation of the Riccati equation for duct impedance, or reflection coefficient, respectively. New exact analytical scattering matrix solutions, which follow as consequences of this connection, are given for two significant duct acoustics problems, namely, the sound transmission in non-uniform ducts carrying an incompressible subsonic low Mach number mean flow and the transmission of sound in uniform ducts with a full quadratic axial mean temperature gradient.

1. INTRODUCTION

This paper presents, for the propagation of plane sound waves in inhomogeneous ducts, a new relationship between the duct impedance matrix, or scattering matrix, and the solution of a Riccati equation for duct impedance, or reflection coefficient. It has been known for a long time that the plane wave impedance or reflection coefficient in an inhomogeneous waveguide satisfies a Riccati equation; however, the relationship to be described in this paper has not appeared in the literature, hitherto. Few authors made use of the Riccati equation for the solution of plane wave duct acoustics problems. Kergomard [1] has described a continued fraction solution of the Riccati equation of the classical horn equation.

In this paper, duct inhomogenities due to axial gradients in the ambient conditions as well as the cross-sectional area variations are considered and the presence of a subsonic low Mach number mean flow is allowed for in the analysis. The mathematical formulation is based on the general equations for one-dimensional isentropic sound propagation and encompasses most of the problems considered in previous papers as special cases.

The general acoustic equations can be recast in a state–space form, which eventually leads to a transfer matrix solution for the wave transfer across a given length of a duct, or as a Riccati equation for duct impedance, or reflection coefficient. It is the aim of this paper to show that the solutions of the equations governing these two formulations are connected. This connection can be

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developed either as a relationship between the duct impedance matrix and the solution of the Riccati equation for duct impedance, or as a relationship between the duct scattering matrix and the solution of the Riccati equation for duct reflection coefficient. The main text of this paper is based on the reflection coefficient formulation and the salient results of the impedance formulation are summarized in the Appendix. The paper will also present two new exact analytical solutions which follow as almost immediate consequences of the formulation in terms of a Riccati equation. In the first of these, the exact scattering matrix of non-uniform ducts carrying an incompressible subsonic low Mach number mean flow is explicitly expressed in terms of the scattering matrix of the same duct with no mean flow. The second gives the exact scattering matrix for sound propagation in a uniform duct with a full quadratic ambient temperature distribution. Both solutions are of some practical importance and are presented here for the first time.

2. FORMULATION OF DUCT ACOUSTICS PROBLEMS

2.1. BASIC ACOUSTIC EQUATIONS

The equations governing the propagation of plane sound waves come from the one-dimensional gas dynamic equations. To first order in the acoustic perturbations, the basic equations can be expressed, upon assuming $e^{-i\omega t}$ time dependence, where i denotes the unit imaginary number, ω is the radian frequency and t is the time, as follows.

The continuity equation is

$$-i\omega\rho + v_o\rho' + v\rho_o' + \rho_o v' + \rho v_o' + (\rho_o v + \rho v_o)(\ln S)' = 0. \tag{1}$$

The momentum equation is

$$\rho_o(-i\omega v + v_o v' + v v_o') + v_o v_o' \rho + p' = 0. \tag{2}$$

For a perfect gas, the energy equation, which is tantamount to the statement that sound propagation is isentropic, can be expressed as [2]

$$-i\omega p + v_o p' + v p_o' + \gamma_o p_o v' + \gamma_o p v_o' + \gamma_o(p_o v + p v_o)(\ln S)' = 0. \tag{3}$$

Here, a prime ($'$) denotes differentiation with respect to x, the duct axis, S denotes the duct cross-sectional area and γ_o is the ratio of specific heat coefficients of the ambient gas. ρ, p and v the acoustic density, pressure and particle velocity, respectively, are first order perturbations, with zero averages, superimposed on the corresponding steady mean flow values ρ_o, p_o and v_o, which are to be understood as cross-sectionally averaged values that satisfy the continuity equation $(\rho_o v_o S)' = 0$ and the state equation $p_o = \rho_o R T_o$, where R denotes the gas constant and T_o is the ambient temperature.

2.2. TRANSFORMATION TO PRESSURE WAVE COMPONENTS

It is convenient to recast the foregoing equations by using the following transformation:

$$p = p^+ + p^-, \qquad \rho_o c_o v = p^+ - p^-, \qquad c_o^2 \rho = p + \epsilon, \tag{4a, b, c}$$

where $c_o = \sqrt{\gamma_o R T_o}$ is the local speed of sound. For a uniform duct with zero ambient gradients, it is well known that p^+ and p^- correspond to plane waves travelling in $+x$ and $-x$ directions, respectively. This interpretation can be extended to the present case; however, equations (4) can also be considered as a purely mathematical transformaton. By using equations (4) in the basic acoustic equations, it can be shown, after some algebra, that p^+ and p^- are given by

$$\begin{bmatrix} p^{+\prime} \\ p^{-\prime} \end{bmatrix} = \begin{bmatrix} A[M_o] & B[M_o] \\ B[-M_o] & A^*[-M_o] \end{bmatrix} \begin{bmatrix} p^+ \\ p^- \end{bmatrix} - \frac{M_o^2(\ln v_o)'}{2(1-M_o^2)} \begin{bmatrix} 1-M_o \\ 1+M_o \end{bmatrix} \varepsilon, \tag{5}$$

where $M_o = v_o/c_o$ denotes the local mean flow Mach number, an asterisk (*) denotes complex conjugate and $A[M_o]$, $B[M_o]$ and ε are given by

$$A[M_o] = (\mathrm{i}2k_o + (1+M_o)(\ln \rho_o c_o)' - M_o(1+\gamma_o+M_o)(\ln v_o)' \\ - (\ln p_o)'/\gamma_o - (1+\gamma_o M_o)(\ln S)')/2(1+M_o), \tag{6}$$

$$B[M_o] = (-(1+M_o)(\ln \rho_o c_o)' - M_o(-1+\gamma_o+M_o)(\ln v_o)' \\ + (\ln p_o)'/\gamma_o + (1-\gamma_o M_o)(\ln S)')/2(1+M_o), \tag{7}$$

$$\varepsilon' - (\mathrm{i}k_o/M_o + (\ln \gamma_o p_o)')\varepsilon = (\ln \gamma_o)' p, \tag{8}$$

where $k_o = \omega/c_o$ denotes the local wavenumber. It should be noted that $A[M_o]$ and $B[M_o]$ are functions of x, that is $A[M_o] = A(x)$ and $B[M_o] = B(x)$. The symbolic notation $A[M_o]$ and $B[M_o]$ is used here to indicate the relationship between the diagonal and the off-diagonal elements of the state–space matrix in equation (5).

2.3. A SUBSONIC LOW MACH NUMBER APPROXIMATION

Equations (5) and (8) have been solved numerically as a coupled system of three first order differential equations in p^+, p^- and ε. This part of the analysis, which has been presented elsewhere [2], shows that for the subsonic low Mach numbers that are of interest here, say, $M_o < 0{\cdot}3$, the second term on the right of equation (5) is negligibly small. Thus, the following simplified form of equation (5) is assumed to be valid throughout the present analysis, namely,

$$\begin{bmatrix} p^{+\prime} \\ p^{-\prime} \end{bmatrix} = \begin{bmatrix} A[M_o] & B[M_o] \\ B[-M_o] & A^*[-M_o] \end{bmatrix} \begin{bmatrix} p^+ \\ p^- \end{bmatrix}. \tag{9}$$

Another approach which is used in a number of papers reporting solutions of duct acoustics problems to decouple the three basic acoustic equations is to neglect the term $v_o' v_o \rho$ as a small term of the second order in the momentum equation, equation (2). This omission decouples equation (1) from equations (2) and (3), and p and v can then be determined by solving only the latter two equations. If this approach were adopted in the present analysis, equation (4c) would not be needed, and in equations (6) and (7) the term M_o in the coefficients of $(\ln v_o)'$ in brackets would be absent. The present approach, which is tantamount to assuming that $v_o' v_o \varepsilon / c_o^2$ is small to second order, is more accurate because, for the subsonic low Mach number ducts, the error in the isentropic relationship $p = c_o^2 \rho$ is negligibly small and, therefore, the term $v_o' v_o \varepsilon / c_o^2$ is some orders of magnitude smaller than the term $v_o' v_o \rho$. Indeed, equation (9) can be derived directly by replacing the energy equation, equation (3), by $p = c_o^2 \rho$.

3. FORMULATION OF DUCT ACOUSTICS PROBLEMS IN TERMS OF A RICCATI EQUATION

3.1. THE RICCATI EQUATION FOR THE REFLECTION COEFFICIENT

The duct reflection coefficient is defined formally by the quotient,

$$r = p^- / p^+. \tag{10}$$

This coincides with the traditional definition of the reflection coefficient if the "incident" wave is taken in the positive direction of the duct axis. Upon introducing r, equation (9) can be recast as

$$(\ln p^+)' = A[M_o] + rB[M_o], \qquad (\ln p^-)' = A^*[-M_o] + B[-M_o]/r. \tag{11a, b}$$

Therefore, as can be shown by differentiating equation (10) with respect to x, the reflection coefficient is given by

$$r' + (A[M_o] - A^*[-M_o])r + r^2 B[M_o] = B[-M_o]. \tag{12}$$

This is an ordinary non-linear differential equation which is known as the general Riccati equation. It is encountered in a number of physical problems and there exists a vast amount of literature on its applications and solutions; however, no one has yet found an exact analytical general solution. For the purpose of the present study, it will suffice to consider the classical mathematical features of the Riccati equation. A Riccati equation can also be derived for the duct impedance, $z = p/v$. This alternative formulation is presented in the Appendix.

The general solution of equation (12) can be expressed as [3]

$$r(x) = (y_1' + Cy_2')/B(x)(y_1 + Cy_2), \tag{13}$$

where C denotes an integration constant, and $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are two independent solutions of the linear differential equation,

$$y'' + (A[M_o] - A^*[-M_o] - (\ln B[M_o])')y' - B[M_o]B[-M_o]y = 0. \tag{14}$$

This equation is called the associated linear differential equation of the Riccati equation, equation (12). The integration constant C is determined from the boundary condition that at $x = 0$ the reflection coefficient is $r(0)$

$$C = -(r(0)B(0)y_1(0) - y_1'(0))/(r(0)B(0)y_2(0) - y_2'(0)). \tag{15}$$

Hence, equation (13) can be expressed as

$$r(x)B(x) = \frac{y_2' y_1'(0) - y_1' y_2'(0) + r(0)B(0)[y_1' y_2(0) - y_2' y_1(0)]}{y_2 y_1'(0) - y_1 y_2'(0) + r(0)B(0)[y_1 y_2(0) - y_2 y_1(0)]}. \tag{16}$$

This formula can be simplified by defining a suitable set of boundary conditions at $x = 0$ for the solutions of the associated linear equation. The preferred set of boundary conditions in the present analysis is: $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$. With these boundary conditions, equation (16) simplifies to

$$r(x)B(x) = \frac{y_2'(x) + r(0)B(0)y_1'(x)}{y_2(x) + r(0)B(0)y_1(x)}. \tag{17}$$

3.2. THE DUCT SCATTERING MATRIX

The solution of the Riccati equation yields the duct reflection coefficient. In most applications, however, what is required is a wave transfer relationship between the ends of a given length of a duct so that it can be incorporated into the complete gas flow system as an acoustic two-port. Therefore, it is desirable to express the solution of the Riccati equation for the reflection coefficient in the form of a duct scattering matrix. A formal general exact expression for the scattering matrix can be derived as follows.

The relationship between the presure wave components $p^+(0)$ and $p^-(0)$ at the origin $x = 0$ and the pressure wave components $p^+(x)$ and $p^-(x)$ at any section x of a duct can be expressed as

$$\begin{bmatrix} p^+(x) \\ p^-(x) \end{bmatrix} = \begin{bmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}(x) & T_{22}(x) \end{bmatrix} \begin{bmatrix} p^+(0) \\ p^-(0) \end{bmatrix}, \tag{18}$$

or, in expanded form, as,

$$p^+(x) = T_{11}(x)p^+(0) + T_{12}(x)p^-(0), \tag{19}$$

$$p^-(x) = T_{21}(x)p^+(0) + T_{22}(x)p^-(0). \tag{20}$$

The 2×2 square matrix in equation (18), which is subsequently denoted by $\mathbf{T}(x, 0)$, is called the scattering matrix of the duct for the interval $(0, x)$.

The elements of the scattering matrix satisfy the following boundary conditions at $x = 0$:

$$T_{11}(0) = 1, \qquad T_{12}(0) = 0, \qquad T_{21}(0) = 0, \qquad T_{22}(0) = 1. \tag{21}$$

The reflection coefficient at any section x, $r(x)$, of the duct can be expressed in terms of the elements of the scattering matrix as

$$r(x) = \frac{T_{21}(x) + r(0)T_{22}(x)}{T_{11}(x) + r(0)T_{12}(x)}. \tag{22}$$

A similar expression for $r(x)$ can be derived also by differentiating equation (19) with respect to x and using equation (11a). Equating the two equations for $r(x)$ gives

$$T_{11}' - A[M_o]T_{11}(x) - B[M_o]T_{21}(x) = 0,$$

$$T_{12}' - A[M_o]T_{12}(x) - B[M_o]T_{22}(x) = 0. \tag{23, 24}$$

This procedure can be repeated by using equation (20) and equation (11b) to obtain

$$T_{22}' - A^*[-M_o]T_{22}(x) - B[-M_o]T_{12}(x) = 0,$$

$$T_{21}' - A^*[-M_o]T_{21}(x) - B[-M_o]T_{11}(x) = 0. \tag{25, 26}$$

The following theorem gives the solution of equations (23)–(26) under the boundary conditions stated in equation (21).

Theorem. The elements of the scattering matrix defined in equation (18) are given by,

$$T_{11}(x) = \eta(x) y_2(x), \qquad T_{12}(x) = B(0)\eta(x) y_1(x), \tag{27a, b}$$

$$T_{21}(x) = \eta(x) y_2'(x)/B(x), \qquad T_{22}(x) = B(0)\eta(x) y_1'(x)/B(x), \tag{27c, d}$$

where $y_1(x)$ and $y_2(x)$ are two independent solutions of equation (14) satisfying the boundary conditions $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$, and $\eta(x)$ is given by

$$\eta(x) = \exp\left(\int_0^x A(x)\,\mathrm{d}x\right). \tag{28}$$

Proof. This theorem can easily be shown to be true by substitution of equations (27) into equations (23)–(26). A causal proof, on the other hand, may start with the observation that the nominator and the denominator of equation (17) can be multiplied by any function of x, say, $\eta(x)$. Then, using this modified form of equation (17) in place of equation (22), one obtains, by using the same procedure described above, four equations that are counterparts of equations (23)–(26), which, upon solving for $\eta(x)$, yield equations (27).

This proves that the general solution of equation (9) can be expressed in the form of equation (18) where the elements of the duct scattering matrix are given by equations (27). Then, for a duct of length L, the scattering matrix can be expressed as

$$\mathbf{T}(L,0) = \begin{bmatrix} y_2(L) & B(0)y_1(L) \\ y_2'(L)/B(L) & y_1'(L)B(0)/B(L) \end{bmatrix} \exp\left(\int_0^L A(x)\,\mathrm{d}x\right), \tag{29a}$$

or, alternatively, in the product form

$$\mathbf{T}(L,0) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{B(L)} \end{bmatrix} \begin{bmatrix} y_2(L) & y_1(L) \\ y_2'(L) & y_1'(L) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B(0) \end{bmatrix} \exp\left(\int_0^L A(x)\,\mathrm{d}x\right). \tag{29b}$$

As far as the author is aware, this result has not been published elsewhere.

Equation (29a) can be checked for the case of plane sound wave propagation in a uniform duct carrying a uniform mean flow. This case is obtained as a limiting case of equations (6) and (7) when all the gradient terms tend to zero. Then, $A[\mp M_o] \to \mathrm{i}k_o^{\mp} = \mp \mathrm{i}k_o/(1 \mp M_o)$, $B(x) \to 0$ with $B(0)/B(L) \to 1$, and equation (14) reduces to $[y' + \mathrm{i}k_o y/(1 - M_o^2)]' = 0$, the two solutions of which that satisfy the boundary conditions stated in the above theorem are $y_1(x) = [1 - \exp(\mathrm{i}(k_o^- - k_o^+)x]/\mathrm{i}(k_o^+ - k_o^-)$ and $y_2(x) = 1$. Then, since $y_2' = 0$ and $B(x) \to 0$, both off-diagonal terms in equation (29a) vanish and the scattering matrix reduces to the correct diagonal form with the diagonal terms given by $T_{11}(x) = \exp(\mathrm{i}k_o^+ x)$ and $T_{22}(x) = \exp(\mathrm{i}k_o^- x)$.

3.3. SOLUTION OF THE RICCATI EQUATION BY SINGLE QUADRATURE

The result of the foregoing section is based on the solution of the associated linear equation of the Riccati equation for the duct reflection coefficient. A case where the two solutions of the associated linear equation of the Riccati equation are found relatively easily is when the following quotients are constants:

$$2\alpha = (A[M_o] - A^*[-M_o])/B[M_o], \qquad \beta = B[-M_o]/B[M_o]. \qquad (30a, b)$$

This follows because the Riccatti equation for the reflection coefficient can be recast as

$$\frac{\mathrm{d}r}{r^2 + 2\alpha r - \beta} = -B(x)\,\mathrm{d}x. \qquad (31)$$

If α and β are both constant, then this equation can be integrated by a single quadrature:

$$r(x) = \frac{(1 - \mathrm{e}^{\phi})\beta + r(0)(r_1 - r_2\,\mathrm{e}^{\phi})}{r_1\,\mathrm{e}^{\phi} - r_2 + r(0)(1 - \mathrm{e}^{\phi})}, \qquad (32)$$

where

$$r_{1,2} = -\alpha \mp \sqrt{(\alpha^2 + \beta)}, \qquad \phi(x) = (r_2 - r_1)\int_0^x B(x)\,\mathrm{d}x. \qquad (33, 34)$$

The two independent solutions of the associated linear differential equation of equation (31), equation (14), which correspond to this quadrature can be derived now by multiplying the nominator and the denominator of equation (32) by an undetermined function of x, say, $\zeta = \zeta(x)$. The resulting expression must be consistent with equation (17). This gives the function ζ as

$$\zeta(x) = \frac{1}{r_1 - r_2}\exp\left(r_1\int_0^x B(x)\,\mathrm{d}x\right). \qquad (35)$$

Hence, the solutions of the associated linear equation corresponding to equation (32) can be expressed as

$$y_1(x) = (1 - \mathrm{e}^{\phi})\zeta(x)/B(0), \qquad y_2(x) = (r_1\,\mathrm{e}^{\phi} - r_2)\zeta(x). \qquad (36a, b)$$

Note that the boundary conditions $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$ are satisfied by these solutions.

4. NEW EXACT ANALYTICAL RESULTS

In this section, exact analytical solutions are presented for two duct acoustics problems of practical interest. These results follow from foregoing general considerations on the formulation of sound propagation in an inhomogeneous duct in terms of a Riccati equation and are presented here for the first time.

4.1. SOUND PROPAGATION IN A NON-UNIFORM DUCT CARRYING A SUBSONIC LOW MACH NUMBER MEAN FLOW

The first problem to be considered is on the transmission of sound in non-uniform ducts carrying a subsonic incompressible low Mach number mean flow. For this problem, the ambient gradients may be taken as $\rho_o' = 0$, $p_o' = 0$. Then, upon assuming further that $\gamma_o' = 0$ and $M_o^2 \ll 1$, equations (6) and (7) reduce to

$$A[M_o] \simeq \frac{\mathrm{i}k_o}{1+M_o} - \frac{(1-M_o)(\ln S)'}{2(1+M_o)}, \qquad B[M_o] \simeq \frac{(1-M_o)(\ln S)'}{2(1+M_o)}. \quad \text{(37a, b)}$$

The linear differential equation associated with the Riccati equation for the reflection coefficient, equation (14), corresponding to equations (37) is, for $M_o^2 \ll 1$, independent of the mean flow Mach number:

$$y'' + (\mathrm{i}2k_o - (\ln S')' + (\ln S)')y' - ((\ln S)'/2)^2 y = 0, \quad (38)$$

Therefore, insofar as the approximation $M_o^2 \ll 1$ is valid, the two independent solutions of equation (38) and their derivatives with respect to x can be expressed, through equations (27), in terms of the elements of the scattering matrix of the same duct with zero mean flow, say, $\mathbf{T}^o(x, 0)$, as

$$y_1(x) = T^o_{12}(x)/\eta^\circ(x)B^o(0), \qquad y_2(x) = T^o_{11}(x)/\eta^o(x), \quad \text{(39a, b)}$$

$$y_1'(x) = T^o_{22}(x)B^o(x)/\eta^o(x)B^o(0), \qquad y_2'(x) = T^o_{21}(x)B^o(x)/\eta^o(x), \quad \text{(39c, d)}$$

where the superscript (o) refers to the case of zero mean flow, $M_o = 0$. Hence, substituting equations (39) into equation (29a) and using equations (37), the scattering matrix with mean flow can be expressed as

$$\mathbf{T}(L, 0) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1+M_o(L)}{1-M_o(L)} \end{bmatrix} \begin{bmatrix} T^o_{11}(L) & T^o_{12}(L) \\ T^o_{21}(L) & T^o_{22}(L) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1-M_o(0)}{1+M_o(0)} \end{bmatrix}$$

$$\times \exp\left(\int_0^L \{A(x) - A^o(x)\}\, \mathrm{d}x \right). \quad (40)$$

Classically, the isentropic sound propagation in a non-uniform duct with no mean flow is given by the horn equation which is known to admit exact analytical solutions for a family of cross-sectional area variations. The foregoing result extends all the known analytical solutions of the horn equation for the presence of incompressible subsonic low Mach number mean flow.

The assumptions underlying the foregoing low Mach number incompressible flow solution coincide with the assumptions made by Easwaran and Munjal [4] in deriving their analytical solutions for conical and exponential ducts carrying subsonic incompressible mean flow. Therefore, for the same duct geometries, the solutions indicated here should be comparable with the solutions presented in reference [4] (see the Postscript). Similar assumptions are also made in the three-dimensional formulation described by Doak [5].

Exact scattering matrix solutions can also be derived directly from the solution of equation (38), if one can be found. As a straightforward application, consider the case of an exponential diverging or converging duct. In this case $S(x) = S(0)\,\mathrm{e}^{2mx}$, where m is a real constant, and equation (40) becomes $y'' + \mathrm{i}2k_o y' - m^2 y = 0$. The two independent solutions of this equation that satisfy the boundary conditions of the theorem of section 3.2 can be expressed as

$$y_1 = (\mathrm{e}^{r_1 x} - \mathrm{e}^{r_2 x})/(r_1 - r_2), \qquad y_2 = (-r_2\,\mathrm{e}^{r_1 x} + r_1\,\mathrm{e}^{r_2 x})/(r_1 - r_2), \qquad (41, 42)$$

where $r_{1,2}$ are given by

$$r_{1,2} = \mathrm{i}(-k_o \pm \sqrt{k_o^2 - m^2}). \qquad (43)$$

The duct scattering matrix can be obtained now by substituting equations (41) and (42) into equation (29a) with $A(x)$ and $B(x)$ given by equations (37).

4.2. SOUND PROPAGATION IN A DUCT WITH A TEMPERATURE GRADIENT

The second problem to be considered is the propagation of plane sound waves in an inhomogeneous duct with zero mean flow, $v_o = 0$, and zero mean pressure gradient, $p_o' = 0$. Under these conditions, equations (6) and (7) simplify to

$$2A(x) = \mathrm{i}2k_o + (\ln \rho_o c_o/S)', \qquad 2B(x) = -(\ln \rho_o c_o/S)'. \qquad (44a, b)$$

Then, from equations (30), $\alpha = -\mathrm{i}\omega/c_o(\ln \rho_o c_o/S)'$ and $\beta = 1$. Therefore, the Riccati equation for the reflection coefficient of this duct can be integrated by single quadrature if $c_o(\ln \rho_o c_o/S)'$ is constant. A further assumption that can usually be made in many practical cases is $\gamma_o' = 0$. Then, the condition for the single quadrature solution of the Riccati equation reduces to $c_o(\ln c_o S)' = \text{constant}$ or, $(S\sqrt{T_o})' \propto S$. For a uniform duct, this condition becomes $\sqrt{T_o} = mx + C$, where m and C denote arbitrary constants. This corresponds to a full quadratic mean temperature distribution $T_o(x) = (mx + C)^2$, which is of some practical significance because the axial temperature distribution in most applications is usually a linear distribution like $T_o(x) = T_o(0)(1 + \tau x)$, where $|\tau x| \ll 1$, and, therefore, can be replaced by the full quadratic distribution $T_o(x) = T_o(0)(1 + \tau x/2)^2$ without loss of accuracy.

For the case of full quadratic temperature distribution, equations (44) become

$$A(x) = \mathrm{i}k_o - m2(mx + C), \qquad B(x) = m/2(mx + C). \qquad (45a, b)$$

Then, equations (33) and (34) can be expressed as

$$r_{1,2} = \mathrm{i}(-2\omega/m \mp \sqrt{4\omega^2/m^2 - \gamma_o R})/\sqrt{\gamma_o R}, \qquad (46)$$

$$\phi(x) = \mathrm{i}\sqrt{4\omega^2/m^2\gamma_o R - 1}\,\ln(mx + C). \qquad (47)$$

$y_1(x)$ and $y_2(x)$ are now determined by equations (36) and the duct scattering matrix then follows from equation (29a). This exact analytical solution can be shown to be approximately valid up to mean flow Mach numbers of $M_o \ll 0{\cdot}25$, say, $M_o < 0{\cdot}025$.

5. CONCLUSION

In this paper, new exact analytical results have been presented for plane sound wave transmission in subsonic ducts in which arbitrary ambient gradients and cross-sectional area variations may be present. It is interesting to note that the general formal transfer matrix solution of the governing acoustic equations is related not to the solution of the Riccati equation, but to the solution of its associated linear equation, equation (14). The present study has provided some solutions to equation (14) and it appears that other analytical solutions may also be developed, but this may require considerably more effort. In the case of zero mean flow, equation (14) reduces to equation (39), which may be looked upon as a subtle form of the classical horn equation.

POSTSCRIPT

After the submission of this paper for publication, the present author became aware of a paper by Zhenlin and Jiazheng [6], in which the authors have derived, from the Helmholtz equation, a relationship between the elements of the impedance matrix of a non-uniform duct with no mean flow and the impedance matrix of the same duct carrying an incompressible low Mach number mean flow with $M_o^2 \ll 1$. To compare this relationship with equation (40), which is based on quite different considerations, it is given here in terms of the elements of the scattering matrix of the duct with mean flow, $\mathbf{T}(L, 0)$, and the elements of the scattering matrix of the same duct with zero mean flow, $\mathbf{T}^o(L, 0)$:

$$T_{11}(L) = T^o_{11}(L)[1 - M_o(L) + M_o(0)]\exp(\cdot), \tag{48a}$$

$$T_{12}(L) = T^o_{12}(L)[1 - M_o(L) - M_o(0)]\exp(\cdot), \tag{48b}$$

$$T_{21}(L) = T^o_{21}(\mathrm{L})[1 + M_o(L) + M_o(0)]\exp(\cdot), \tag{48c}$$

$$T_{22}(L) = T^o_{22}(L)[1 + M_o(L) - M_o(0)]\exp(\cdot). \tag{48d}$$

Here, the exponential factor is given by $(\cdot) = \mathrm{i}k_o(\Phi(0) - \Phi(L))/c_o$, where $\Phi(x)$ denotes the mean flow velocity potential function [5, 6]. The present counterparts of these relations are given by equation (40):

$$T_{11}(L) = T^o_{11}(L)\exp(\times), \tag{49a}$$

$$T_{12}(L) = T^o_{12}(L)[1 - M_o(0)]\exp(\times)/[1 + M_o(0)], \tag{49b}$$

$$T_{21}(L) = T^o_{21}(L)[1 + M_o(L)]\exp(\times)/[1 - M_o(L)], \tag{49c}$$

$$T_{22}(L) = T^o_{22}(L)[1 - M_o(0)][1 + M_o(L)]\exp(\times)/[1 - M_o(L)][1 + M_o(0)], \tag{49d}$$

where the argument of the exponential factor, $(\times)$, is the integral in equation (40).

With equations (48) holding, there is not any variation in the transmission loss with mean flow [6], that is, $TL = TL^o$, where TL denotes transmission loss, which is defined as the level difference between the incident sound power and the transmitted sound power for an anechoic termination. With equations (49) holding, the transmission loss is given by the following relationship: $TL = TL^{o} + 20\,\mathrm{Log}\,[1 + M_o(0)]/[1 + M_o(L)]$.

Thus, equation (40), is akin, but not identical, to the corresponding relationship presented in reference [6]. Zhenlin and Jiazheng also show that, in the case of hyperbolic and parabolic ducts, their relationship agrees with the exact solutions presented previously by other authors. This raises the question: Is it possible that different exact analytical solutions of the problem can exist under the assumption $M_o^2 \ll 1$? So far, the present author has not succeeded in providing a rigorous analytical answer to this question. However, equation (40) has been checked numerically for several duct geometries by the exact solutions of the governing acoustic equations, equation (9), which are obtained by using a numerical

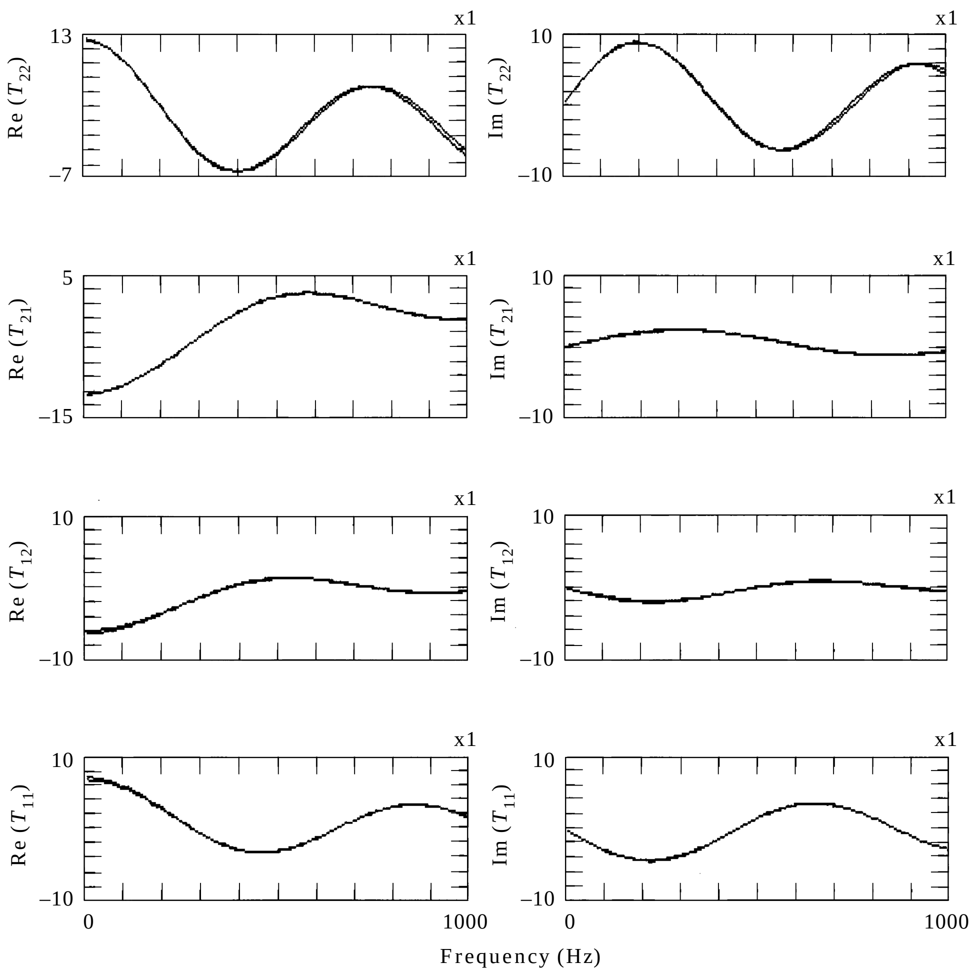

Figure 1. Elements of the inverted scattering matrix of a 0·444 m long diverging conical duct with inlet diameter 0·0246 m, truncated cone length 0·141 m and inlet mean flow Mach number of $M_o(0) = 0{\cdot}3$ at 25°C. Compared in the figure are the solutions obtained by using the numerical matrizant method and by using equation (40).

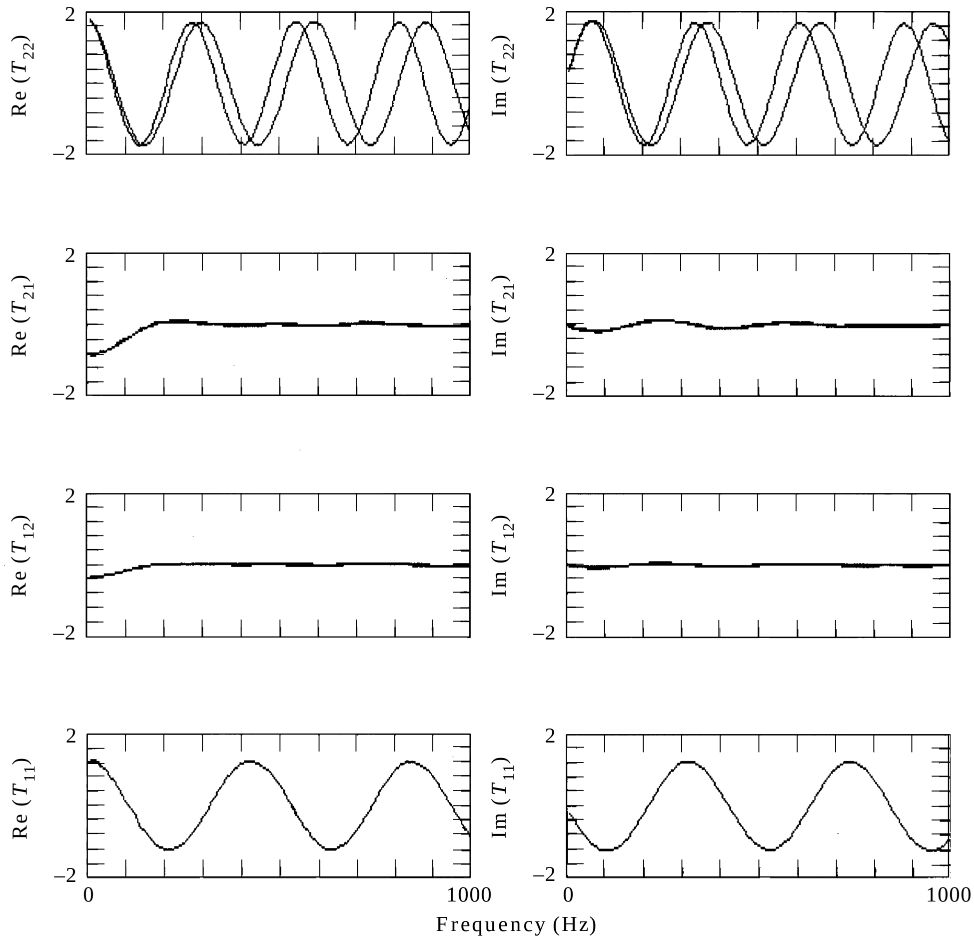

Figure 2. Elements of the inverted scattering matrix of a 1-m long diverging exponential duct with inlet cross-sectional area of 0·01 m^2, outlet cross-sectional area of 0·02 m^2 and inlet mean flow Mach number of $M_o(0) = 0{\cdot}15$ at 25°C. Compared in the figure are the solutions obtained by using the numerical matrizant method and by using equation (40).

matrizant method [7] which does not require the small Mach number assumption. A typical case is shown in Figure 1. This is a 0·444 m long diverging conical duct with inlet diameter 0·0246 m, truncated cone length 0·141 m and inlet mean flow Mach number of $M_o(0) = 0{\cdot}3$ at 25°C. Compared in Figure 1 are the elements of the inverse of the scattering matrix of this duct, which are still denoted by T_{ij}, as determined by using the numerical matrizant method and by using equation (40). In the latter case, the duct scattering matrix with zero mean flow is also computed by using the numerical matrizant method. As can be seen, equation (40) yields almost exact results. A similar comparison is presented in Figure 2 for a 1-m long diverging exponential duct with inlet cross-sectional area of 0·01 m^2, outlet cross-sectional area of 0·02 m^2, inlet mean flow Mach number of $M_o(0) = 0{\cdot}3$ at 25°C. In this case, equation (40) gives almost exact results except for the element

T_{22}. The observed phase error in T_{22} almost dissappears when $M_o(0) = 0{\cdot}15$. Thus, it is seen that equation (40) is a valid one and, therefore, that the answer to the above posed question can be stated to be an affirmative one.

The numerical matrizant solution of equation (9) is described briefly in a companion paper [7]. It may be of interest to note that, equations (41) and (42), and equations (46) and (47) have also been checked numerically by the numerical matrizant solutions of equation (9).

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APPENDIX: THE IMPEDANCE MATRIX FORMULATION

Equation (9) can be backtransformed, by using equations (4a) and (4b), to the impedance variables p and v. The resulting equations can be expressed in state-space form as

$$\begin{bmatrix} p' \\ v' \end{bmatrix} = \begin{bmatrix} E(x) & F(x) \\ G(x) & H(x) \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}, \tag{A1}$$

where

$$E = v_o[-\mathrm{i}\omega + (\gamma_o - 1)v_o' + \gamma_o v_o(\ln S)']/(c_o^2 - v_o^2), \tag{A2}$$

$$F = \rho_o c_o^2[\mathrm{i}\omega - v_o' + v_o p_o'/\rho_o c_0^2 + v_o(\ln S)']/(c_o^2 - v_o^2), \tag{A3}$$

$$G = [\mathrm{i}\omega - (\gamma_o - v_o^2/c_o^2)v_o' - \gamma_o v_o(\ln S)']/\rho_o(c_o^2 - v_o^2), \tag{A4}$$

$$H = [-\mathrm{i}\omega v_o + v_o v_o' - p_o'/\rho_o - c_o^2(\ln S)']/(c_o^2 - v_o^2). \tag{A5}$$

Upon introducing the duct impedance, $z = p/v$, equation (A1) can be written as

$$(\ln p)' = E + F/z, \qquad (\ln v)' = G + Hz. \tag{A6}$$

Therefore, z must satisfy the Riccati equation

$$z' + (G - E)z + Hz^2 = F. \tag{A7}$$

The analysis of section 3 can now be repeated by using equation (A7) instead of equation (12). As is evident from the similarity of these equations, the impedance formulation can be obtained from the expressions given in the main text of the paper simply by making notational changes such as $p^+ \Rightarrow p$, $p^- \Rightarrow v$, $r \Rightarrow z$, $A[M_o] \Rightarrow E$, $B[M_o] \Rightarrow F$, $B[-M_o] \Rightarrow G$ and $A^*[-M_o] \Rightarrow H$, etc. Some of the salient relationships of the impedance formulation are summarized in the following.

The linear differential equation associated with equation (A7) is

$$u'' + (E - H - (\ln G)')u' - GFu = 0. \tag{A8}$$

The solution of equation (A1) can be expressed as

$$\begin{bmatrix} p(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} Z_{11}(x) & Z_{12}(x) \\ Z_{21}(x) & Z_{22}(x) \end{bmatrix} \begin{bmatrix} p(0) \\ v(0) \end{bmatrix}. \tag{A9}$$

Here, the 2×2 square matrix, say, $\mathbf{Z}(x, 0)$, is called the duct impedance matrix. For a duct of length L, the impedance matrix is given by

$$\mathbf{Z}(L, 0) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{F(L)} \end{bmatrix} \begin{bmatrix} u_2(L) & u_1(L) \\ u_2'(L) & u_1'(L) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & F(0) \end{bmatrix} \exp\left(\int_0^L E(x)\,\mathrm{d}x\right), \tag{A10}$$

where u_1 and u_2 are two independent solutions of equation (A8) satisfying the boundary conditions $u_1(0) = 0$, $u_1'(0) = 1$, $u_2(0) = 1$, $u_2'(0) = 0$.

The Riccati equation for the duct impedance, equation (A7), can be integrated by single quadrature if the quotients $a = (E - H)/2F$ and $b = G/F$ are constants. Then, the two independent solutions of equation (A8) are:

$$u_1(x) = (1 - \mathrm{e}^{\psi})\xi(x)/F(0), \qquad u_2(x) = (z_1\,\mathrm{e}^{\psi} - z_2)\xi(x), \tag{A11a, b}$$

where

$$z_{1,2} = -a \mp \sqrt{(a^2 + b)}, \qquad \psi(x) = (z_2 - z_1)\int_0^x F(x)\,\mathrm{d}x,$$

$$\xi(x) = \frac{1}{z_1 - z_2}\exp\left(z_1 \int_0^x F(x)\,\mathrm{d}x\right). \tag{A12–A14}$$