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RESONANCE FREQUENCIES OF VISCOUSLY DAMPED STRUCTURES

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1. INTRODUCTION

The analysis of viscously damped structures is becoming increasingly important in the fields of smart structures and rotating machinery. In smart structures passive and active damping are used to reduce the vibration response of the structure to disturbances. Journal bearings in rotating machinery add significant damping to the system. Usually this damping is non-classical, in the sense that the modes are complex. Caughey and O'Kelly [1] gave the conditions on the mass, damping and stiffness matrices for the modes to be classical (or real). A number of authors have given an overview of complex modes [2–6]. Other interesting work has concerned calculating bounds on the damped response [7], the derivation of conditions to determine whether the response is over, under or critically damped [8], and the estimation of the errors involved in neglecting the coupling between the undamped modes through the damping matrix [9, 10].

This letter is concerned with the resonance frequencies of viscously damped structures, defined as the frequency at which the response attains a local maximum. For a single-degree-of-freedom system with viscous damping it is well known that this resonance frequency is neither the undamped nor the damped natural frequency. For multi-degree-of-freedom systems the resonance frequency can change depending on which degree of freedom is considered. This occurs for well separated complex modes and also for systems with real modes, due to the influence of neighbouring modes. In rotating structures these frequencies are termed the critical speeds, and one standard definition of a critical speed is based on the frequency at which the response attains a local maximum [11]. Unfortunately, as demonstrated here, this definition of the critical speeds of a machine leads to different critical speeds depending on which degree of freedom is considered.

2. MODES OF A MULTI-DEGREE-OF-FREEDOM DAMPED SYSTEM

The standard equations of motion in structural dynamics in second order form are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \tag{1}$$

where the response is $\mathbf{x}$, the force is $\mathbf{f}$, and the mass, damping and stiffness matrices are $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$. The eigenvalues, λ_i, and corresponding eigenvectors, $\boldsymbol{\phi}_i$, are given by

$$[\lambda_i^2\mathbf{M} + \lambda_i\mathbf{D} + \mathbf{K}]\boldsymbol{\phi}_i = 0. \tag{2}$$

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Since the structural matrices are symmetric, the left and right eigenvectors are equal. Also the eigenvalues and eigenvectors must occur in complex conjugate pairs, because the structural matrices are real. The direct solution of equation (2), for the general damping case, is very difficult, and the equations of motion are conveniently rewritten in the state space form as,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{Bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D}\end{bmatrix}\begin{Bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{Bmatrix} + \begin{Bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}\end{Bmatrix} = [\mathbf{A}]\begin{Bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{Bmatrix} + \begin{Bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}\end{Bmatrix} \tag{3}$$

where **A** is the state space matrix. Transformations may be introduced to generate different formulations of the state space equations (in particular a symmetric state space matrix) but this is not required for the development in this paper. The eigenvalues, λ_i, and the associated right eigenvectors, $\mathbf{\Psi}_{Ri}$, of **A** are related by

$$\mathbf{A}\mathbf{\Psi}_{Ri} = \lambda_i \mathbf{\Psi}_{Ri}. \tag{4}$$

Notice that the eigenvalues are the same as those from the second order form, and that from the definition of the state space matrix

$$\mathbf{\Psi}_{Ri} = \begin{Bmatrix}\boldsymbol{\phi}_i\\ \lambda_i\boldsymbol{\phi}_i\end{Bmatrix}. \tag{5}$$

Similarly the left eigenvectors, $\mathbf{\Psi}_{Li}$, are obtained from

$$\mathbf{\Psi}_{Li}^{\mathrm{T}}\mathbf{A} = \lambda_i\mathbf{\Psi}_{Li}^{\mathrm{T}} \qquad \text{or} \qquad \mathbf{A}^{\mathrm{T}}\mathbf{\Psi}_{Li} = \lambda_i\mathbf{\Psi}_{Li}. \tag{6}$$

Although relating the left eigenvectors of the state space matrix to the eigenvectors of the second order form is more difficult than for the right eigenvectors, from equation (6) and the definition of **A**,

$$\mathbf{\Psi}_{Li} = \begin{Bmatrix}-(1/\lambda_i)\mathbf{K}\boldsymbol{\phi}_i\\ \mathbf{M}\boldsymbol{\phi}_i\end{Bmatrix}. \tag{7}$$

The eigenvalues and eigenvectors of **A** must occur in complex conjugate pairs, because **A** is real. Thus far no normalisation of the eigenvectors has been enforced. The eigenvectors may be multiplied by any complex scalar, and the result is also an eigenvector. A convenient scaling for the development of the forced response is

$$\mathbf{\Psi}_{Li}^{\mathrm{T}}\mathbf{\Psi}_{Ri} = 1. \tag{8}$$

With the definitions of equations (5) and (7) this forces the following normalisation on the eigenvectors of the second order form,

$$\boldsymbol{\phi}_i^{\mathrm{T}}[\lambda_i\mathbf{M} - (1/\lambda_i)\mathbf{K}]\boldsymbol{\phi}_i = \boldsymbol{\phi}_i^{\mathrm{T}}[2\lambda_i\mathbf{M} + \mathbf{D}]\boldsymbol{\phi}_i = 1. \tag{9}$$

The definition of the left eigenvectors, equation (7), and the normalisation, equation (8), agree with those of Lancaster [2]. The eigenvectors are also orthogonal, in the sense that,

$$\boldsymbol{\Psi}_{Li}^{\mathrm{T}}\boldsymbol{\Psi}_{R\ell} = 0 \qquad \text{and} \qquad \boldsymbol{\Psi}_{Li}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Psi}_{R\ell} = 0 \qquad \text{if } i \neq \ell. \tag{10}$$

The normalisation in equation (8) also implies that

$$\boldsymbol{\Psi}_{Li}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Psi}_{Ri} = \lambda_i. \tag{11}$$

3. FORCED RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM

The response of the single degree of freedom system

$$m\ddot{x} + d\dot{x} + kx = f. \tag{12}$$

may be written in the frequency domain (for zero initial conditions) as

$$X(\omega) = [(1/m)/(-\omega^2 + 2\mathrm{j}\zeta\omega_n\omega + \omega_n^2)]F(\omega), \tag{13}$$

where ζ and ω_n are the natural frequency and damping ratio, and X and F are the Fourier transforms of the response and the force. The maximum amplitude of the response to a white noise force input, occurs at a frequency

$$\omega_{max} = \omega_n\sqrt{1 - 2\zeta^2}. \tag{14}$$

Interestingly, this frequency is not the damped natural frequency of the system, nor is it the natural frequency.

4. FORCED RESPONSE OF MULTI-DEGREE-OF-FREEDOM SYSTEMS

It is now required to determine the maximum amplitude response for a multi-degree-of-freedom system. The approach will be to obtain the receptance matrix as a sum of contributions from each mode. Then near a maximum response it will be assumed that a single mode dominates, and determines the frequency at which the response is maximum. The receptance matrix is obtained by transforming to modal co-ordinates, $\mathbf{p}$, using the modal matrix, thus,

$$\begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} = [\boldsymbol{\Psi}_{R1} \quad \boldsymbol{\Psi}_{R2} \quad \cdots \quad \boldsymbol{\Psi}_{R2n}]\mathbf{p} = [\boldsymbol{\Psi}_R]\mathbf{p}. \tag{15}$$

Substituting into the state space equations (3), pre-multiplying by the transpose of the matrix of left eigenvectors $[\boldsymbol{\Psi}_L] = [\boldsymbol{\Psi}_{L1} \ \boldsymbol{\Psi}_{L2} \ \cdots \ \boldsymbol{\Psi}_{L2n}]$, and using the orthogonality and normalisation conditions, equations (8), (10) and (11),

$$\dot{\mathbf{p}} = [\boldsymbol{\Lambda}]\mathbf{p} + \boldsymbol{\Psi}_L^{\mathrm{T}}\begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f} \end{Bmatrix} \tag{16}$$

where $[\boldsymbol{\Lambda}] = \mathrm{diag}\,[\lambda_1, \lambda_2, \ldots \lambda_{2n}]$. In the frequency domain,

$$\mathbf{P}(\omega) = [\mathrm{j}\omega\mathbf{I} - \boldsymbol{\Lambda}]^{-1}\boldsymbol{\Psi}_L^{\mathrm{T}}\begin{Bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{F}(\omega)\end{Bmatrix}. \tag{17}$$

Transforming back to physical co-ordinates and using the definitions of the state space modal matrices, equations (5) and (7),

$$\mathbf{X}(\omega) = \boldsymbol{\Phi}[\mathrm{j}\omega\mathbf{I} - \boldsymbol{\Lambda}]^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{F}(\omega) \tag{18}$$

where $\boldsymbol{\Phi} = [\phi_1, \phi_2, \ldots, \phi_{2n}]$ is the $(n, 2n)$ modal matrix of the second order form. Equation (18) may be conveniently written as the summation [2]

$$\mathbf{X}(\omega) = \left[\sum_{i=1}^{2n} \frac{\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}}{\mathrm{j}\omega - \lambda_i}\right]\mathbf{F}(\omega). \tag{19}$$

Assuming the first n eigenvalues have positive imaginary part, and the second n are the corresponding complex conjugates, then

$$\mathbf{X}(\omega) = \sum_{i=1}^{n} \left[\frac{\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}}{\mathrm{j}\omega - \lambda_i} + \frac{\bar{\boldsymbol{\phi}}_i\bar{\boldsymbol{\phi}}_i^{\mathrm{T}}}{\mathrm{j}\omega - \bar{\lambda}_i}\right]\mathbf{F}(\omega), \tag{20}$$

where the overbar denotes the complex conjugate. Combining the complex conjugate terms gives

$$\mathbf{X}(\omega) = \sum_{i=1}^{n} \left[\frac{\mathbf{G}_i\mathrm{j}\omega + \mathbf{H}_i}{-\omega^2 + 2\zeta_i\omega_i\mathrm{j}\omega + \omega_i^2}\right]\mathbf{F}(\omega), \tag{21}$$

where ω_i and ζ_i are the ith natural frequency and damping ratio, and the definition of the eigenvalues in terms of the natural frequency and damping ratio, $\lambda_i = -\zeta_i\omega_i + \mathrm{j}\omega_i\sqrt{1 - \zeta_i^2}$, has been used. The numerator terms are

$$\mathbf{G}_i = 2\,\mathrm{Re}\,[\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}] \quad \text{and} \quad \mathbf{H}_i = -2\,\mathrm{Re}\,[\bar{\lambda}_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i^{\mathrm{T}}]. \tag{22}$$

Note that for real modes $\mathbf{G}_i$ is not automatically zero, because of the scaling of the modes, equation (9). For a particular frequency response function, near to the ith natural frequency, the magnitude is maximum when

$$\alpha(\omega) = \left|\frac{g_i\mathrm{j}\omega + h_i}{-\omega^2 + 2\zeta_i\omega_i\mathrm{j}\omega + \omega_i^2}\right|^2 = \frac{g_i^2\omega^2 + h_i^2}{(\omega_i^2 - \omega^2)^2 + 4\zeta_i^2\omega_i^2\omega^2} \tag{23}$$

is maximum. The scalars g_i and h_i are the relevant elements of $\mathbf{G}_i$ and $\mathbf{H}_i$. The maximum of α is computed by taking the derivative of equation (23) with respect to ω^2 and setting the result to zero. The optimum frequency, ω, is then the solution of

$$g_i^2\omega^4 + 2h_i^2\omega^2 - (2h_i^2\omega_{max}^2 + g_i^2\omega_i^4) = 0. \tag{24}$$

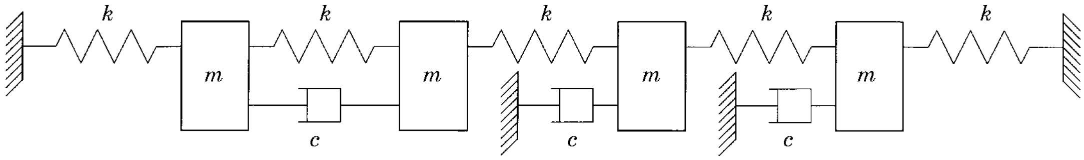

Figure 1. The four degree of freedom discrete example.

where ω_{max} is given by equation (14). The solution for the single-degree-of-freedom case, ω_{max}, is recovered if $g_i = 0$. Also note that there is only one positive solution of this quadratic equation in ω^2, equation (24), and thus only one real, positive solution for the frequency at which the response is a maximum, for each receptance. Note, however, that because the values of g_i and h_i change depending the force and response degrees of freedom considered, the frequency at which the response is maximum also changes. Thus even with a single complex mode approximation the resonance frequency changes with the measurement and forcing location.

Residuals from neighbouring modes may be incorporated into the receptance, instead of the single mode assumption of equation (23). For the higher modes this residual is almost constant and for the $(i+1)$th mode the equivalent to equation (23) is

$$\alpha(\omega) = |(g_i \mathrm{j}\omega + h_i)/(-\omega^2 + 2\zeta_i\omega_i\mathrm{j}\omega + \omega_i^2) + h_{i+1}/\omega_{i+1}^2|^2. \tag{25}$$

Maximising this receptance produces an equation equivalent to equation (24), but where

$$g_i^2 \rightarrow g_i^2 - 2h_i h_{i+1}/\omega_{i+1}^2 + 4g_i h_{i+1}\zeta_i\omega_i/\omega_{i+1}^2 \tag{26}$$

and

$$h_i^2 \rightarrow h_i^2 + 2h_i h_{i+1}\omega_i^2/\omega_{i+1}^2. \tag{27}$$

5. REAL MODES

For proportional damping, that is when the modes of the second order system are real, then $\mathbf{G}_i = \mathbf{0}$, and the maxima of the receptance amplitudes occur at the same point for all frequency response functions. This frequency may be computed

TABLE 1

Natural frequencies for the discrete example

Mode	Natural frequency (rad/s)	Damping ratio (%)	Damped natural frequency (rad/s)	Frequency at maximum for SDOF system (rad/s)
1	0·6407	23·2	0·6233	0·6053
2	1·1404	11·3	1·1332	1·1259
3	1·6734	23·8	1·6252	1·5755
4	1·8287	17·7	1·7997	1·7703

TABLE 2

Numerator terms for the discrete example

Mode 1	Mode 2
$\mathbf{G}_1 = \begin{bmatrix} 0{\cdot}1454 & 0{\cdot}1533 & 0{\cdot}0899 & 0{\cdot}0369 \\ 0{\cdot}1533 & 0{\cdot}1036 & -0{\cdot}0130 & -0{\cdot}0418 \\ 0{\cdot}0899 & -0{\cdot}0130 & -0{\cdot}1445 & -0{\cdot}1270 \\ 0{\cdot}0369 & -0{\cdot}0418 & -0{\cdot}1270 & -0{\cdot}1028 \end{bmatrix}$	$\mathbf{G}_2 = \begin{bmatrix} -0{\cdot}1471 & -0{\cdot}1308 & -0{\cdot}0869 & 0{\cdot}0096 \\ -0{\cdot}1308 & -0{\cdot}1066 & -0{\cdot}0336 & 0{\cdot}0405 \\ -0{\cdot}0869 & -0{\cdot}0336 & 0{\cdot}1451 & 0{\cdot}1494 \\ 0{\cdot}0096 & 0{\cdot}0405 & 0{\cdot}1494 & 0{\cdot}1045 \end{bmatrix}$
$\mathbf{H}_1 = \begin{bmatrix} 0{\cdot}1387 & 0{\cdot}2399 & 0{\cdot}2596 & 0{\cdot}1653 \\ 0{\cdot}2399 & 0{\cdot}3897 & 0{\cdot}4025 & 0{\cdot}2510 \\ 0{\cdot}2596 & 0{\cdot}4025 & 0{\cdot}3996 & 0{\cdot}2448 \\ 0{\cdot}1653 & 0{\cdot}2510 & 0{\cdot}2448 & 0{\cdot}1486 \end{bmatrix}$	$\mathbf{H}_2 = \begin{bmatrix} 0{\cdot}3709 & 0{\cdot}2209 & -0{\cdot}2708 & -0{\cdot}3828 \\ 0{\cdot}2209 & 0{\cdot}1265 & -0{\cdot}1838 & -0{\cdot}2444 \\ -0{\cdot}2708 & -0{\cdot}1838 & 0{\cdot}0964 & 0{\cdot}2053 \\ -0{\cdot}3828 & -0{\cdot}2444 & 0{\cdot}2053 & 0{\cdot}3408 \end{bmatrix}$

using the single modal approximation, and is the equivalent to equation (14). Although the mode is real, that is the relationship between the degrees of freedom is real, the mode will be multiplied by a complex constant because of the normalisation imposed on the $\boldsymbol{\phi}_i$ by equation (9). Let η_i be the real, mass normalised mode shape. Then the normalised complex mode shape is

$$\boldsymbol{\phi}_i = \beta_i \boldsymbol{\eta}_i \tag{28}$$

for some complex scalar β_i. The normalisation constant is obtained from equation (9), since

$$\boldsymbol{\phi}_i^{\mathrm{T}}\left[\lambda_i \mathbf{M} - \frac{1}{\lambda_i}\mathbf{K}\right]\boldsymbol{\phi}_i = \beta_i^2 \boldsymbol{\eta}_i^{\mathrm{T}}\left[\lambda_i \mathbf{M} - \frac{1}{\lambda_i}\mathbf{K}\right]\boldsymbol{\eta}_i = \beta_i^2\left(\lambda_i - \frac{\omega_i^2}{\lambda_i}\right) = 1. \tag{29}$$

Then, from the definition of $\mathbf{G}_i$, equation (21),

$$\mathbf{G}_i = 2[\boldsymbol{\eta}_i \boldsymbol{\eta}_i^{\mathrm{T}}]\ \mathrm{Re}\,(\beta_i^2) = 2[\boldsymbol{\eta}_i \boldsymbol{\eta}_i^{\mathrm{T}}]\ \mathrm{Re}\left(\frac{\lambda_i}{\lambda_i^2 - \omega_i^2}\right). \tag{30}$$

But,

$$\begin{aligned}
&\mathrm{Re}\,(\lambda_i/(\lambda_i^2 - \omega_i^2)) \\
&\quad = \mathrm{Re}\,[(-\zeta_i\omega_i + \mathrm{j}\omega_i\sqrt{1-\zeta_i^2})/(-2\omega_i^2(1-\zeta_i^2) - 2\mathrm{j}\zeta_i\omega_i^2\sqrt{1-\zeta_i^2})] \\
&\quad = \mathrm{Re}\,[(-\zeta_i\omega_i + \mathrm{j}\omega_i\sqrt{1-\zeta_i^2})((1-\zeta_i^2) - \mathrm{j}\zeta_i\sqrt{1-\zeta_i^2})/-2\omega_i^2(1-\zeta_i^2)] = 0
\end{aligned} \tag{31}$$

Thus for systems with real eigenvectors of the second order form, the maximum response based on a single mode approximation occurs at the same frequency for all degrees of freedom. For closely spaced modes, or systems with high damping, this single mode assumption is invalid, and even with real modes the frequency

TABLE 3

Frequencies for maximum response amplitude for the first mode

Forced DoF	Response DoF	Full order model (rad/s)	Single mode assumption (rad/s)	Single mode with residual from mode 2 (rad/s)	Single mode/single DoF assumption (rad/s)
1	1	0·6099	0·6163	0·5933	0·6053
1	2	0·6164	0·6103	0·5973	0·6053
1	3	0·6421	0·6069	0·6285	0·6053
1	4	0·6632	0·6060	0·6571	0·6053
2	2	0·5972	0·6063	0·6001	0·6053
2	3	0·6216	0·6053	0·6155	0·6053
2	4	0·6384	0·6057	0·6277	0·6053
3	3	0·5842	0·6071	0·6015	0·6053
3	4	0·5995	0·6088	0·5893	0·6053
4	4	0·5320	0·6110	0·5578	0·6053

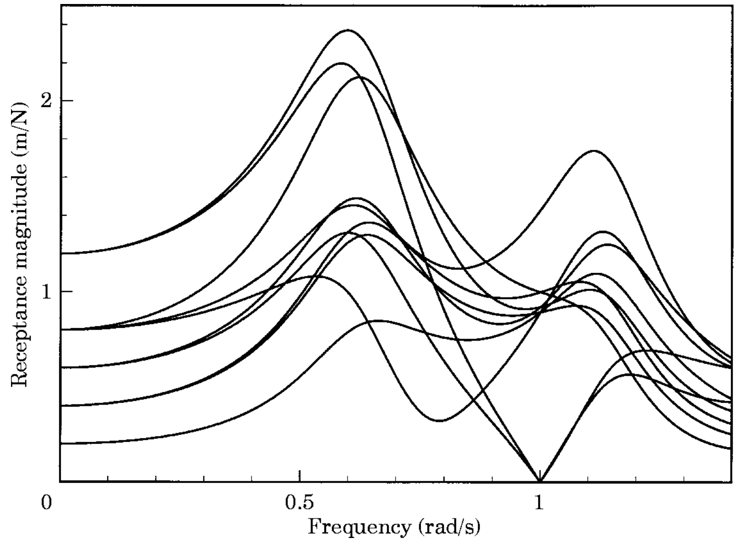

Figure 2. The receptance magnitude for the four-degree-of-freedom discrete example.

at which the response is maximum can change with the position of the measurement.

6. A DISCRETE EXAMPLE

Consider the four-degree-of-freedom example shown in Figure 1, with $m = 1$ kg, $c = 0{\cdot}5$ Ns/m and $k = 1$ N/m. There are 16 frequency response functions associated with this model, although by reciprocity six of them are repeated. Table 1 shows the natural frequencies, damping ratios and the damped natural frequencies. Also shown are the frequency at which the response would be maximum if each mode was considered as a single-degree-of-freedom system (i.e., the frequencies given by equation (14)). Table 2 shows the values of g_i and h_i.

TABLE 4

Frequencies for maximum response amplitude for the second mode

Forced DoF	Response DoF	Full order model (rad/s)	Single mode assumption (rad/s)	Single mode/single DoF assumption (rad/s)
1	1	1·1107	1·1283	1·1259
1	2	1·1399	1·1304	1·1259
1	3	1·0812	1·1276	1·1259
1	4	1·1175	1·1259	1·1259
2	2	1·1033	1·1329	1·1259
2	3	–	1·1265	1·1259
2	4	1·0744	1·1264	1·1259
3	3	1·1871	1·1368	1·1259
3	4	1·2228	1·1318	1·1259
4	4	1·1302	1·1275	1·1259

Tables 3 and 4 show the frequency of maximum amplitude for the 10 independent frequency response curves, obtained numerically from the four-degree-of-freedom equations of motion. Clearly the maximum response occurs at significantly different frequencies. Figure 2 shows the absolute frequency response functions for all 10 independent responses, for a frequency range covering the first two modes. Tables 3 and 4 also show the frequencies of maximum response obtained by solving equation (24), based on a single mode assumption. Although these estimated frequencies for maximum response change for different responses, clearly the estimates are very inaccurate. The problem is that for highly damped systems the single mode assumption is invalid. Table 3 also shows that even allowing for a constant residual for the second mode (equations (26) and (27)) doesn't help very much. The effect of neighbouring modes is higher than that of the mode complexity. Thus, in general, the accurate determination of frequencies of maximum response for highly damped systems must be obtained by numerical optimisation.

7. CONCLUSIONS

It is well known that the resonance frequencies of a multi degree of freedom viscously damped structure change with the measurement (and forcing) location. This phenomena has been investigated using the complex mode expansion of the structure's receptance matrix. A single mode approximation to a real mode of a system does have the same resonance frequency at all degrees of freedom. However, the resonance frequencies of a single mode approximation to a complex mode of a system are different at different degrees of freedom. For multi-degree-of-freedom systems the resonance frequencies change with the measurement location, even for systems with real modes, because of the interaction between modes.

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