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EIGENFREQUENCIES OF A TWO-MASS OSCILLATOR UNIFORMLY MOVING ALONG A STRING ON A VISCO-ELASTIC FOUNDATION

H. Kruse and K. Popp

Faculty of Mechanical Engineering, Institute of Mechanics, University of Hannover, Appelstrasse 11, D-30167 Hannover, Germany

and

A. V. Metrikine

Mechanical Engineering Institute Russian Academy of Sciences, Belinskogo 85, 603024 Nizhny Novgorod, Russian Federation

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The eigenfrequencies of a two-mass oscillator moving uniformly along a string on a visco-elastic foundation are analysed. It is shown that in the case of purely elastic foundation, the oscillator has either one or two real positive eigenfrequencies dependent on the system parameters. Taking into account the viscosity of the foundation, the complex eigenfrequencies of the oscillator are investigated. The study shows that eigenfrequencies, which are related to attenuating vibrations of the oscillator, are not uniquely determined. It is found that the existence of an eigenfrequency $\omega = \omega_0 + i\delta$ with a small imaginary part $\delta \ll \omega_0$ is not a sufficient condition for resonance under an external force $P \exp(i\omega_0 t)$.

1. INTRODUCTION

The development of high-speed trains initiated a number of investigations dealing with the dynamic behaviour of elastic systems interacting with moving loads. These investigations are important for both the interaction between train and track and between pantograph and catenary. Concerning the modelling of the load, two possibilities exist. First a load can be described as a given external force acting on an elastic system and having no internal degrees of freedom. Such an approach allows the determination of displacements and stresses of the elastic system. The second possibility is to take into account the internal degrees of freedom of the moving object (train bogie or pantograph). Then the contact force has to be determined from the condition that the displacements of the elastic system and the load are equal at the loading point. The first approach is simpler to be analysed and frequently used by investigators [1–5]. It is clear, however, that the inertial and elastic properties of the moving object must be considered if vibrations take

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place with a frequency close to one of the resonance frequencies of the moving object, interacting with the elastic system [6].

This paper is devoted to the analysis of the eigenfrequencies of a two-mass oscillator uniformly moving along a string. This simple model is chosen to investigate the problem analytically and to be able to interpret the results physically in a relatively clear way. Particular attention is paid to the following questions: (1) How many real eigenfrequencies does the oscillator have, as it interacts with the elastic system? (2) How does the number of these eigenfrequencies depend on the parameters of the system? (3) Are all eigenfrequencies of the oscillator on the elastic system uniquely determined? (4) Is it always possible to declare that if the system has an eigenfrequency with a small imaginary part $\omega = \omega_0 + i\delta$, $\delta \ll \omega_0$, then an external harmonic force with frequency ω_0 leads to resonance?

The analysis of the system shows that the oscillator has either one or two real eigenfrequencies. The lower real eigenfrequency exists for all parameters of the system. The higher one disappears in a range of the system parameters. Further it is demonstrated that the characteristic equation uniquely determines only real eigenfrequencies and eigenfrequencies related to unstable vibrations of the oscillator. Eigenfrequencies, related to attenuating vibrations, are not uniquely determined. A range of system parameters is found where a complex eigenfrequency $\omega = \omega_0 + i\delta$, $\delta \ll \omega_0$ (slightly attenuating vibrations) exists which does not give the resonance due to an external force with frequency ω_0.

Though the results obtained have mostly an academic interest, parameters are chosen in such a way that to some extent the results can be used for the interaction of a railroad catenary and a moving pantograph.

2. MODEL AND EQUIVALENT STIFFNESS OF THE STRING

A two-mass oscillator moving uniformly along an infinite string on a visco-elastic foundation is considered; see Figure 1. It is assumed that the lower mass of the oscillator is always in contact with the string.

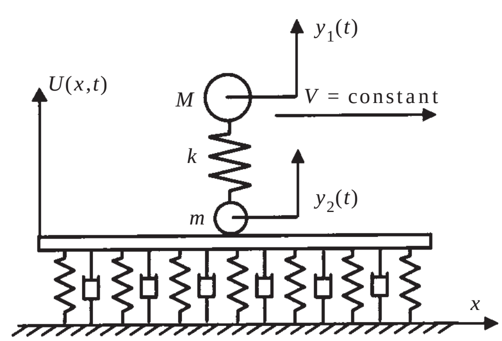

Figure 1. Oscillator motion along string on foundation.

The governing equations, describing the vertical vibrations of the string and the oscillator are

$$\mu U_{tt} - NU_{xx} + \beta U_t + \gamma U = -(m\ddot{y}_2 + k(y_2 - y_1))\delta(x - Vt),$$

$$M\ddot{y}_1 + k(y_1 - y_2) = 0, \qquad U(Vt, t) = y_2(t),$$

$$U(x, t) \to 0 \qquad \text{for } x - Vt \to \pm\infty, \tag{1}$$

where $U(x, t)$ is the vertical displacement of the string, μ is the mass per unit length and N is the tension of the string, β and γ are the viscosity and the stiffness of the foundation per unit length, $y_1(t)$ and $y_2(t)$ are the vertical displacements of the upper mass M and lower mass m of the oscillator, k is the stiffness of the oscillator spring and $\delta(\cdots)$ denotes the Dirac delta function.

For the analysis it is convenient to introduce a moving co-ordinate system $\{\xi = x - Vt, \tau = t\}$. In this system, equations (1) take the form

$$U_{\tau\tau} - 2VU_{\xi\tau} - (c^2 - V^2)U_{\xi\xi} + \tilde{\varepsilon}U_t + h^2 U = -\frac{1}{\mu}\left(m\frac{\mathrm{d}^2 y_2}{\mathrm{d}\tau^2} + k(y_2 - y_1)\right)\delta(\xi),$$

$$M\frac{\mathrm{d}^2 y_1}{\mathrm{d}\tau^2} + K(y_1 - y_2) = 0, \qquad U(0, \tau) = y_2(\tau),$$

$$U(\xi, \tau) \to 0 \qquad \text{for } \xi \to \pm\infty, \tag{2}$$

where $c = \sqrt{N/\mu}$ is the velocity of waves in the string, $h = \sqrt{\gamma/\mu}$ characterises the cut-off frequency and $\tilde{\varepsilon} = \beta/\mu$. For a catenary, for example, the wave velocity c is about $c = 100$ m/s. Although a catenary has discrete supports, it has a cut-off frequency due to the support stiffness and the weight of the cables. As an approximation the cut-off frequency can be taken in the range 1 Hz $< h <$ 10 Hz. The damping coefficient $\tilde{\varepsilon}$ is low. We choose a value of $\tilde{\varepsilon} = 0{\cdot}01\ \mathrm{s}^{-1}$. The eigenfrequencies of the moving oscillator on the string can be determined from equation (2) using the following Fourier transforms

$$V_\omega(\xi, \omega) = \int_{-\infty}^{+\infty} U(\xi, \tau)\exp(-\mathrm{i}\omega\tau)\,\mathrm{d}\tau,$$

$$W_{\kappa,\omega}(\kappa, \omega) = \int_{-\infty}^{+\infty} V_\omega(\xi, \omega)\exp(-\mathrm{i}\kappa\xi)\,\mathrm{d}\xi. \tag{3}$$

Applying these transforms, one gets

$$D(\kappa, \omega)\cdot W_{\kappa,\omega}(\kappa, \omega) = \frac{1}{\mu}(m\omega^2\cdot z_2(\omega) - k\cdot(z_2(\omega) - z_1(\omega))),$$

$$-M\omega^2\cdot z_1(\omega) + k\cdot(z_1(\omega) - z_2(\omega)) = 0, \qquad V_\omega(0, \omega) = z_2(\omega), \tag{4}$$

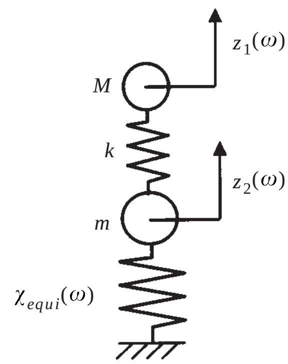

Figure 2. Equivalent model in frequency domain.

where $D(\kappa, \omega) = -\omega^2 + 2V\omega\kappa + i\tilde{\varepsilon}(\omega - V\kappa) - (V^2 - c^2)\kappa^2 + h^2$ is the dispersion relation of the string on visco-elastic foundation and

$$z_i(\omega) = \int_{-\infty}^{+\infty} y_i(\tau) \exp(-i\omega\tau)\, d\tau, \qquad i = 1, 2$$

are the Fourier displacements of the masses. To find the Fourier displacement of the string we apply the inverse Fourier transform with respect to κ to the first equation of (4). This yields

$$V_\omega(\xi, \omega) = (m\omega^2 \cdot z_2(\omega) - k \cdot (z_2(\omega) - z_1(\omega))) \cdot \frac{1}{2\pi\mu} \cdot \int_{-\infty}^{+\infty} \frac{\exp(i\kappa\xi)}{D(\kappa, \omega)}\, d\kappa. \quad (5)$$

Letting $\xi = 0$ and using the condition of the contact between the string and the oscillator ($V_\omega(0, \omega) = z_2(\omega)$), one obtains

$$z_2(\omega) = (m\omega^2 \cdot z_2(\omega) - k \cdot (z_2(\omega) - z_1(\omega))) \cdot \frac{1}{2\pi\mu} \cdot \int_{-\infty}^{+\infty} \frac{1}{D(\kappa, \omega)}\, d\kappa. \quad (6)$$

Expression (6) and the second of equations (4) describe the eigenvibrations of the oscillator in the Fourier domain. These two equations can be written as

$$\begin{bmatrix} -M\omega^2 + k & -k \\ -k & -m\omega^2 + \chi_{equi}(\omega) + k \end{bmatrix} \cdot \begin{bmatrix} z_1(\omega) \\ z_2(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (7)$$

where

$$\chi_{equi}(\omega) = \left(\frac{1}{2\pi\mu} \cdot \int_{-\infty}^{+\infty} \frac{d\kappa}{D(\kappa, \omega)} \right)^{-1} \quad (8)$$

is the equivalent stiffness of the string. Introduction of this equivalent stiffness allows replacing the string under the oscillator by an equivalent spring. Now the oscillator vibrations can be considered in frames of the discrete model depicted in Figure 2. The special property of this model is that the equivalent stiffness of the lower spring is not a constant, but a complex function of the frequency ω and the velocity V of the oscillator.

To analyse the equivalent stiffness we have to evaluate the integral in equation (8). The integration can be performed with the help of the Residue theorem. For the application of this theorem it is convenient to rewrite expression (8) in the form

$$\chi_{equi}(\omega) = \left(\frac{1}{2\pi\mu(c^2 - V^2)} \cdot \int_{-\infty}^{+\infty} \frac{d\kappa}{(\kappa - \kappa_1) \cdot (\kappa - \kappa_2)}\right)^{-1}, \tag{9}$$

where

$$\kappa_{1,2} = \frac{1}{c^2 - v^2}((-\omega V + \tfrac{1}{2}i\tilde{\varepsilon}V) \pm \sqrt{(-\omega V + \tfrac{1}{2}i\tilde{\varepsilon}V)^2 - (c^2 - V^2)(-\omega^2 + h^2 + i\tilde{\varepsilon}\omega)}). \tag{10}$$

The location of the poles $\kappa_{1,2}$ of the integrand in equation (9) depends on the relationship between the velocity V of the oscillator and the velocity c of waves in the string. We will further consider the sub-critical motion $V < c$ (if $V > c$, the equivalent stiffness is infinite and the string behaves as a rigid support). In this case the poles lie at different sides of the real axis of the complex κ-plane. Fixing the branch of the square root in equation (10) by inequality Im $(\sqrt{\ldots .}) > 0$, one obtains that the pole κ_1 (with '+' sign in equation (10)) is located in the upper half-plane and κ_2 in the lower one. Closing the contour of integration over the upper half-plane and, therefore, taking into account the pole κ_1, one obtains the following expression for the equivalent stiffness,

$$\chi_{equi}(\omega) = \left(\frac{1}{2\pi\mu(c^2 - V^2)} \cdot 2\pi i \cdot \frac{1}{(\kappa_1 - \kappa_2)}\right)^{-1}. \tag{11}$$

Substitution of expressions for $\kappa_{1,2}$ yields

$$\chi_{equi}(\omega) = \frac{2\mu}{i} \cdot \sqrt{(-\omega V + \tfrac{1}{2}i\tilde{\varepsilon}V)^2 - (c^2 - V^2)(-\omega^2 + h^2 + i\tilde{\varepsilon}\omega)} \quad \text{with } \operatorname{Im}(\chi_{equi}) > 0. \tag{12}$$

This expression can be evaluated to give

$$\chi_{equi}(\omega) = 2\mu \cdot \sqrt{-c^2\omega^2 + (c^2 - V^2)h^2 + ic^2\tilde{\varepsilon}\omega + 1/4\tilde{\varepsilon}^2 V^2} \quad \text{with } \operatorname{Re}(\chi_{equi}) > 0. \tag{13}$$

It is important to underline, that the requirement Re $(\chi_{equi}) > 0$ should be fulfilled only for real ω (see the definition of inverse Fourier transform). Physically this requirement implies that the displacement of the string, related to harmonic vibrations of the oscillator, should decrease as the distance from the oscillator increases.

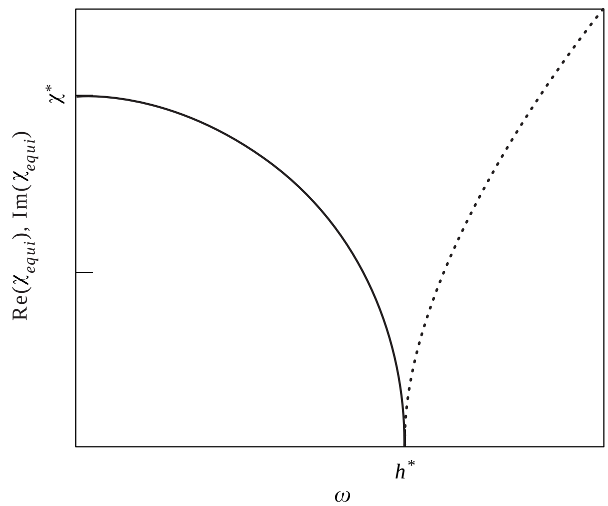

Figure 3. Real part (continuous line) and imaginary part (dotted line) of the equivalent stiffness χ_{equi} as a function of frequency ω.

Figure 3 shows the real and the imaginary part of the equivalent stiffness as a function of the frequency ω for zero viscosity of the foundation. One can see from the figure that for frequencies lower than the cut-off frequency $h^* = h\sqrt{1 - V^2/c^2}$ in the moving co-ordinate system, the string acts like a spring, which stiffness starts for zero frequency with $\chi^* = 2\mu hc\sqrt{1 - V^2/c^2}$ and decreases with increasing frequency. The reason for the pure real stiffness is that vibrations of the oscillator with frequencies $|\omega| < h^*$ excite no waves in the string. These vibrations correspond to an eigenfield, which moves with the oscillator, and do not extract energy from it.

Reaching the cut-off frequency h^*, the string does not give any reaction ($\chi_{equi} = 0$), since a moving load of this frequency causes resonance in the string. Therefore, a limited excitation leads to infinite displacements under the load.

For frequencies higher than h^* waves propagate in the string. This wave radiation results in a pure imaginary equivalent stiffness of the string. The string acts in a similar way as a damper.

The velocity V of the oscillator influences both, the equivalent stiffness χ^* for zero frequency and the cut-off frequency h^*, in the same way (see formulas above). An increase in V leads to a reduction in these two parameters.

3. EIGENFREQUENCIES OF THE OSCILLATOR

Having analysed the equivalent stiffness, we come back to the system of equations (7). Letting the determinant of this system be equal to zero, one obtains the following characteristic equation

$$k \cdot \chi_{equi} - k(m + M)\omega^2 - \chi_{equi} M\omega^2 + mM\omega^4 = 0. \tag{14}$$

Roots of equation (14) determine eigenfrequencies of the two-mass oscillator moving along the string.

First the real solutions of characteristic equation (14) will be studied, neglecting the viscosity of the string foundation. Substituting expression (13) for the equivalent stiffness and letting $\tilde{\varepsilon} = 0$, one can rewrite characteristic equation (14) as follows

$$2\mu c\sqrt{(h^*)^2 - \omega^2}(k - M\omega^2) = \omega^2(k(m + M) - mM\omega^2), \tag{15}$$

where $h^* = h\sqrt{1 - V^2/c^2}$. In equation (15) ω always occurs to the second power. Therefore, if a positive solution $\omega = +a$ exists, then $\omega = -a$ is a solution as well. Furthermore, one can see that equation (15) can possess real roots only in the range $|\omega| < h^*$ (physically it implies that the oscillator can vibrate harmonically if it does not radiate waves). In this range the square root $\sqrt{(h^*)^2 - \omega^2}$ is positive; see expression (13). This fact allows us to find out domains of possible existence of real eigenfrequencies of the considered system. Indeed, it follows from equation (15) that its real roots have to satisfy the inequality

$$\frac{(m + M - m\omega^2/\omega_0^2)}{(1 - \omega^2/\omega_0^2)} > 0 \qquad \text{with } \omega_0^2 = \frac{k}{M},$$

which gives

$$\frac{\omega^2}{\omega_0^2} < 1 \qquad \text{or} \qquad \frac{\omega^2}{\omega_0^2} > 1 + \frac{M}{m}. \tag{16}$$

In the case of a pantograph, magnitudes of the masses are located in the range $1\text{ kg} < m_i < 10\text{ kg}$. As the springs of this structure have different stiffnesses, the eigenfrequencies lie in a wide range, $1\text{ rad/s} < \omega_{0,i} < 50\text{ rad/s}$. Combining inequality (16) with the condition $|\omega| < h^*$, one can draw the following conclusions: 1. If $h < \omega_0$, then the domain of possible existence of the real eigenfrequencies is determined by the inequality $|\omega| < h^*$. This domain is shaded in Figure 4(a). 2. If $\omega_0 < h < \omega_0\sqrt{1 + M/m}$, then this domain is determined by the inequality $|\omega| < \omega_0$ for velocities $V < c\sqrt{1 - \omega_0^2/h^2}$ and by the inequality $|\omega| < h^*$ in the interval of velocities $c\sqrt{1 - \omega^2/h^2} < V < c$. This domain is shaded in Figure 4(b). 3. If $h > \omega_0\sqrt{1 + M/m}$, then there are two domains of existence which are plotted in Figure 4(c). The lower domain is determined by the same expressions as in the previous case and the upper one exists for velocities smaller than $V^* = c\sqrt{1 - (\omega_0/h)^2(1 + M/m)}$ in the interval $\omega_0\sqrt{1 + M/m} < |\omega| < h^*$.

Solid lines in Figures 4(a)–(c) show qualitatively the real positive eigenfrequencies of the oscillator in dependency of the velocity. The eigenfrequencies are obtained by squaring equation (15) and further numerical determination of roots of a polynom of the eighth order, located in the domains of possible existence. One can see from the figures that the moving oscillator has one real positive eigenfrequency if $h < \omega_0\sqrt{1 + M/m}$ (Figures 4(a) and (b)) and two such eigenfrequencies if $h > \omega_0\sqrt{1 + M/m}$ (Figure 4(c)). The lower eigenfrequency is always smaller than ω_0, while the higher one is always larger than $\omega_0\sqrt{1 + M/m}$. This situation is exactly the same as for a classical two-mass oscillator (Figure 2 with constant χ_{equi}) [7]. The only difference is that in our case the higher eigenfrequency cannot be larger than h^* since in this case the oscillator begins to generate waves in the string.

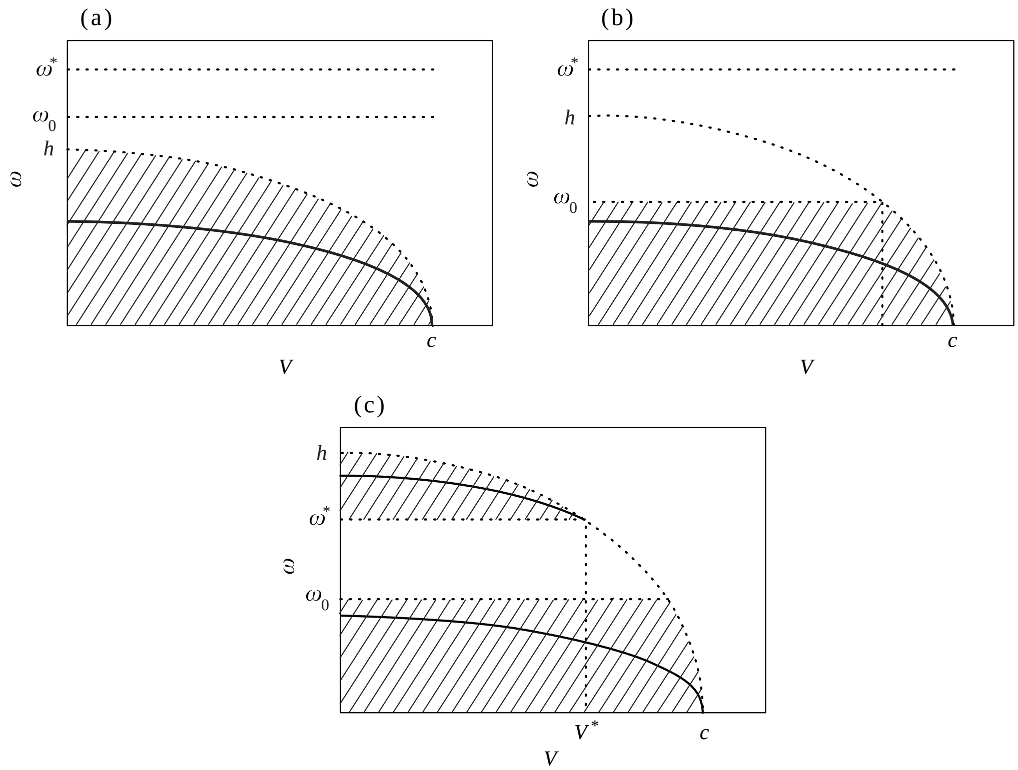

Figure 4. Domains of possible existence of real eigenfrequencies (shaded) for three different cases of the relation between h, $\omega_0 = \sqrt{k/M}$ and $\omega^* = \omega_0\sqrt{1 + M/m}$. Solid lines show real eigenfrequencies as functions of the velocity.

Thus, for one set of parameters the oscillator has one real and positive eigenfrequency and for another set two of them. Here the question might arise whether in the case of one real eigenfrequency the oscillator has an additional complex one or not. To answer this question one has to look for complex roots of the characteristic equation (14).

A particular feature of this equation is that it contains the radical given by expression (13). The sign of the real part of this radical is fixed, but only for real frequencies ω. Therefore, to analyse complex eigenfrequencies, the radical has to be analytically extended into the complex frequency domain [8]. It can be done

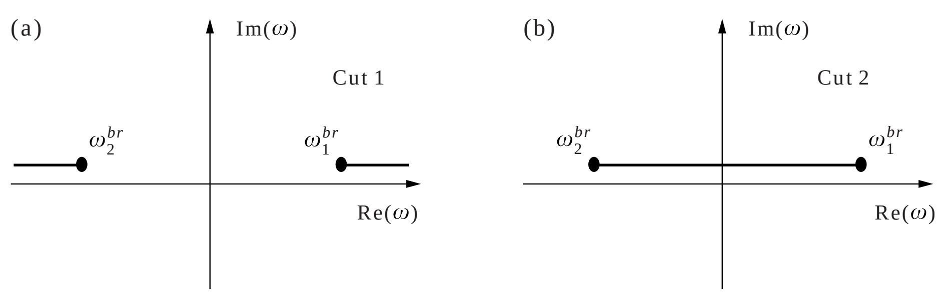

Figure 5. Branch cuts in the complex ω-plane.

in different ways choosing different branch cuts. To investigate whether this choice influences the number and location of the complex eigenfrequencies, two possible cuts are considered. They are depicted in Figures 5(a) and (b) for a small viscosity of the string foundation ($\tilde{\varepsilon} \ll h$). The branch points are determined in this case by the following approximate expression

$$\omega_{1,2}^{br} \approx \frac{\mathrm{i}\tilde{\varepsilon}}{2} \pm h^*.$$

It is seen from the figures that brance cuts are chosen to be parallel to the real axis of the ω-plane. The difference is that in Figure 5(a) the ω-plane is cut for $|\mathrm{Re}\,(\omega)| > h^*$ and in Figure 5(b) for $|\mathrm{Re}\,(\omega)| < h^*$. The brance cuts depicted in Figure 5(a) will be further referred to as "Cut 1" and the cut in Figure 5(b) as "Cut 2".

Analytical extension of the square root $\sqrt{-c^2\omega^2 + \cdots}$ with positive real part along the real axis gives: Cut 1: $\mathrm{Re}\,(\sqrt{\ldots.}) > 0$ in the entire ω-plane; Cut 2: $\mathrm{Re}\,(\sqrt{\ldots.}) > 0$ for $\mathrm{Im}\,(\omega) < \tilde{\varepsilon}/2$ and $\mathrm{Re}\,(\sqrt{\ldots.}) < 0$ for $\mathrm{Im}\,(\omega) > \tilde{\varepsilon}/2$.

Physically Cut 1 implies that for an arbitrary complex frequency of the oscillator vibrations, i.e., for harmonic, unstable and attenuating vibrations, the string displacement gets smaller as the distance from the oscillator increases. In contrast, Cut 2 allows the string displacement to grow for attenuating vibrations with the decrement larger than $\tilde{\varepsilon}/2$. It should not be considered as strange that the string displacement can grow with the distance from the oscillator. It is due to the fact that the attenuating eigenvibrations are substantially a transient process. In the transient process a wave pattern in the string exists only between the fronts of waves propagating from the oscillator. Therefore, the string displacement can grow in space with the distance from the oscillator, but only till the fronts of waves (not till infinity). Along with this spatial grow the vibrations are attenuated in time and the total energy kept in the string increases in time.

There are three qualitatively different possible locations of the eigenfrequencies in the complex domain: 1. The lower half-plane $\mathrm{Im}\,(\omega) < 0$. Vibrations with such frequencies would be unstable. However, in the considered case of the sub-critical motion $V < c$ the instability cannot occur; see reference [9]. 2. The band $0 < \mathrm{Im}\,(\omega) < \tilde{\varepsilon}/2$. In this band the oscillator vibrates with decreasing amplitude. The string displacement decreases with the distance from the oscillator. 3. The half-plane $\mathrm{Im}\,(\omega) > \tilde{\varepsilon}/2$. The amplitude of the oscillator vibration decreases in this domain as well. The string behaviour is, however, not uniquely determined. According to Cut 1 the string displacement decreases with the distance from the oscillator. In contrast, Cut 2 implies that this displacement grows.

The complex solutions of equation (14) can be found in the same way as the real ones. After squaring equation (15), one finds the roots of a polynom of the eighth order numerically. Subsequently it has to be checked, whether the roots fulfil the conditions for the square root given above.

Performing this procedure, one can find out that the cuts lead to different solutions. Choosing Cut 1, only the real eigenfrequencies depicted in Figures 4(a)–(c) are obtained. Due to the viscosity in the string foundation, they have now

TABLE 1

	μ (kg/m)	N (kN)	γ (N/m²)	β (Ns/m²)	m (kg)	M (kg)	k (N/m)
Parameter set 1	1	0·1	100	0·01	40	20	500
Parameter set 2	1	10	100	0·01	2	20	500

a small imaginary part $\mathrm{Im}(\omega) < \tilde{\varepsilon}/2$. In the half-plane $\mathrm{Im}(\omega) > \tilde{\varepsilon}/2$ no eigenfrequencies occur.

In contrast to Cut 1, Cut 2 gives not only the same solutions as Cut 1, but also additional ones, which lie in the half-plane $\mathrm{Im}(\omega) > \tilde{\varepsilon}/2$. For all possible cases the total number of eigenfrequencies is the same, namely five. One of these eigenfrequencies is pure imaginary. It is not investigated in detail since it cannot lead to resonance. The other four eigenfrequencies are complex. They are situated symmetrically with respect to the imaginary axis. Thus, in the case when Cut 1 gives only one eigenfrequency with positive real part, Cut 2 leads to an additional one. Dependent on the system parameters, this additional eigenfrequency is located in different areas of the half plane $\mathrm{Im}(\omega) > \tilde{\varepsilon}/2$. We will investigate this additional eigenfrequency for two different sets of system parameters, which are given in Table 1.

The first set corresponds to the relationship of $h > \omega^*$ depicted in Figure 4(c). The situation is redrawn in Figure 6, where both, real and imaginary part of the eigenfrequencies, are shown as a function of oscillator velocity V.

The real eigenfrequencies shown in Figure 4(c) exist for Cut 1 and Cut 2. They are depicted in Figure 6 by continuous lines. As already mentioned, they have a small positive imaginary part $\mathrm{Im}(\omega_i) < \tilde{\varepsilon}/2$ because of the small viscosity in the string foundation and correspond to the string vibrations, decreasing with the distance from the oscillator. The interesting additional eigenfrequency occurs in the velocity range $V > V^*$ for the Cut 2. It is depicted in Figure 6 by the dashed

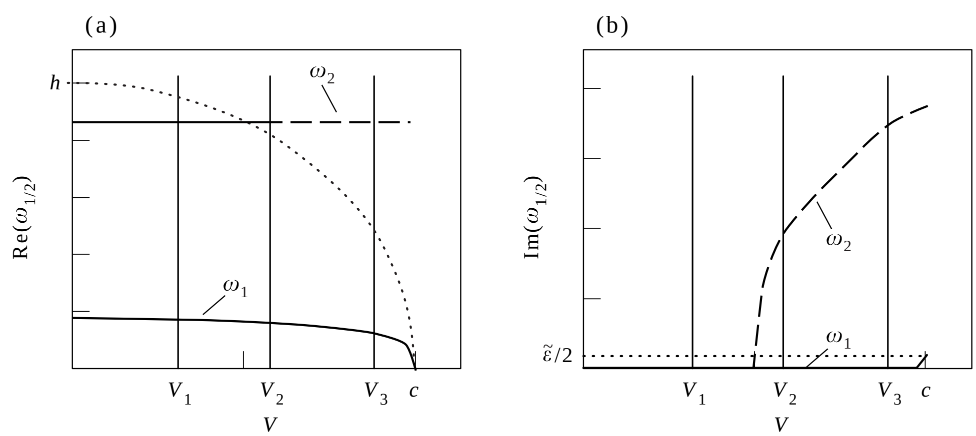

Figure 6. Real part and imaginary part of eigenfrequencies, which exist for Cut 1 and Cut 2 (continuous line) and eigenfrequencies, which only occur with Cut 2 (dashed line). For system parameters, see set 1 in Table 1 resulting in $h = 10$ rad/s; $c = 10$ m/s; $\tilde{\varepsilon} = 0.01\ \mathrm{s}^{-1}$. The system parameters fulfil the inequality $\omega^* < h$; see Figure 4(c).

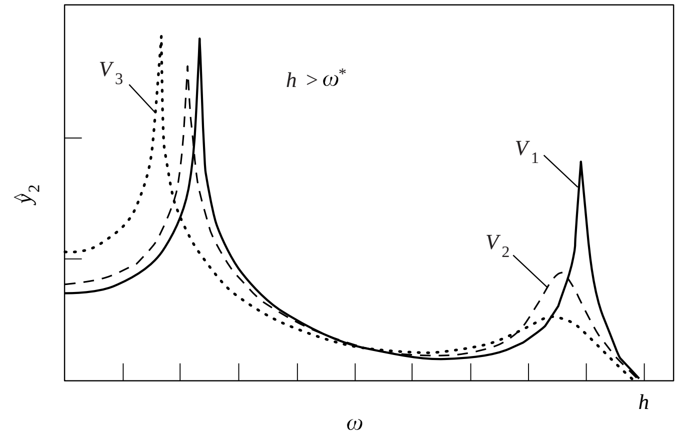

Figure 7. Amplitude of vibration of the lower mass m as function of frequency ω for three different oscillator velocities: $V = 0{\cdot}3 \cdot c$ (bold continuous line); $V = 0{\cdot}6 \cdot c$ (dashed line); $V = 0{\cdot}9 \cdot c$ (thin dotted line). For system parameters, see set 1 in Table 1 resulting in $h = 10$ rad/s; $c = 10$ m/s. The amplitude is given in logarithmic scale.

line. Since its real part is always larger than h^*, this eigenfrequency is related to an eigenmotion with wave radiation in the string. Due to these waves there exists an energy flow away from the oscillator. Therefore, oscillator vibrations attenuate in time and the eigenfrequency has an imaginary part. With growing oscillator velocity this energy flow becomes higher. Consequently, the imaginary part of the eigenfrequency increases as well.

Now the question arises, whether this not uniquely determined second eigenfrequency leads to resonance. To answer this question, we investigate forced vibrations of the oscillator. Suppose, that a harmonically varying vertical force of frequency ω acts on the upper mass M. Figure 7 shows the amplitude of steady state vibrations of the lower mass m as a function of frequency for three different oscillator velocities. These velocities are drawn in Figure 6 as vertical lines.

It is seen from the figure, that with $V = V_1$ (at this velocity two real eigenfrequencies exist for zero viscosity) the frequency response function is qualitatively the same as for a usual two-mass oscillator on a spring of constant stiffness. Both eigenfrequencies lead to strong resonances. Increasing V up to $V = V_2$ and further to $V = V_3$ (at these frequencies Cut 1 gives only one eigenfrequency), one still observes a resonance peak at the additional eigenfrequency. However, this peak decreases as the velocity grows. This is due to the powerful radiation at higher velocities, which leads to the larger imaginary part of the eigenfrequency; see Figure 6.

Now we come to the second parameter set, which corresponds to Figures 4(a) and (b). Here, Cut 2 leads to an additional eigenfrequency (with positive real part) which appears in the whole velocity range $V \geqslant 0$. The situation is depicted in Figure 8.

It is seen from the figure that for low velocities the real part of the additional eigenfrequency (dashed line) is smaller than h. The imaginary part starts with small values, which increase in the higher velocity range.

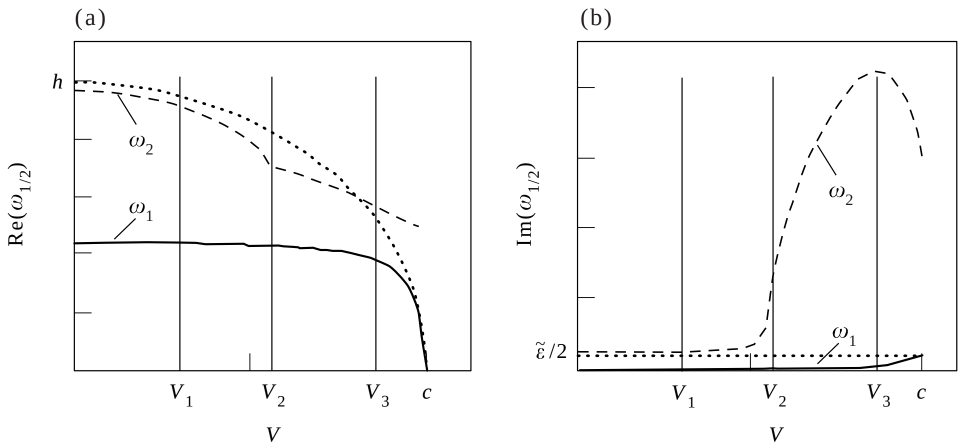

Figure 8. Real part and imaginary part of eigenfrequencies, which exist for Cut 1 and Cut 2 (continuous line) and eigenfrequencies, which only occur with Cut 2 (dashed line). For system parameters, see set 2 in Table 1 resulting in $h = 10$ rad/s; $c = 100$ m/s; $\tilde{\varepsilon} = 0{\cdot}01$ s^{-1}. The system parameters fulfil the inequality $\omega^* > h_1$; see Figures 4(a) and (b).

The effect of this additional eigenfrequency on the resonances of the oscillator is investigated in the same way as for the first system parameter set. The frequency response functions for the three different velocities are shown in Figure 9.

An interesting result is related to the solid line depicted in this figure ($V = 0{\cdot}3 \cdot c$). Namely, to the high-frequency resonance which is almost absent. It seems to be against the expectations, since the higher eigenfrequency in this case is almost real; see Figure 8. To understand this phenomenon one has to remember that we deal with a system which consists of both a concentrated (oscillator) and a continuous (string) part. In such systems the condition of resonance should be extended: not only the eigenfrequency has to be close to the excitation frequency, but the corresponding patterns of the forced and eigenvibrations should be close.

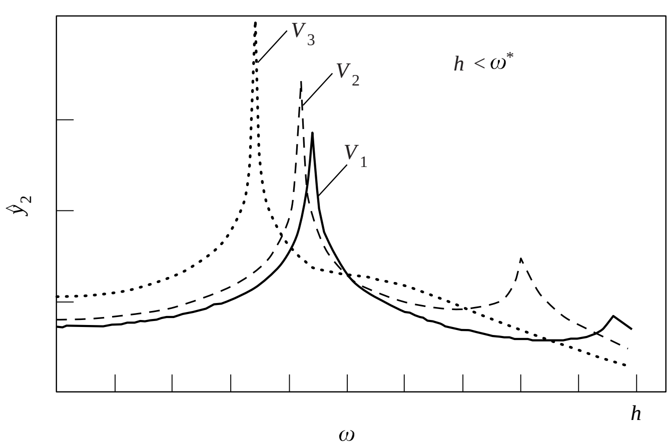

Figure 9. Amplitude of vibration of the lower mass m as function of frequency ω for three different oscillator velocities: $V = 0{\cdot}3 \cdot c$ (bold continuous line); $V = 0{\cdot}6 \cdot c$ (dashed line); $V = 0{\cdot}9 \cdot c$ (thin dotted line). For system parameters, see set 2 in Table 1 resulting in $h = 10$ rad/s; $c = 100$ m/s. The amplitude is given in logarithmic scale.

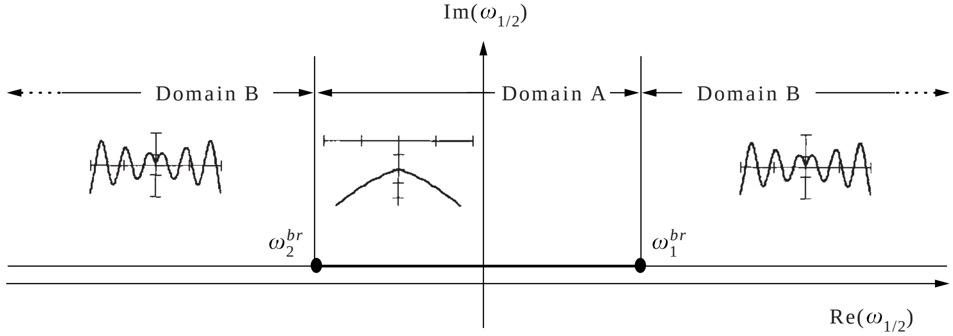

Figure 10. Domains of possible location of the additional eigenfrequency. Diagrams show corresponding patterns of the string.

In the discussed case this is impossible. Indeed, the external harmonically varying load, applied to the oscillator, can only generate a displacement field in the string, which *decrease* as the distance from the load increases. In contrast, the higher eigenfrequency depicted in Figure 8 corresponds to a string displacement with *growing* amplitude; see the diagram depicted in Figure 10, domain A. Therefore, although the frequency of excitation can be close to the eigenfrequency, the load will never excite the string pattern corresponding to this eigenfrequency. Thus, eigenfrequencies, located in domain A (Figure 10), do not lead to resonance even if they have a very small imaginary part. This is due to quite a fast grow of the corresponding string displacment making the forced and the eigenvibrations of the string completely different. Note, that the string displacement grows faster as the real part of the eigenfrequency gets smaller. Eigenfrequencies with a small imaginary part, situated in domain B, can give a limited resonance since they are related to a very slow growing wave pattern in the string, which can be similar to the force-induced shape.

4. CONCLUSIONS

The eigenfrequencies of a two-mass oscillator moving uniformly along a string on a visco-elastic foundation have been analysed. First the case of the purely elastic foundation has been investigated showing that the oscillator has either one or two real positive eigenfrequencies depending on the system parameters. When the oscillator has only one real eigenfrequency, the second one does not exist due to radiation of waves in the high frequency range.

Complex eigenfrequencies of the oscillator have been further studied taking into account a small viscosity of the foundation. It has been shown that eigenfrequencies related to attenuating vibrations of the oscillator are not uniquely determined. Mathematically, this is due to the characteristic equation, containing a radical. This radical can be extended into the complex frequency domain by different branch cuts, which lead to different eigenfrequencies. Physically it can be understood as follows. The considered system consists of two parts: the oscillator (concentrated system) and the string (distributed system). Consequently, the spectrum of such a system should contain a discrete part and a continuous one. The discrete part of the spectrum is formed by solutions of the characteristic equation (the eigenfrequencies) and the continuous part is related to the branch cuts. The non-uniqueness of the eigenfrequencies means that the division of the spectrum into the discrete and the continuous part is not unique. Any division of the spectrum provided in the frequency domain will, however, lead to the same unique expression for the steady state displacement in the time domain. It is important to emphasise again that only eigenfrequencies related to attenuating vibrations of the oscillator are not uniquely determined. Eigenfrequencies, describing harmonic or unstable vibrations, are unique.

It has been found that the existence of an eigenfrequency $\omega = \omega_0 + i\delta$ with a small imaginary part $\delta \ll \omega_0$ is not a sufficient condition for resonance under an external force $P \exp(i\omega_0 t)$. This is still the necessary condition, but additionally the shape of the string, corresponding to the eigenvibration with this eigenfrequency, should be similar to that which occurs under the harmonic excitation. The more similar both displacement fields are, the higher are the resonance amplitudes.

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