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METHOD OF MULTIPLE SCALES FOR VIBRATION ANALYSIS OF ROTOR–SHAFT SYSTEMS WITH NON-LINEAR BEARING PEDESTAL MODEL

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The method of multiple scales is developed to analyze the free and forced vibration of non-linear rotor-bearing systems. The rotating shaft is described by the Timoshenko beam theory which considers the effect of the rotary inertia and shear deformation. A non-linear bearing pedestal model is assumed which has a non-linear spring and linear damping characteristics. Numerical simulations are carried out to illustrate the non-linear effect on the free and forced vibrations of the system. It is shown that for free vibrations, the amplitude has a one-to-one relationship with the non-linear natural frequency. For steady-state response, however, multi-valued displacements occur, indicating the existence of bifurcation points in the system.

1. INTRODUCTION

Bearings in rotor–shaft systems possessing non-linear behavior have received considerable attention. The non-linearity in ball bearings is due to Coulomb friction and the angular clearance between the roller and the ring. Yamamoto *et al*. [1] showed that the non-linear force in single-row deep groove ball bearing is the third power of deflection. Ishida *et al*. [2] obtained the bearing force being the fourth power of deflection in a double-row angular contact ball bearing.

The existence of the non-linear bearing forces makes the dynamic analysis of such rotor-bearing systems complicated. Yamamoto *et al*. [1, 3, 4] used the harmonic balance method to study the subharmonic and superharmonic vibrations of a two-degree-of-freedom rotor mounted on non-linear bearings. For the same rotor system, Ishida *et al*. [2] theoretically and experimentally discussed non-linear forced oscillation caused by quartic non-linearity in angular contact ball bearings. They showed a good agreement between analytical results and experimental results. Using the transfer matrix method in conjunction with the harmonic balance method, Lee *et al*. [5, 6] performed the steady-state analysis of a shaft–rotor system supported by linear and power non-linear bearing. In all the work mentioned above, only forced vibration analysis for steady-state response was performed. Free vibration analysis to calculate non-linear natural frequencies has not been reported for non-linear rotor-bearing systems.

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Dowell [7] examined the free vibration of a simply supported beam with an attached non-linear spring–mass system by using component modal analysis. Pakdemirli and Nayfeh [8] extended the work of Dowell [7] by including stretching, damping and an external primary resonance excitation using the method of multiple scales and the time-averaged Lagrangian method.

In this paper, the method of multiple scales is adopted for free vibration analysis and forced vibration analysis of shaft–rotor systems with a non-linear bearing pedestal model. The shaft is modelled based on the Timoshenko beam theory. A typical roller bearing model is assumed, which has cubic non-linear spring and linear damping characteristics. Non-linear natural frequency response and steady-state response are obtained using the third order perturbation expansion. A typical non-linear rotor bearing system is simulated to show the effectiveness of the analysis method and to illustrate the non-linear effect on the free and forced vibrations of the system.

2. EQUATIONS OF MOTION

Consider a continuous shaft–rotor system mounted on non-linear bearings, as shown in Figure 1. The shaft is modelled by the Timoshenko beam theory so that any shaft, either slender or stubby, can be treated, and the rotor is considered as a rigid disk.

For a continuous shaft depicted in an inertial frame $oxyz$, there are four generalized displacements when considering the shear deformation and rotary inertia. u_x and u_y are the two transverse displacements along the x and y directions and ψ_x, ψ_y are the corresponding bending angles. Introducing the complex variables, $u = u_x + iu_y$, $\psi = \psi_x + i\psi_y$ and the non-dimensional space variable, $\xi = z/l$, the equations of motion of the rotating shaft based on the Timoshenko beam theory are given by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\kappa G}{\rho l^2}\left[l\frac{\partial \psi}{\partial \zeta} - \frac{\partial^2 u}{\partial \zeta^2}\right] = 0, \tag{1}$$

$$\frac{\partial^2 \psi}{\partial t^2} - i\frac{\Omega J_z}{\rho I}\frac{\partial \psi}{\partial t} - \frac{E}{\rho l^2}\frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\kappa A G}{\rho I l}\left(l\psi - \frac{\partial u}{\partial \zeta}\right) = 0, \tag{2}$$

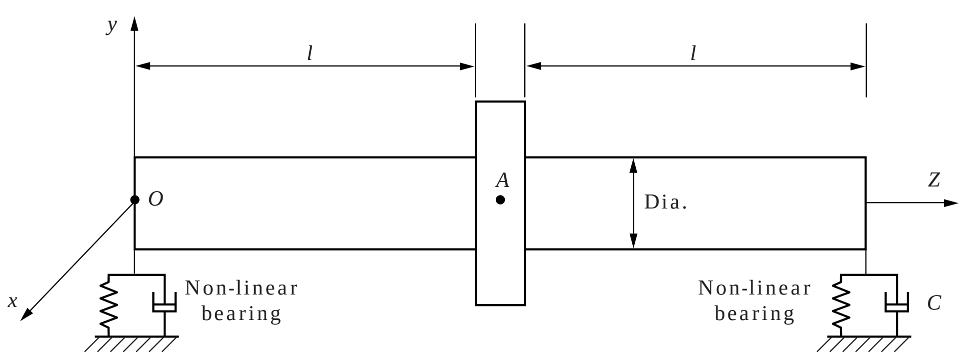

Figure 1. A shaft–rotor system.

where l, A, I and J_z are the length of the beam, cross-sectional area, transverse moment of inertia and polar mass moment of inertia; ρ is the mass density and Ω is the rotating speed of the shaft; E, G and κ are Young's modulus, shear modulus and shear coefficient, respectively. The relationship between the stress resultants and displacements in a complex form can be written as

$$M(\zeta, t) = M_x(\zeta, t) + iM_y(\zeta, t) = EI\psi'(\zeta, t)/l,$$

$$Q(\zeta, t) = Q_x(\zeta, t) + iQ_y(\zeta, t) = \kappa AG\left(\frac{1}{l}u'(\zeta, t) - \psi(\zeta, t)\right), \tag{3}$$

where $M(\zeta, t)$ and $Q(\zeta, t)$ are the transverse bending moment and shear force at each cross-section along the shaft.

The non-linear bearings are assumed to have cubic non-linear and linear damping characteristics. Thus, the bearing force is given as

$$F_x = K_1 u_x + K_3 u_x^3 + C\dot{u}_x, \qquad F_y = K_1 u_y + K_3 u_y^3 + C\dot{u}_y, \tag{4}$$

where K_1 and K_3 are the linear and non-linear stiffness coefficients and C is the linear damping coefficient.

Consider the symmetry, only half of the system is analyzed. The boundary conditions can be written as

$$M(0, t) = 0, \qquad Q(0, t) = F_l(0, t) + F_n(0, t),$$

$$\psi(1, t) = 0, \qquad Q(1, t) = (-M_D\ddot{u}(1, t) + M_D e\Omega^2 \exp(i\Omega t))/2, \tag{5}$$

where M_D is the total mass of the disk and e is the eccentricity of the unbalanced mass of the disk. Note there is no influence from the gyroscopic moment and the moment of inertia of the disk because the disk is located at the symmetric point of the system. F_l and F_n are the linear and non-linear terms in equation (4), respectively. F_l is given by

$$F_l = F_{xl} + iF_{yl} = K_1 u + C\dot{u} \tag{6}$$

and F_n can be written as

$$F_n = K_3(u_x^3 + iu_y^3) = K_3(\tfrac{3}{4}u^2\bar{u} + \tfrac{1}{4}\bar{u}^3) \tag{7}$$

where $\bar{u}$ is the complex conjugate of u. The dimensionless quantities are introduced as follows:

$$t^* = \sqrt{\frac{\kappa G}{\rho l^2}}\, t, \qquad u^* = \frac{u}{l}, \qquad \psi^* = \psi, \qquad \Omega^* = \sqrt{\frac{\kappa G}{\rho l^2}}\, \Omega. \tag{8}$$

Substituting equation (8) into equations (1) and (3), non-dimensional equations of motion can be obtained.

$$\frac{\partial^2 u^*}{\partial t^{*2}} + \left(\frac{\partial \psi^*}{\partial \zeta} - \frac{\partial^2 u^*}{\partial \zeta^2}\right) = 0, \tag{9}$$

$$\frac{\partial^2 \psi^*}{\partial t^{*2}} - i\alpha_1 \Omega^* \frac{\partial \psi^*}{\partial t^*} - \alpha_2 \frac{\partial^2 \psi^*}{\partial \zeta^{*2}} + \alpha_3\left(\psi^* - \frac{\partial u}{\partial \zeta}\right) = 0, \tag{10}$$

where the coefficients α_1, α_2 and α_3 are defined as

$$\alpha_1 = \frac{J_z}{\rho I}, \qquad \alpha_2 = \frac{E}{\kappa G}, \qquad \alpha_3 = \frac{Al^2}{I}. \tag{11}$$

The non-dimensional moment and shear force are given by

$$M^* = \alpha_2 \frac{\partial \psi^*}{\partial \zeta}, \qquad Q^* = \alpha_3\left(\frac{\partial u^*}{\partial \zeta} - \psi^*\right). \tag{12}$$

The relationships between M and M^*, Q and Q^* can be expressed as

$$M = \frac{\kappa G I}{l} M^*, \qquad Q = \frac{\kappa G I}{l^2} Q^*. \tag{13}$$

For a shaft with circular cross-section, $J_z = 2\rho I$. In this case, the coefficient $\alpha_1 = 2$.

Using equations (6) to (8) and equations (11) to (13), the boundary conditions in equation (5) can be rewritten in terms of the non-dimensional variables as

$$M^*(0, t^*) = 0,$$

$$Q^*(0, t^*) = K_1^* u^*(0, t^*) + C^* \frac{\partial u^*(0, t^*)}{\partial t^*} + K_3^*\left(\tfrac{3}{4} u^{*2}(0, t^*)\bar{u}^*(0, t^*) + \tfrac{1}{4}\bar{u}^{*3}(0, t^*)\right),$$

$$Q^*(1, t^*) = \tfrac{1}{2} M_D^* \frac{\partial u^{*2}(1, t^*)}{\partial t^* t^*} + \tfrac{1}{2} F^* \Omega^{*2} \exp(i\Omega^* t^*),$$

$$\psi^*(1, t^*) = 0, \tag{14}$$

where the dimensionless quantities K_1^*, K_3^*, C^*, M_D^* and F^* are defined as

$$K_1^* = \frac{l^3}{\kappa I G} K_1, \qquad K_3^* = \frac{l^5}{\kappa I G} K_3, \qquad C^* = \frac{l^2}{I\sqrt{\kappa G \rho}} C,$$

$$M_D^* = \frac{l}{\rho I} M_D, \qquad F^* = \frac{e M_D}{\rho I}. \tag{15}$$

3. METHOD OF MULTIPLE SCALES

The method of multiple scales is used in this section to solve for the free and forced vibration of the system. The expansions for the displacement u^* and the bending angle ψ^* are assumed in the form

$$u^* = \varepsilon u_1(\zeta, T_0, T_2) + \varepsilon^3 u_3(\zeta, T_0, T_2) + \cdots,$$

$$\psi^* = \varepsilon \psi_1(\zeta, T_0, T_2) + \varepsilon^3 \psi_3(\zeta, T_0, T_2) + \cdots, \tag{16}$$

where u_n and ψ_n ($n = 13, \ldots$) are $O(1)$; ε is a small dimensionless parameter; $T_0 = t^*$ is a fast-time scale characterizing motions occurring at the spin-rate Ω and natural frequency ω_n of the rotor bearing system; $T_2 = \varepsilon^2 t^*$ is a slow-time scale, characterizing the modulation of the amplitude and phase due to damping, non-linearity and possible resonance. In this analysis, only the primary resonance is considered. Hence, the damping, the forcing term and the excitation frequency are assumed as

$$C^* = \varepsilon^2 C_\varepsilon, \qquad F^* = \varepsilon^3 F_\varepsilon, \qquad \Omega^* = \omega_n^* + \varepsilon^2 \sigma, \tag{17}$$

where $\omega_n^* = \omega_n \sqrt{\rho l^2/\kappa G}$ and $\sigma = O(1)$ is a detuning parameter. For linear systems, $\sigma = 0$, thus $\Omega^* = \omega_n^*$ at the primary resonance. For non-linear systems, however, ω_n^* is slightly deviated from Ω^* and this deviation is reflected by the value of σ.

Using the chain rule, the time derivatives in terms of T_0 and T_2 become

$$\frac{\partial}{\partial t^*} = D_0 + \varepsilon^2 D_2 + \cdots, \qquad \frac{\partial^2}{\partial t^{*2}} = D_0^2 + 2\varepsilon^2 D_2 D_0 + \cdots, \tag{18}$$

where $D_n = \partial/\partial T_n$ ($n = 0, 2$). Substituting equations (16) to (18) into equations (9), (10) and (14) and equating coefficients of the same powers of ε, the following equations are obtained.

Order ε:

$$\frac{\partial^2 u_1}{\partial T_0^2} + \frac{\partial \psi_1}{\partial \zeta} - \frac{\partial^2 u_1}{\partial \zeta^2} = 0,$$

$$\frac{\partial^2 \psi_1}{\partial T_0^2} - i\alpha_1 \omega_n^* \frac{\partial \psi_1}{\partial T_0} - \alpha_2 \frac{\partial \psi_1}{\partial \zeta^2} + \alpha_3\left(\psi_1 - \frac{\partial u_1}{\partial \zeta}\right) = 0, \tag{19}$$

$$\left.\begin{aligned} &\psi_1' = 0, \\ &\alpha_3\left(\frac{\partial u_1}{\partial \zeta} - \psi_1\right) - K_1^* u_1 = 0, \end{aligned}\right\} \quad \text{at} \quad \zeta = 0; \tag{20}$$

$$\left.\begin{aligned} &\psi_1 = 0, \\ &\alpha_3\left(\frac{\partial u_1}{\partial \zeta} - \psi_1\right) + \frac{M_D^*}{2} \frac{\partial^2 u_1}{\partial T_0^2} = 0, \end{aligned}\right\} \quad \text{at} \quad \zeta = 1. \tag{21}$$

Order ε^3:

$$\frac{\partial^2 u_3}{\partial T_0^2} + \frac{\partial \psi_3}{\partial \zeta} - \frac{\partial^2 u_3}{\partial \zeta^2} = -2 \frac{\partial^2 u_1}{\partial T_2\, \partial T_0},$$

$$\frac{\partial^2 \psi_3}{\partial T_0^2} - i\alpha_1 \omega_n^* \frac{\partial \psi_3}{\partial T_0} - \alpha_2 \frac{\partial^2 \psi_3}{\partial \zeta^2} + \alpha_3\left(\psi_3 - \frac{\partial u_3}{\partial \zeta}\right)$$

$$= -2 \frac{\partial^2 u_1}{\partial T_2\, \partial T_0} + i\alpha_1\left(\omega_n^* \frac{\partial \psi_1}{\partial T_2} + \sigma \frac{\partial \psi_1}{\partial T_0}\right), \tag{22}$$

$$\left.\begin{aligned}&\psi_3' = 0,\\&\alpha_3\left(\frac{\partial u_3}{\partial \zeta} - \psi_3\right) - K_1^* u_3 = K_3^*\left(\tfrac{3}{4}u_1^2\bar{u}_1 + \tfrac{1}{4}\bar{u}_1^3\right) + C_\varepsilon \frac{\partial u_1}{\partial T_0}0,\end{aligned}\right\} \quad \text{at} \quad \zeta = 0; \tag{23}$$

$$\left.\begin{aligned}&\psi_3 = 0,\\&\alpha_3\left(\frac{\partial u_3}{\partial \zeta} - \psi_3\right) + \frac{M_D^*}{2}\frac{\partial^2 u_3}{\partial T_0^2} = -M_D^* \frac{\partial^2 u_1}{\partial T_0\, \partial T_2} + \tfrac{1}{2}F_\varepsilon \omega_n^{*2}\, \mathrm{e}^{\mathrm{i}\Omega^* t^*},\end{aligned}\right\} \quad \text{at} \quad \zeta = 1. \tag{24}$$

The equations and boundary conditions are linear at order ε, hence, a solution of the form

$$u_1 = A(T_2)\,\mathrm{e}^{i\omega_n^* T_0} Y_u(\zeta), \qquad \psi_1 = A(T_2)\,\mathrm{e}^{i\omega_n^* T_0} Y_\psi(\zeta) \tag{25}$$

can be assumed. Substituting equation (25) into equation (19) and decoupling the subsequent set of ordinary differential equations, the following equations are obtained:

$$L_2 \frac{\mathrm{d}^4 Y_u}{\mathrm{d}\zeta^4} + L_1 \frac{\mathrm{d}^2 Y_u}{\mathrm{d}\zeta^2} + L_0 Y_u = 0, \qquad L_2 \frac{\mathrm{d}^4 Y_\psi}{\mathrm{d}\zeta^4} + L_1 \frac{\mathrm{d}^2 Y_\psi}{\mathrm{d}\zeta^2} + L_0 Y_\psi = 0. \tag{26}$$

The coefficients in equation (26) are given by

$$L_0 = \omega_n^{*4}(1 - \alpha_1) - \alpha_3 \omega_n^{*2}, \qquad L_1 = (1 + \alpha_2 - \alpha_1)\omega_n^{*2}, \qquad L_2 = \alpha_2. \tag{27}$$

The solution to equation (26), when $\sqrt{L_1^2 - 4L_2L_1} > L_1$, is

$$\begin{aligned}Y_u(\zeta) &= A_1 \cosh(s_1\zeta) + A_2 \sinh(s_1\zeta) + A_3 \cos(s_1\zeta) + A_4 \sin(s_1\zeta),\\ Y_\psi(\zeta) &= A_1' \sinh(s_1\zeta) + A_2' \cosh(s_1\zeta) + A_3' \sin(s_1\zeta) + A_4' \cos(s_1\zeta),\end{aligned} \tag{28}$$

where

$$s_1 = \sqrt{\frac{-L_1 + \sqrt{L_1^2 - 4L_2L_1}}{2L_2}}, \qquad s_2 = \sqrt{\frac{L_1 + \sqrt{L_1^2 - 4L_2L_1}}{2L_2}}. \tag{29}$$

$A_1 \sim A_4$ and $A_1' \sim A_4'$ in equation (28) are arbitrary complex constants. Of the eight constants, only four of them are independent. The relationship between $A_1' \sim A_4'$ and $A_1 \sim A_4$ can be obtained from either one of the equations in equation (19) as

$$A_1' = C_1 A_1, \qquad A_2' = C_1 A_2, \qquad A_3' = C_2 A_3, \qquad A_4' = -C_2 A_4, \tag{30}$$

where

$$C_1 = \frac{1}{s_1}(\omega_n^{*2} + s_1^2), \qquad C_2 = \frac{1}{s_1}(\omega_n^{*2} - s_1^2). \tag{31}$$

Substituting equations (25) and (28) into the boundary conditions, equations (20) and (21), the eigenmatrix **B** for solving the eigenvalue ω_n^* and the eigenconstants $A_1 \sim A_4$ are obtained as

$$
\mathbf{B} = \begin{bmatrix} -K_1^* & n_1 & -K_1^* & n_2 \\ c_1 s_1 & 0 & c_2 s_2 & 0 \\ c_1 \sinh(s_1) & c_1 \cosh(s_1) & c_2 \sin(s_2) & -c_2 \cos(s_2) \\ n_1 \sinh(s_1) - m \cosh(s_1) & n_1 \cosh(s_1) - m \sinh(s_1) & -n_2 \sin(s_2) - m \cos(s_2) & n_2 \cos(s_2) - m \sinh(s_2) \end{bmatrix}, \quad (32)
$$

where

$$n_1 = \alpha_3(s_1 - c_1), \qquad n_2 = \alpha_3(s_2 - c_2), \qquad m = \frac{M_D^*}{2}\,\omega_n^{*2}. \tag{33}$$

From $|\mathbf{B}| = 0$, the natural frequency ω_n^* can be obtained. Considering that the coefficient function $A(T_2)$ in equation (25) is arbitrary and only three constants in $A_1 \sim A_4$ are independent, the constant A_1 is assumed as 1. $A_2 \sim A_4$ can be derived from the eigenmatrix $\mathbf{B}$ and thus, the normal modes $Y_u(\zeta)$ and $Y_\psi(\zeta)$ in equation (28) are uniquely defined by the coefficients $A_1 \sim A_4$.

The displacements u_3 and ψ_3 in equation (22) can be assumed as

$$u_3(\zeta, t) = \phi_u(\zeta, T_2)\,\mathrm{e}^{i\omega_n^* T_0}, \qquad \psi_3(\zeta, t) = \phi_\psi(\zeta, T_2)\,\mathrm{e}^{i\omega_n^* T_0}. \tag{34}$$

Substituting equation (34) into equations (22)–(24) and equating the coefficient $\exp(i\omega_n^* T_0)$ on both sides of the equations leads to

$$-\omega_n^{*2}\phi_u + \frac{\partial\phi_\psi}{\partial\zeta} - \frac{\partial^2\phi_u}{\partial\zeta^2} = -2i\omega_n^* A'(T_2)Y_u(\zeta),$$

$$(\alpha_1 - 1)\omega_n^{*2}\phi_\psi - \alpha_2\frac{\partial^2\phi_\psi}{\partial\zeta^2} + \alpha_3\left(\phi_\psi - \frac{\partial\phi_u}{\partial\zeta}\right) = -i(2\omega_n^* - \alpha_1\omega_n^*)A'(T_2)Y_\psi(\zeta)$$
$$-\alpha_1\omega_n^*\sigma A'(T_2)Y_\psi(\zeta), \tag{35}$$

$$\left.\begin{aligned} &\phi'_\psi = 0,\\ &\alpha_3\left(\frac{\partial\phi_u}{\partial\zeta} - \phi_\psi\right) - K_1^*\phi_u = \tfrac{3}{4}K_3^*\overline{A}(T_2)A^2(T_2)Y_u^3(\zeta) + C_\varepsilon i\omega_n^* A(T_2)Y_u(\zeta), \end{aligned}\right\} \quad \text{at } \zeta = 0; \tag{36}$$

$$\left.\begin{aligned} &\phi_\psi = 0,\\ &\alpha_3\left(\frac{\partial\phi_u}{\partial\zeta} - \phi_\psi\right) - \frac{M_D^*}{2}\,\omega_n^*\phi_\psi = -M_D^* i\omega_n^* A'(T_2)Y_u(\zeta) - \tfrac{1}{2}F_\varepsilon i\omega_n^{*2}\,\mathrm{e}^{i\varepsilon^2\sigma t^*}, \end{aligned}\right\} \quad \text{at } \zeta = 1. \tag{37}$$

As pointed out by Nayfeh [10] for the homogeneous part of equation (22) to have a non-trivial solution, the inhomogeneous equation (22) has a solution only if a solvability condition is satisfied. The solvability condition demands that the right side of equation (35) be orthogonal to every solution of the homogeneous problem. Thus, the solvability condition can be derived by

$$\int_0^1 [\tilde{u}\alpha_3(-2i\omega_n^* A'(T_2))Y_u(\zeta) + \tilde{\psi}(-i(2\omega_n^* - \alpha_1\omega_n^*))A'(T_2)Y_\psi(\zeta)$$
$$-\alpha_1\omega_n^*\sigma A(T_2)Y_\psi(\zeta)]\,\mathrm{d}\zeta = \tilde{u}(0)(\tfrac{3}{4}K_3^*\overline{A}(T_2)A^2(T_2)Y_u^3(0) + C_\varepsilon i\omega_n^* A(T_2)Y_u(0))$$
$$+\tilde{u}(1)(M_D^* i\omega_n^* A'(T_2))Y_u(1)\,\mathrm{e}^{i\omega_n^* T_0} + \tfrac{1}{2}F_\varepsilon^*\omega_n^*\,\mathrm{e}^{i\sigma T_2}), \tag{38}$$

where $\tilde{u}$ and $\tilde{\psi}$ are the solutions of the adjoint equation of the homogeneous part of equation (35) with the following boundary conditions

$$\left.\begin{aligned}&\tilde{\psi}=0,\\&\alpha_3\left(\frac{\partial\tilde{u}}{\partial\zeta}-\tilde{\psi}\right)-K_1^*\tilde{u}=0,\end{aligned}\right\}\quad \text{at}\quad \zeta=0; \tag{39}$$

$$\left.\begin{aligned}&\tilde{\psi}=0,\\&\alpha_3\left(\frac{\partial\tilde{u}}{\partial\zeta}-\tilde{\psi}\right)+\frac{M_D^*}{2}\frac{\partial^2\tilde{u}}{\partial T_0^2}=0,\end{aligned}\right\}\quad \text{at}\quad \zeta=1. \tag{40}$$

It can be proven that the homogeneous part of equation (35) is a set of self-adjoint equations. Comparing equations (39) and (40) with equations (20) and (21), the solutions for $\tilde{u}$ and $\tilde{\psi}$ should have the same form as u_1 and ψ_1. Thus,

$$\tilde{u}=\tilde{A}(T_2)\,\mathrm{e}^{i\omega_n^*T_0}Y_u(\zeta),\qquad \tilde{\psi}=\tilde{A}(T_2)\,\mathrm{e}^{i\omega_n^*T_0}Y_\psi(\zeta), \tag{41}$$

Substituting equation (41) into equation (38), the solvability condition can be rewritten as

$$\begin{aligned}&-2i\omega_n^*A'(T_2)b_1-\omega_n^*\sigma A(T_2)b_2\\&\quad=\tfrac{3}{4}K_3^*\bar{A}(T_2)A^2(T_2)Y_u^4(0)+C_\varepsilon i\omega_n^*A(T_2)Y_u^2(0)+M_D^*i\omega_n^*A'(T_2)Y_u^2(1)\,\mathrm{e}^{i\omega_n^*T_0}\\&\quad\quad+\tfrac{1}{2}F_\varepsilon^*\omega_n^*\,\mathrm{e}^{i\sigma T_2}Y_u(1),\end{aligned} \tag{42}$$

where

$$b_1=\alpha_3\int_0^1Y_u^2(\zeta)\,\mathrm{d}\zeta+\int_0^1\left(1-\frac{\alpha_1}{2}Y_\psi^2(\zeta)\right)\mathrm{d}\zeta,\qquad b_2=\alpha_1\int_0^1Y_\psi^2(\zeta)\,\mathrm{d}\zeta. \tag{43}$$

Express A in a polar form

$$A=\tfrac{1}{2}a(T_2)\,\mathrm{e}^{\mathrm{i}\theta(T_2)}, \tag{44}$$

where $a(T_2)$ and $\theta(T_2)$ represent the amplitude and phase angle of the response, respectively. Substituting equation (44) into equation (42) and separating the real and imaginary part, the modulation can be written as

$$\begin{aligned}\omega_n^*b_3a'&=-\frac{C_\varepsilon}{2}\omega_n^*aY_u^2(0)-\tfrac{1}{2}F_\varepsilon\omega_n^{*2}\sin(\gamma)Y_u(1),\\\omega_n^*b_3a\gamma'&=b_4\omega_n^*a\sigma-\tfrac{3}{32}K_3^*a^3Y_u^4(0)-\tfrac{1}{2}F_\varepsilon\omega_n^{*2}\cos(\gamma)Y_u(1),\end{aligned} \tag{45}$$

where

$$b_3=b_1+\frac{M_D^*}{2}Y_u^2(1),\qquad b_4=b_3-b_2/2,\qquad \gamma=\sigma T_2-\theta. \tag{46}$$

Non-linear frequencies can be calculated from equation (45) by considering free undamped vibrations. Letting dimensionless quantities C_ε, F_ε defined in equation (17) be zero in equation (45) leads to

$$\omega_n^* b_3 a' = 0, \qquad \omega_n^* b_3 a\theta' = \tfrac{1}{2}\omega_n^* \sigma b_2 + \tfrac{3}{32} K_3^* a^3 Y_u^4(0). \tag{47, 48}$$

It is clear from equation (47) that a is a constant. It is understood that for undamped free vibration, the amplitude remains constant. The angular derivative θ' can be calculated from equation (48). Adding θ' and the linear natural frequency ω_n^*, the non-linear natural frequency ω_{nt}^* can be obtained:

$$\omega_{nt}^* = \omega_n^* + \varepsilon^2\theta' = \omega_n^* \sigma b_2 + (\tfrac{1}{2}\omega_n^* \sigma b_2 + \tfrac{3}{32} K_3^* a^2 Y_u^4(0))\varepsilon^2. \tag{49}$$

Note that ω_{nt}^* depends on the amplitude of the response

The forced vibration of the system can be obtained by including the damping and forcing-terms in equation (45). For steady-state response with periodic motion, a' and γ' should be equal to zero. Thus, the detuning parameter σ can be solved in terms of a by eliminating γ from both equations in equation (45):

$$\sigma = \frac{3a}{32\omega_n^* b_4} K_3^* Y_u^4(0) \pm \frac{1}{2ab_4}\sqrt{F_\varepsilon^2 \omega_n^{*2} Y_u^2(1) - C_\varepsilon^2 a^2 Y_u^4(0)}. \tag{50}$$

4. NUMERICAL SIMULATIONS

Numerical simulations of free and forced vibrations of a typical non-linear rotor-bearing system are presented in this section to show the effectiveness of the analysis method. The system consists of a shaft supported by two bearings at two ends and there is an intermediate rotor at the middle, as shown in Figure 1. The half length and diameter of the shaft are chosen as $l = 1$ m and $d = 0{\cdot}2$ m. The dimensionless quantities in equation (12) are assumed as $\alpha_1 = 2$, $\alpha_2 = 4$, $\alpha_3 = 400$. The dimensionless quantities K_1^* and M_D^* in equation (15) are chosen as 100 and 500, respectively.

In Figure 2, the steady-state response at the mid-shaft versus the excitation frequency is shown near the synchronous resonance for $F^* = 1$, $C^* = 100$ and for various K_3^*. For the linear system, as $K_3^* = 0$, the response curve goes up, peaks at the resonance and come down. For non-linear systems, as $K_3^* > 0$, the response curve near the peak is distorted and bends towards the right side. The higher the K_3^* value is, the more bending the curve displays. This curve bending causes multiple valued displacements at the same excitation frequency. In other words, there exists a jump corresponding to the multi-valued phenomenon, indicating the bifurcation in the system. This is expected for non-linear systems.

The loci of the first and the second saddle-node bifurcation points of the frequency-response curve are shown in Figures 3 and 4, respectively. Different values of F^* are chosen representing various excitation levels. The detuning frequency $\Delta\omega_{nt}$ in Figures 3 and 4 denotes the difference between the non-linear natural frequency ω_{nt} and the linear natural frequency ω_n at the saddle-node

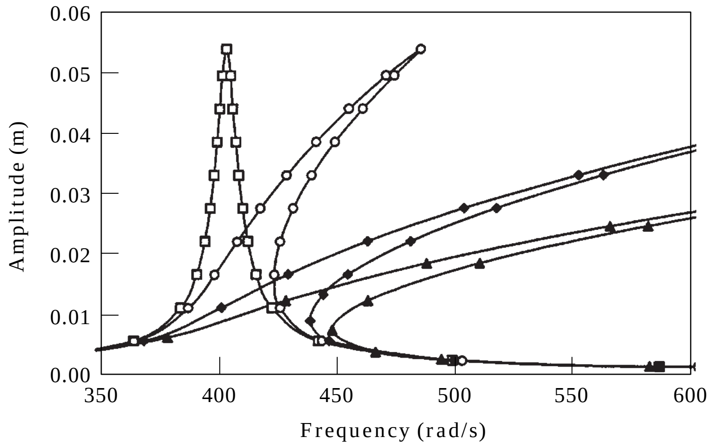

Figure 2. Frequency response curves for various K_3^*: □, $K_3^* = 0$; ○, $K_3^* = 1{\cdot}0\text{e}7$; ◆, $K_3^* = 5{\cdot}0\text{e}7$; ▲, $K_3^* = 1{\cdot}0\text{e}8$.

bifurcation point. The non-linear stiffness coefficient K_3^* is chosen as $1{\cdot}0 \times 10^7$. It is seen from Figures 3 and 4 that the detuning frequency $\Delta\omega_{nt}$ increases as the excitation force F^* increases for both the first and second saddle-node bifurcation points. However, the increase in $\Delta\omega_{nt}$ gets larger as F^* increases for the first saddle-node bifurcation point, while the increase in $\Delta\omega_{nt}$ gets smaller as F^* increases for the second saddle-node bifurcation point.

5. SUMMARY

The free and forced vibration analysis of shaft–rotor systems with non-linear bearings are performed analytically in the paper. The shaft is described by the Timoshenko beam model and the rotor is considered as a rigid disk. The bearings possess linear stiffness and damping and cubic non-linear stiffness. The method of multiple scales is used to determine the non-linear natural frequency and

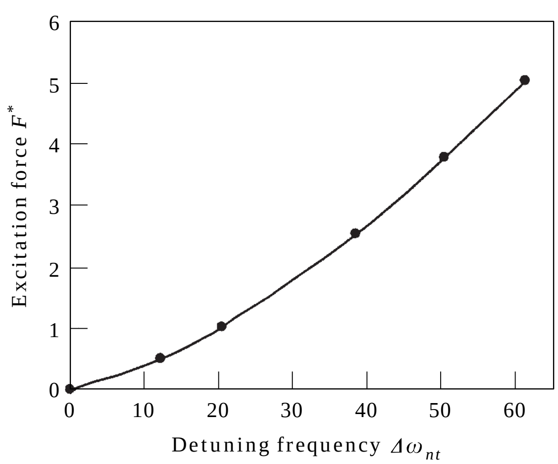

Figure 3. Loci of the first saddle-node bifurcation point.

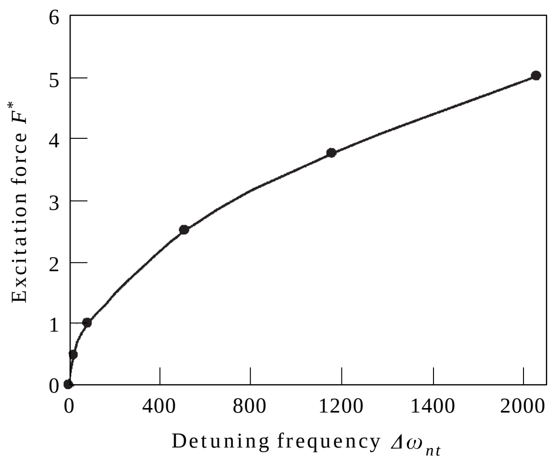

Figure 4. Loci of the second saddle-node bifurcation point.

steady-state response. A typical non-linear rotor bearing system is simulated to illustrate the non-linear effect on the free and forced vibrations of the system. It is shown that for free vibrations, the amplitude has a one-to-one relationship with the non-linear natural frequency. For steady-state response, multi-valued displacements occur, indicating the existence of bifurcation points in the systems.

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