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VAN DER POL'S OSCILLATOR UNDER DELAYED FEEDBACK

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The effect of delayed feedback on oscillatory behaviour is investigated for the classical van der Pol oscillator. It is shown how the presence of delay can change the amplitude of limit cycle oscillations, or suppress them altogether. The result is compared to the conventional proportional-and-derivative feedback. The derivative-like effect of delay is also demonstrated in a modified equation where a delayed term provides the damping.

1. INTRODUCTION

One of the classical equations of non-linear dynamics was formulated by and bears the name of the Dutch physicist van der Pol [1]. Originally it was a model for an electrical circuit with a triode valve, and was later extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasiperiodicity, elementary bifurcations, and chaos [2–4]. Nevertheless, it is perhaps best known as a prototype system exhibiting limit cycle oscillations. This celebrated equation has a non-linear damping term,

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = f(t), \qquad x \in \mathbf{R},\ \varepsilon > 0, \tag{1}$$

which is responsible for energy generation at low amplitudes and dissipation at high amplitudes. The unforced case ($f \equiv 0$) is an equation of Liénard type, and thus can be shown to have a unique periodic solution which attracts all other orbits except the origin, which is an unstable equilibrium point [4]. Limit cycle oscillations with such strong stability properties are important in applications; hence, being able to modify their behavior through feedback is a question of interest. On the other hand, most practical implementations of feedback have inherent delays, the presence of which results in an infinite-dimensional system, thus complicating the analysis. In this study averaging methods are used to investigate the behavior of the limit cycle of equation (1) when the forcing f is a delayed feedback of the position x.

Since the limit cycle disappears when ε is zero, it is convenient to scale the parameters by ε. Hence, equation (1) will be considered with

$$f(t) = \varepsilon k x(t - \tau), \tag{2}$$

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where τ is a positive quantity representing the delay and k is the feedback gain. It is shown in section 2 that both the amplitude and frequency of the oscillations can be modified by changing the delay and the gain. In particular, it is possible to reduce the amplitude to zero, thereby preventing oscillations and stabilizing the origin. To gain more insight into the mechanism of delayed feedback, the results are compared to the conventional feedback of the state co-ordinates $x(t)$ and $\dot{x}(t)$. While derivative feedback can change the amplitude of oscillations (but not the frequency), position feedback without delay affects only the frequency. Hence, if the derivative is not available for measurement, delayed feedback of position can be used to modify the amplitude of oscillations or to stabilize the equilibrium solution. The derivative-like effect of the delay is further illustrated in section 3 for a modified van der Pol oscillator where delayed position is used to provide the damping that leads to self-sustained oscillations.

2. DELAY IN FEEDBACK

In what follows, delayed quantities are denoted with the subscript τ, e.g., $x_\tau = x(t - \tau)$. Thus, the van der Pol oscillator (1) under delayed feedback (2) is written in the form

$$\ddot{x} + x = \varepsilon g(x, \dot{x}, x_\tau), \tag{3}$$

where

$$g(x, \dot{x}, x_\tau) = (1 - x^2)\dot{x} + kx_\tau. \tag{4}$$

For small values of ε, equation (3) can be viewed as a perturbation of the harmonic oscillator, and can be analyzed by the method of averaging for delay-differential equations [5]. The nature of oscillations is given by the following result.

Proposition 1. *Suppose* $k \sin \tau < 1$. *Then for each sufficiently small and positive value of* ε, (3) *has an attracting periodic solution given by*

$$x(t) = 2\sqrt{1 - k \sin \tau} \cos \left(1 - \frac{\varepsilon}{2} k \cos \tau\right)t + \mathcal{O}(\varepsilon^2),$$

while the zero solution is unstable. If $k \sin \tau > 1$, *then the zero solution is stable.*

Proof. To prove the proposition, an amplitude-phase transformation is introduced

$$x(t) = r(t) \cos (t + \theta(t)), \qquad \dot{x}(t) = -r(t) \sin (t + \theta(t)), \tag{5}$$

so that equation (3) becomes

$$\begin{aligned} \dot{r} &= -\varepsilon \sin (t + \theta)g(r \cos (t + \theta), -r \sin (t + \theta), r_\tau \cos (t - \tau + \theta_\tau)), \\ r\dot{\theta} &= -\varepsilon \cos (t + \theta)g(r \cos (t + \theta), -r \sin (t + \theta), r_\tau \cos (t - \tau + \theta_\tau)). \end{aligned} \tag{6}$$

In the classical method of averaging, one replaces the right sides of this system of equations with their time averages on an interval of 2π, while treating r and

θ as constants on this interval since they are slowly varying. But since they are slowly changing, it also makes sense to take $r(t-\tau) \approx r(t)$ and $\theta(t-\tau) \approx \theta(t)$. This heuristic argument can be made rigorous by an application of the averaging method developed for delay-differential equations [5]. Hence averaging equations (6) with g given by equation (4) leads to the pair of decoupled ordinary differential equations

$$\dot{r} = -\frac{\varepsilon}{2} r\left(\frac{r^2}{4} - 1 + k \sin \tau\right), \qquad \dot{\theta} = -\frac{\varepsilon}{2} k \cos \tau. \tag{7}$$

The equation for r has an equilibrium point at the origin, with eigenvalue $(1 - k \sin \tau)\varepsilon/2$. There is another equilibrium at $r = 2\sqrt{1 - k \sin \tau}$, provided that the quantity $(1 - k \sin \tau)$ is positive, with the corresponding eigenvalue $-\varepsilon(1 - k \sin \tau)$. Hence, this equilibrium is stable if it exists. Applying Theorem 3.3 of reference [5] now concludes the proof of the proposition. ■

Of course when $k = 0$ the theorem recovers the familiar limit cycle $2 \cos t$ of the unforced van der Pol oscillator. By modifying k, the amplitude of the oscillations can be set arbitrarily, provided $\sin \tau \neq 0$. In particular, the limit cycle oscillations can be prevented by choosing $k \sin \tau > 1$. Examples are given in Figure 1, where the limit cycle has an amplitude of 3, and in Figure 2, where the limit cycle is annihilated and the zero solution is stable. The numerical calculations are done with a fourth order Runge–Kutta integrator adapted for delay equations, and they

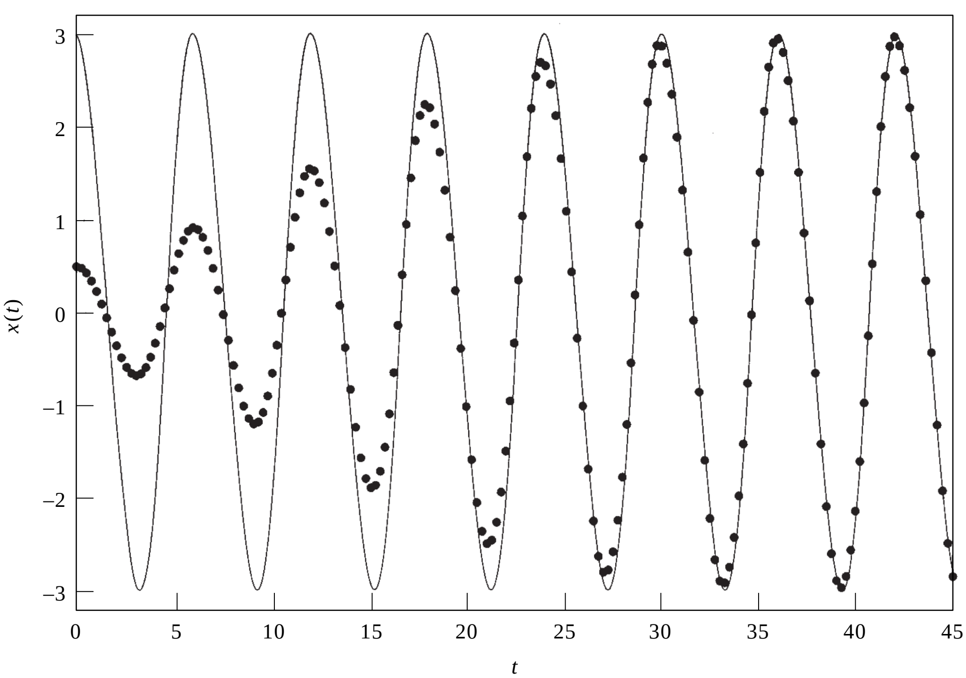

Figure 1. The convergence of the solutions to the attracting limit cycle. The solid line is the limit cycle given in Proposition 1, and the dots depict the numerical solution of equation (3) calculated from an arbitrary initial condition. The parameter values are $\tau = 1$, $\varepsilon = 0{\cdot}1$ and $k = -1$.

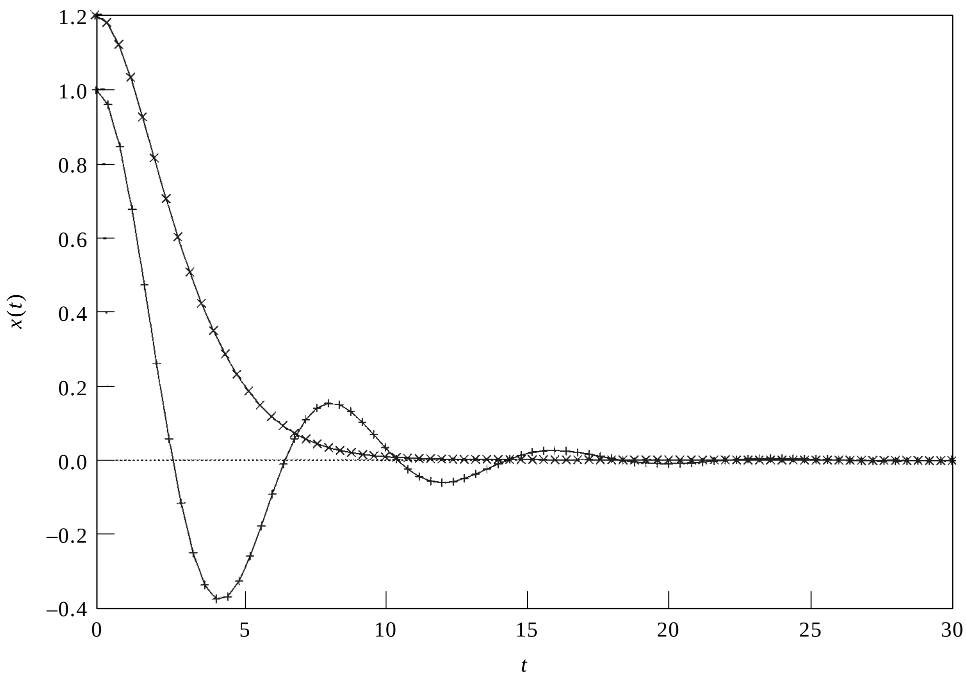

Figure 2. The stability of the zero solution. Numerically computed solutions of equation (3) are shown for $k = 5$ (+) and $k = 7{\cdot}8$ (×). The other parameter values are $\tau = 1$ and $\varepsilon = 0{\cdot}1$. For these values the limit cycle does not exist, and all solutions die down to zero.

are seen to agree with the analytical results given by Proposition 1. The global nature of the results are also worth emphasizing. The limit cycle, when it exists, attracts all trajectories except the origin. Similarly, if the zero solution is stable, all trajectories are attracted by it.

It is instructive to compare this result to the case of conventional state feedback. Thus, letting

$$f(t) = \varepsilon(k_1 x(t) + k_2 \dot{x}(t)) \tag{8}$$

in equation (1), making the change of variables (5), and carrying out the averaging as usual, one obtains the equations

$$\dot{r} = -\frac{\varepsilon}{2} r\left(\frac{r^2}{4} - 1 - k_2\right), \qquad \dot{\theta} = -\frac{\varepsilon}{2} k_1. \tag{9}$$

It is seen that the feedback of the position affects only the frequency, while the feedback of the derivative affects only the amplitude of oscillations (up to second order in ε). In particular, it is not possible to change the amplitude with undelayed position feedback. The presence of the delay allows one to obtain the effects of derivative feedback using only the position x. The comparison of the two types of feedback (2) and (8) can be made precise by equating the right sides of equations (7) and (9). This gives $k_1 = k \cos \tau$ and $k_2 = k \sin \tau$. Hence, the delayed feedback of $x(t - \tau)$ acts like the state-feedback $\cos \tau \cdot x(t) - \sin \tau \cdot \dot{x}(t)$ in modifying the

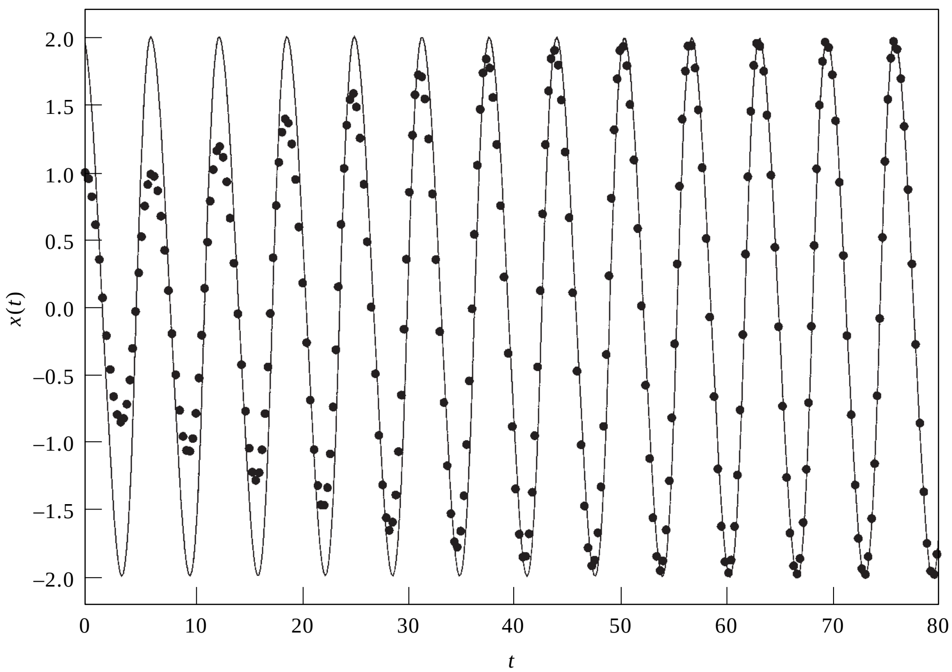

Figure 3. The limit cycle of the modified van der Pol oscillator (10). The solid line is the limit cycle given in Proposition 2, and the dots depict the numerical solution calculated from an arbitrary initial condition. The parameter values are $\varepsilon = 0{\cdot}1$ and $\tau = 4{\cdot}6$, for which the limit cycle is attracting.

properties of the limit cycle. Note that the naïve estimate $x(t-\tau) \approx x(t) - \tau\dot{x}(t)$ based on the Newton quotient for the derivative can lead to totally erroneous conclusions, except for small values of the delay τ.

3. DELAY AS DAMPING

As the delayed position $x(t-\tau)$ is shown to incorporate some characteristics of the derivative $\dot{x}(t)$, one is tempted to explore further the relationship between the two quantities. It is of interest if, for instance, $x(t-\tau)$ can provide the damping in van der Pol's oscillator that results in self-sustained oscillations. Hence, consider the modified equation

$$\ddot{x} + \varepsilon(x^2 - 1)x(t-\tau) + x = 0, \qquad x \in \mathbf{R},\ \varepsilon > 0. \tag{10}$$

The limit cycle oscillations for this case are described by the following proposition.

Proposition 2. *For each sufficiently small positive value of* ε, (10) *has a periodic solution given by*

$$x(t) = 2\cos(1 + \varepsilon\cos\tau)t + \mathcal{O}(\varepsilon^2),$$

which is stable if $\sin\tau < 0$ *and unstable if* $\sin\tau > 0$.

Proof. Equation (10) is again of the form (3) with g given by

$$g(x, \dot{x}, x_\tau) = (1 - x^2)x_\tau. \tag{11}$$

Averaging the amplitude-phase equations (6) with this choice of g results in

$$\dot{r} = \frac{\varepsilon}{2} r \sin \tau \left(\frac{r^2}{4} - 1 \right), \qquad \dot{\theta} = \frac{\varepsilon}{2} \cos \tau \left(\frac{3}{4} r^2 - 1 \right).$$

The first of these equations has an equilibrium at $r = 2$ whose stability is determined by the sign of $\sin \tau$. Integrating the second equation at $r = 2$ leads to the conclusion of the proposition. ■

Thus, the limit cycle oscillations for equation (10) are very similar to those of the classical van der Pol oscillator. The stability of the limit cycle depends on the amount of delay, and a sequence of stability switches occur as τ is increased. The origin is stable whenever the limit cycle is unstable, and *vice versa*.

4. CONCLUSION

It is shown how to modify or quench the limit cycle oscillations in the van der Pol oscillator by using delayed position feedback. Without delay, position feedback cannot change the amplitude of oscillations. The delay thus provides the effect of a derivative feedback in changing the amplitude. This effect is also seen in a modified oscillator, where the damping is achieved by a delayed term. In situations where the derivative is not available for measurement, or its use is not desirable due to high-frequency noise, delayed position can be a viable alternative to conventional state feedback in controlling oscillations. For a discussion in the context of stability of second-order systems, the reader is referred to reference [6].

Delayed feedback has also been used in the control of chaos, where the aim is to stabilize one of an infinite number of periodic solutions embedded in a chaotic attractor [7]. There, the existing solution is not changed, but the other trajectories are led to it by the action of delayed feedback. For this, one needs to have some knowledge of the periodic solutions (or at least their periods) beforehand. Although the principle is simple to understand, the analysis of the resulting closed loop system is difficult and is usually limited to numerical or experimental simulations. (See reference [8] for the extent of the numerical work required.) For instance, too small and too large values of the feedback gain can lead to instability, and these values are not known analytically. The domain of attraction of the stabilized periodic solution is not known either, and the methods rely on the recurrence property of the attractor to enter the domain of attraction. To complicate matters further, the closed loop system can be multistable, and one may end up with a different regular behaviour than the one intended. All this indicates the importance of analytical results concerning non-linear systems controlled by delayed feedback. By illustrating via the specific example of the van der Pol oscillator how the stability of the resulting oscillations is affected by the feedback parameters, this paper is hoped to contribute to the efforts in this area.

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