Journal of Sound and Vibration (1998) **218**(2), 350–360
Article No. sv981812

THE APPLICATION OF FLOQUET METHODS IN THE ANALYSES OF ROTORDYNAMIC SYSTEMS

G. T. Flowers, D. B. Margithu and G. Szasz
Department of Mechanical Engineering, Auburn University, Auburn, AL 36849-5341 *U.S.A.*

(*Received* 16 *September* 1997, *and in final form* 22 *June* 1998)

1. BACKGROUND

Techniques for the analysis and characterization of linear dynamical systems with constant parameters are well established. However, investigating systems with non-linearities and time-varying parametric effects can be a much more difficult undertaking, requiring new methodologies and analysis techniques. An important area in this regard is that of rotating machinery. An important non-linearity results from the rotor interacting with a clearance bearing and there have been a number of important studies concerned with this area. Some of the earliest work was performed by Yamamoto [1], who conducted a systematic study of rotor responses involving bearing clearance effects. Notable studies in this area also include Ehrich [2], who reported the first identification of a second order subharmonic vibration phenomenon in a rotor system associated with bearing clearance and the work of Black [3], which concluded that rotor/stator interactions may occur in a variety of forms and circumstances, including jump phenomena.

Rotordynamic researchers have also investigated other non-linear effects. In particular, Muszynska [4] investigated the non-linear phenomena of oil whirl and oil whip in rotor bearing support systems and concluded that significant stability problems can occur due to the self-excited responses that result. Also, Padovan and Choy [5] investigated the non-linear dynamical behavior excited by blade–casing rub interactions.

A discussion of the dynamics of systems with variable parameters is given by Dimentberg [6]. Analysis strategies for such systems, centered around the use of Floquet and Lyapunov techniques, have been used by a number of researchers. Sinha, in a series of papers [7, 8], has developed a technique for the analysis of dynamical systems governed by equations with time-varying coefficients and non-linear effects. In this regard, a system specific to rotating machinery was considered by Wettergren and Olsson [9], who presented a study of the stability characteristics of an asymmetric rotor with internal damping and supported by anisotropic bearings, using Floquet methods. Childs [10] described a number of non-linear effects that can significantly influence rotordynamical responses. His text serves as an excellent source of information on studies of non-linear rotordynamics. Choi and Noah [11] and Lawen and Flowers [12] investigated the influence of bearing clearances on rotordynamic responses. These papers give additional review material and lists of further references on this topic.

0022–460X/98/470350 + 11 $30.00/0 © 1998 Academic Press

2. INTRODUCTION

Most rotordynamic motions have an inherent periodicity related to the rotor speed. This characteristic lends itself well to study using Fourier analysis, which is the most widely used approach for the vibratory analysis of rotating machinery. While the information obtained is very useful in diagnosing rotordynamic problems, very little insight into the stability characteristics of the system is obtained. Although stable motion is one of the main factors in any successful design, a dynamic stability index for rotordynamic motion has not previously been addressed. There is no commonly accepted quantitative way to judge rotordynamic stability.

Floquet theory, which is commonly used to investigate the stability of periodic systems, offers a possible methodology for quantifying stability in rotordynamic systems. The method proposed below allows for kinematic data collected from experimental and/or simulation codes and captured at specified instants in time to be efficiently analyzed. Complex dynamics under various configurations can be accommodated by this approach. In the following sections, the methodology is described and applied to an example rotor system. This approach is inherently concerned with quantifying the relative stability of systems oscillating about a stable equilibrium orbit. Unstable vibration modes cannot be analyzed with the proposed method.

3. ANALYSIS PROCEDURE

The fundamental aspects of the theory used in the present investigation are first outlined. Then, stability measures to quantify the robustness of periodic patterns subject to perturbations are developed. This development is based on Floquet theory, which is used to investigate the local stability of critical points of periodic motions. In simple terms, it is desired to quantify the robustness of the periodic patterns when subjected to disturbances. The proposed approach is based on an interactive review of phase plane portraits and Poincaré sections at critical transitional points [13].

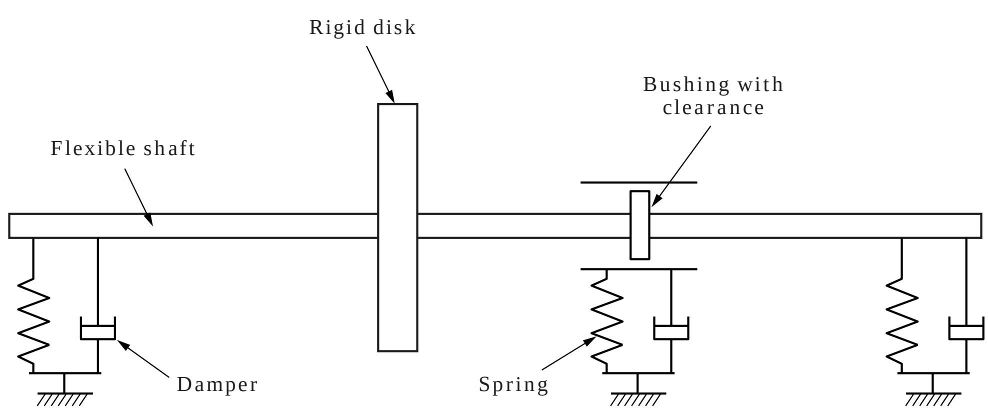

Figure 1. Schematic diagram of simplified rotor model.

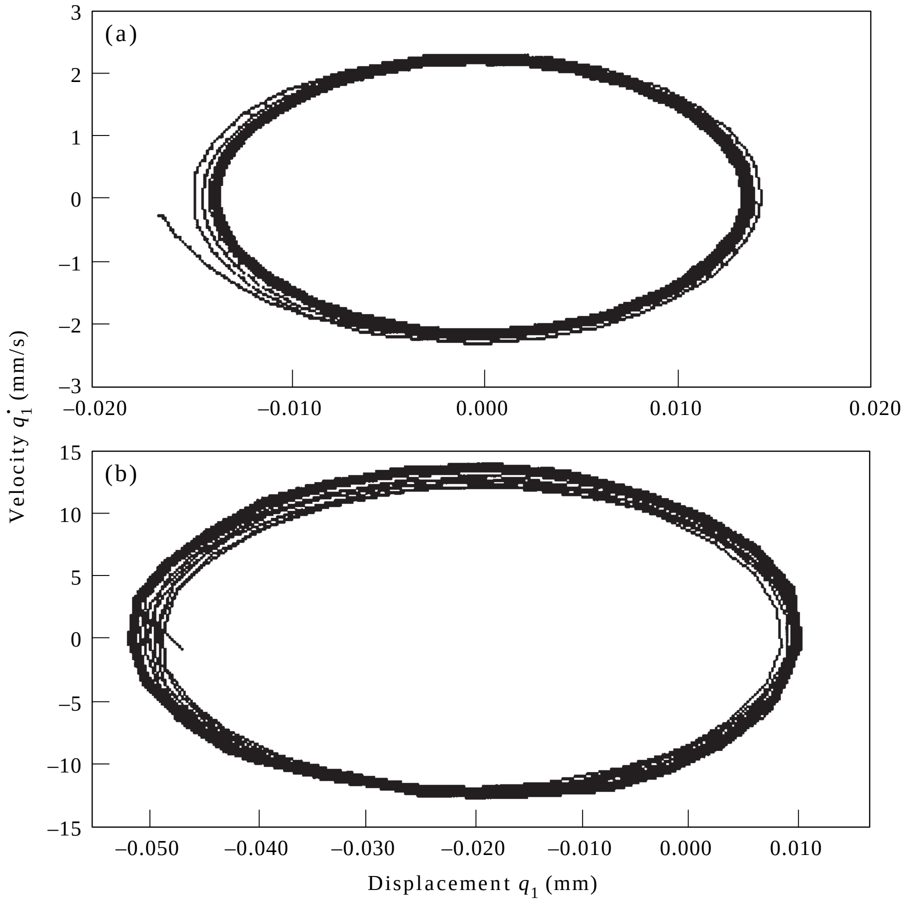

Figure 2. Phase plane portraits of experimental rotor responses: frequency of (a) 25·25 Hz; (b) 70·00 Hz.

As an illustrative example, a simple two-degree-of-freedom (Jeffcott) rotor model, as shown in Figure 1, is considered. The motion of this system evolves in the four-dimensional state space

$$\mathbf{x} = \{q_1, q_2, \dot{q}_1, \dot{q}_2\}, \tag{1}$$

where q_1 is the horizontal displacement and q_2 is the vertical displacement. Dots denote differentiation with respect to time.

The motion of the model is considered to be a periodic function related to the spin speed of the rotor shaft. For such systems, one can identify periodic motion from appropriate phase plane portraits. For example, Figure 2 depicts the motion evolution of the horizontal displacement on the phase plane q_1 versus $\dot{q}_1$ for two cases with different rotor speeds. When the motion becomes periodic, the trajectory is a closed loop. In general, one expects to see several transient steps before the system attains periodicity when the system starts from rest. The evolution of the system's motion can be represented by using projections of the orbits onto the phase plane of a particular displacement and velocity. This requires knowledge of the displacements and velocities at the specific points that are being considered.

A classical technique to analyze dynamical systems was developed by the nineteeth century French mathematician, Henri Poincaré. A formal mathematical description of Poincaré theory can be found in many previously published books and articles, such as Moon [14], Guckenheimer and Holmes [15], and Wiggins [16]. An informal presentation that describes the main features of the approach, its advantages, and its relevance to rotordynamic studies is given in the following discussion.

Periodic motions of a system can be represented by closed orbits in phase space. The points of the trajectory that coincide with a specific chosen instant, T, are labeled as ξ_i and the point of the closed orbit is labeled as ξ_e. The first return map for the generalized co-ordinate q_1 at the instant can now be obtained by plotting the values of q_1 at ξ_i versus the values at ξ_{i+1}, as shown in Figure 3 for two values of rotor speed. The iteration point that corresponds to the closed orbit, ξ_e, is on the 45° line because, when the motion is periodic, ξ_e at T is identical for all successive steps. On the other hand, it is observed that points on the map accumulate at ξ_e as the system takes successive steps. The same construction can also be performed for q_2 or for any other kinematic quantities. Discrete first return

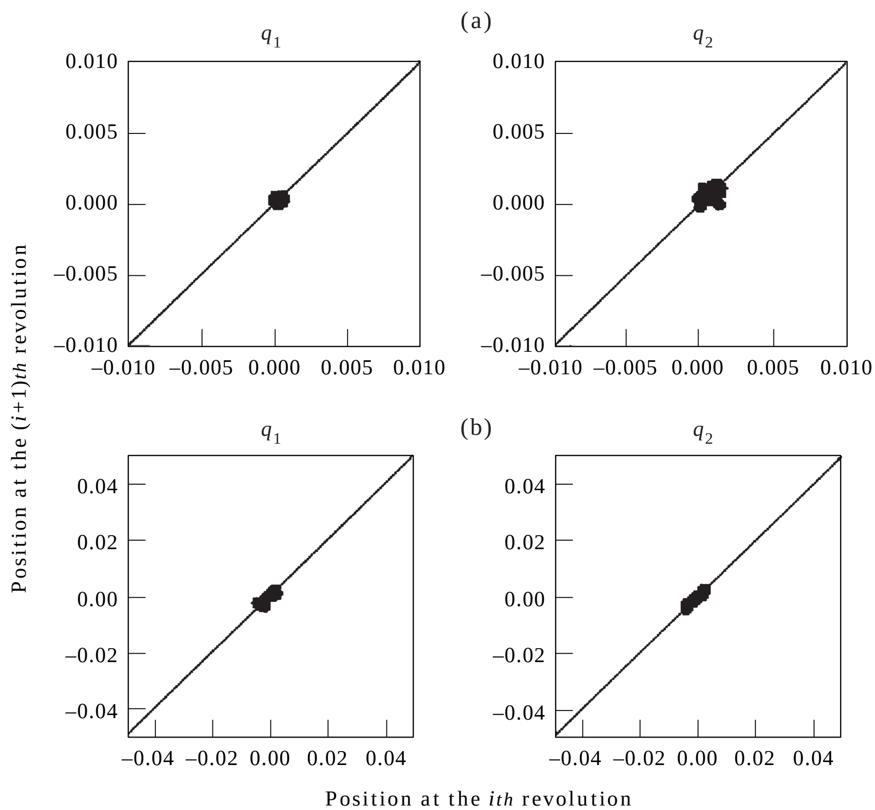

Figure 3. First return maps of experimentally observed responses: frequency of (a) 22·25 Hz; (b) 55·00 Hz.

maps for particular co-ordinates can be constructed and used to investigate their evolution with successive steps.

4. APPLICATION OF FLOQUET THEORY

For the simplified system that was considered above, the state variables are the two generalized co-ordinates (q_1 and q_2) and two generalized velocities ($\dot{q}_1$ and $\dot{q}_2$). Accordingly, the analytical representation of the graphical construction described in the previous section can be written as

$$\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i), \tag{2}$$

where $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ represent the state space vector at the Poincaré section during the *ith* and $(i+1)th$ steps, respectively. The function f is the discrete map that represents the dynamics of the non-linear model. Now, periodic motions of the system become the critical points of this map. For the equilibrium point x_e for the non-linear systems, one can write,

$$x_e = f(x_e) \tag{3}$$

The local stability criteria can be established through discussion of the linearized equations governing small motion about the equilibrium points of the map. If $\mathbf{y}_i$ is defined as the perturbation vector, then

$$\mathbf{x}_i = x_e + \mathbf{y}_i \quad \text{and} \quad \mathbf{x}_{i+1} = x_e + \mathbf{y}_{i+1}. \tag{4}$$

Substituting equation (4) in equation (2), one obtains

$$x_e + \mathbf{y}_{i+1} = f(x_e + \mathbf{y}_i). \tag{5}$$

Linearizing the map f about the equilibrium point x_e and retaining only the linear terms yields

$$\mathbf{y}_{i+1} = \mathbf{J}\mathbf{y}_i, \tag{6}$$

where $\mathbf{J}$ is the Jacobian matrix. In this case, the matrix $\mathbf{J}$ is a 4×4 constant coefficient matrix. The eigenvalues of this matrix are called the Floquet multipliers and can be used to determine the dynamic stability of the periodic motions of the system. The theory states that the motion is stable when the magnitudes of all Floquet multipliers are less that unity. The eigenvalues can be real or complex numbers. If at least one eigenvalue has magnitude equal to one, then the equilibrium point is marginally or simply stable.

Because of the complexity of the rotating machinery systems, the discrete map f cannot be derived analytically. Yet, one can obtain the Jacobian matrix of the linearized map by measuring the data numerically. Non-linear dynamical theory dictates that the Floquet multipliers do not depend on the choice of Poincaré section [12]. One should be able to compute the same multipliers when the section is taken at any well defined instant during the dynamic cycle. However, there are difficulties associated with tracking the motion of individual multipliers on the complex plane when they vary as a result of changing the Poincaré section. According the Floquet theory, the largest multiplier dictates the stability

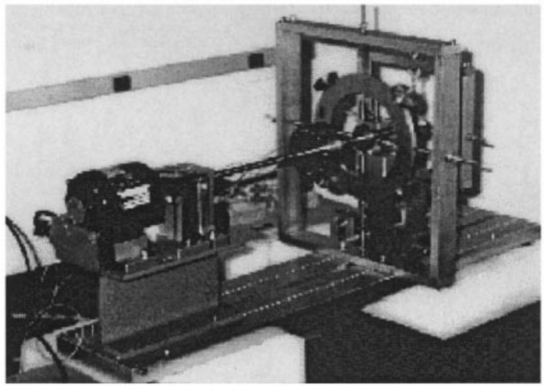

Figure 4. Experimental rotor system with clearance bearing apparatus.

characteristics of the system. Using this idea, an overall measure of stability can be defined that eliminates the need for tracking the individual eigenvalues. A measure can be defined as

$$\rho = \max |\lambda_i|, \tag{7}$$

where the λ_i are the Floquet multipliers. Smaller values of this measure represent periodic motion that is more robust to perturbations and, hence, more stable.

For constant coefficient systems, the real parts of the eigenvalues, μ_i, are related to the Floquet multipliers, λ_i, as

$$|\lambda_i| = e^{(2\pi\mu_i/\omega)}, \tag{8}$$

where the λ_i are the Floquet multipliers. An appropriate stability index can be defined as

$$\kappa = -\frac{\omega \ln \rho}{2\pi}. \tag{9}$$

The minus sign $(-)$ serves to make κ a positive quantity for stable systems and a negative quantity for unstable systems.

5. EXAMPLE SYSTEM

Several studies were performed to demonstrate the analysis method described in the previous sections. These studies used data generated with an experimental test rig with a clearance bearing non-linearity and a corresponding simplified simulation model.

A photograph of the experimental test rig is shown in Figure 4 and the associated schematic diagram is shown in Figure 5. It has two basic components: a flexible shaft and a clearance bearing assembly. The shaft is made of steel and is 9·5 mm in diameter and 0·46 m in length. It is supported at 25·4 mm from the right end by ball bearings suspended in a frame by four springs and at 25·4 mm from the left end by a bushing with a tight clearance. A rigid disk with holes for placing imbalance screws is positioned at the midpoint of the bearing span.

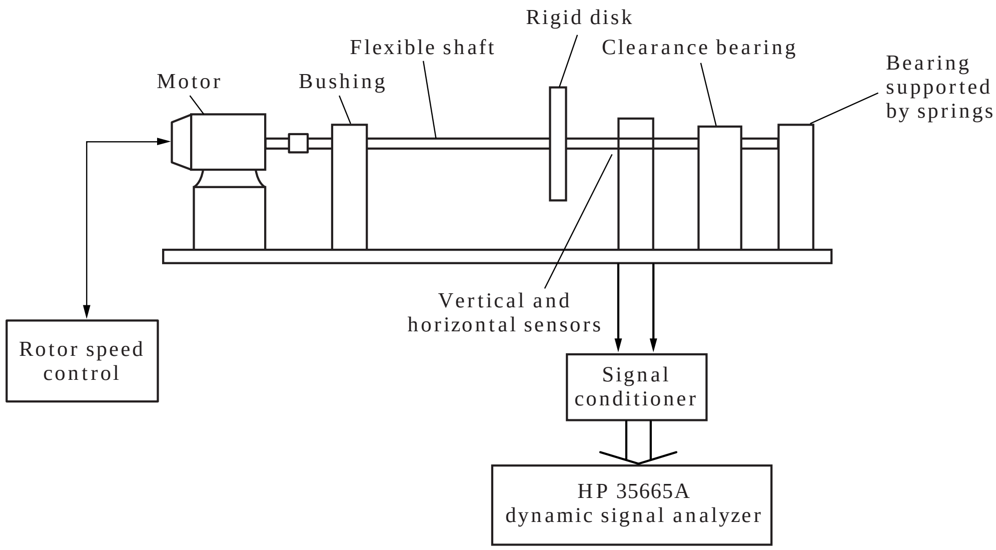

Figure 5. Schematic diagram of experimental rotor system.

The rotor is driven by an adjustable speed motor with a feedback speed controller. Shaft vibration is measured using eddy current proximity sensors fixed to measure displacement in the vertical and horizontal directions. The displacement signals of the shaft are sent to a signal analyzer.

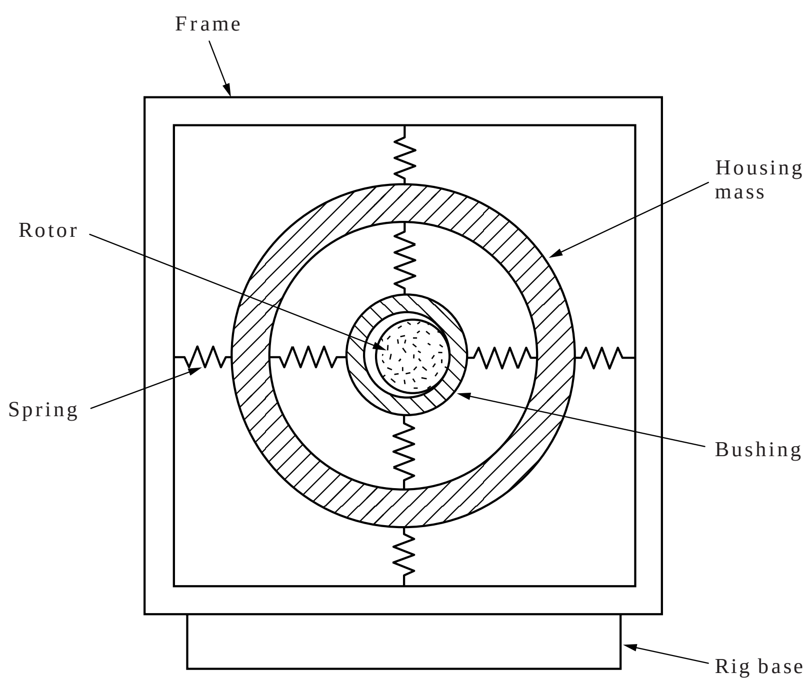

Figure 6. Schematic diagram of clearance bearing apparatus.

TABLE 1

Nominal parametric configuration for simulation model

Parameter	Magnitude	Units
c_1	54·6	1/s
k_1	49023·52	1/s^2
c_2	54·6	1/s
k_2	9804·704	1/s^2

A diagram of the clearance bearing/housing assemble is shown in Figure 6. It consists of a bushing suspended in a frame by springs and is situated at the right end of the rotor. For this study, the housing mass is rigidly connected to the frame and does not play a role in the system dynamic behavior.

The simplified two-degree-of-freedom Jeffcott model used in the simulation studies is as shown in Figure 1. The governing equations are

$$\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + (c_2\dot{q}_1 + k_2 q_1)\left(1 - \frac{\Delta}{r}\right)\Phi = e\omega^2 \cos \omega t, \tag{10}$$

$$\ddot{q}_2 + c_1\dot{q}_2 + k_1 q_2 + (c_2\dot{q}_2 + k_2 q_2)\left(1 - \frac{\Delta}{r}\right)\Phi = e\omega^2 \sin \omega t, \tag{11}$$

$$\Phi = \begin{cases} 1, & \Delta > r, \\ 0, & \Delta < r; \end{cases} \quad r = \sqrt{q_1^2 + q_2^2}. \tag{12, 13}$$

The nominal parametric configuration shown in Table 1 was used for all of the results presented in the following discussion, unless otherwise indicated. Phase plane portraits for each trial were constructed for the system. The phase plane portraits are a closed loop indicating periodic motions of the system, as shown in Figure 2. The first return maps were constructed by plotting values of the displacements or velocities obtained at a particular instance in one pass against the same values in the immediate next pass, as shown in Figure 3. The tendency of the points to be near the diagonal justified our assumption of observing steady state motion after a few rotations.

Figures 7(a) and (b) illustrate the results obtained by varying the bearing clearance for three values of imbalance (using simulated data) and for selected experimental data ($e = 0{\cdot}5$ mm). Note that there is excellent agreement between experimental and simulated results. For zero clearance, all three configurations have the largest subunit Floquet multiplier of 0·38 and a stability index of 0·94. This is the configuration with the maximum stability. This is expected, since for zero clearance, the damping from both the linear bearings and the clearance bearing are effective. As the clearance increases, the damping from the clearance bearing plays a decreasing role in the effective system damping, resulting

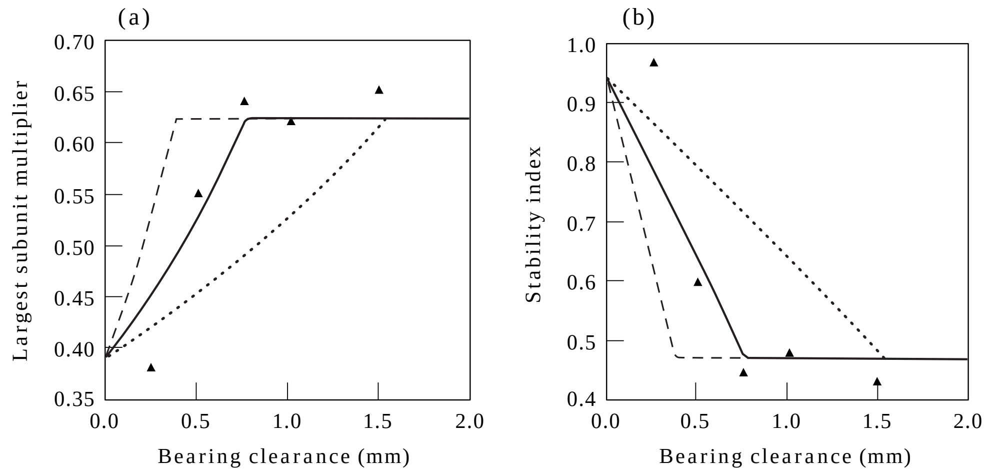

Figure 7. (a) Floquet multiplier values and (b) stability index for varying clearances (ω = 58 Hz); lines represent simulation results: ▲, experiment; ——, $e = 0{\cdot}5$mm; – – –, 0·25 mm; ------, 1·0 mm.

in an increasing largest subunit Floquet multiplier (and correspondingly, a decrease in system stability). For sufficiently large values of clearance, contact between the rotor and the clearance bearing ceases and the stability index goes to a constant value of 0·47. The rotor whirl radius is correspondingly larger for larger values of imbalance. Subsequently, the rotor is in contact with the clearance bearing for larger values of the clearance, resulting in additional effective damping to the system and increased stability.

Figures 8(a) and (b) illustrate the results obtained by varying the rotor speed for a set value of bearing clearance. Again, the results are presented for three values of imbalance (using simulated data) and for selected experimental data

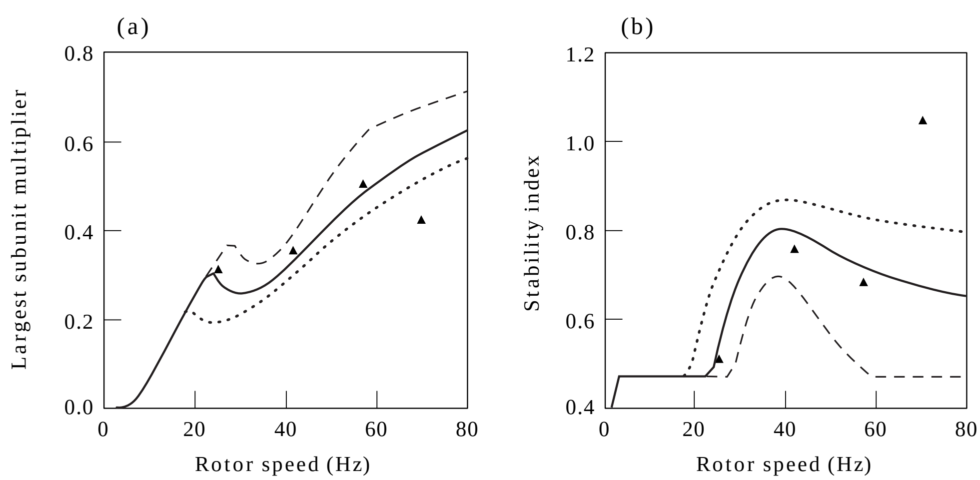

Figure 8. (a) Floquet multiplier values and (b) stability index for varying rotor speed (Δ = 0·381 mm); lines represent simulation results: ▲, experiment; ——, $e = 0{\cdot}5$ mm; – – –, 0·25 mm; -----, 1·0 mm.

($e = 0{\cdot}5$ mm). As the rotor speed increases, the largest subunit Floquet multiplier tends to increase, until a speed of about 25 Hz is achieved. Then, there is a brief decrease for a narrow range of rotor speeds, followed by a continuation of the previous increasing trend for the Floquet multiplier. Interestingly, this range of rotor speeds is in the neighborhood of the first critical speed, where the highest amplitude responses occur. Again, the damping from the clearance bearing is most effective in this range, resulting in an overall increase in system damping and stability. The experimental and simulated results shown in Figures 8(a) and (b) generally have excellent agreement. The most substantial difference occurs for the highest values of the rotor speed. The divergence is due to the limitations of the simulation model. The second critical speed for the rotor system is at about 100 Hz and its effects are evident for speeds above 60 Hz. The influence of this additional critical speed (which is not accounted for by the simple Jeffcott rotor model) is to raise the rotor whirl radius, resulting in more contact with the clearance bearing, increased effective damping, and a larger stability index.

6. CONCLUSIONS

A method of performing stability analyses for rotordynamic systems has been presented. Theoretical aspects of the method were developed. Principally, these consist of using Poincaré plots to obtain the nominal equilibrium periodic solution and applying Floquet analysis in a numerical fashion to the measured data, based upon this equilibrium solution, to obtain the stability measure.

Example studies were performed with a clearance bearing rotor rig to illustrate the method. The vibratory motion of the rotor model was considered to be a periodic function related to the spin speed of the shaft, enabling the use of the Floquet method. An important aspect of this analysis procedure is that the necessity for any information on the complex internal mechanisms of rotating systems is eliminated. The developed Floquet multipliers method is reliable for small data sets, is fast and easy to implement.

REFERENCES

1. T. YAMAMOTO 1954 *Memoirs of the Faculty of Engineering*, *Nagoya University*, *Japan* **6**. On critical speeds of a shaft.
2. F. F. EHRICH 1966 *ASME Paper* 66-*MD*-1. Subharmonic vibration of rotors in bearing clearance.
3. H. F. BLACK 1968 *Journal of Engineering Science* **10**, 1–12. Interaction of a whirling rotor with a vibrating stator across a clearance annulus.
4. A. MUSZYNSKA 1986 *Journal of Sound and Vibration* **110**, 443–462. Whirl and whip—rotor/bearing stability problems.
5. J. PADOVAN and F. K. CHOY 1987 *ASME Journal of Turbomachinery* **109**, 527–534. Nonlinear dynamics of rotor/blade/casing rub interactions.
6. M. F. DIMENTBERG 1988 *Statistical Dynamics of Nonlinear and Time–Varying Systems*. New York: Wiley.
7. S. C. SINHA, N. R. SENTHILNATHAN and R. PANDIYAN 1993 *Nonlinear Dynamics*, *and International Journal of Nonlinear Dynamics and Chaos* **4**, 483–498. A new numerical technique for the analysis of parametrically excited nonlinear systems.

8. S. C. SINHA and R. PANDIYAN to appear in *International Journal on Non-Linear Mechanics*. Analysis of quasilinear dynamical systems with periodic coefficients via Liapunov-Floquet Transformation.
9. H. L. WETTERGREN and K.-O. OLSSON 1996 *Journal of Sound and Vibration* **195**, 75–84. Dynamic instability of a rotating asymmetric shaft with internal viscous damping.
10. D. CHILDS 1993 *Turbomachinery Rotordynamics: Phenomena, Modeling, and Analysis*. New York: Wiley.
11. S.-K. CHOI and S. T. NOAH 1994 *Journal of Applied Mechanics* **61**, 131–138. Mode locking and chaos in a Jeffcott rotor with bearing clearances.
12. J. L. LAWEN JR. and G. T. FLOWERS 1995 *ASME Journal of Engineering for Gas Turbines and Power* **119**, 430–435. Synchronous dynamics of a coupled shaft/bearing/housing system with auxiliary support from a clearance bearing: analysis and experiment.
13. A. H. NAYFEH and B. BALACHANDRAN 1995 *Applied Nonlinear Dynamics*. New York: Wiley.
14. F. C. MOON 1992 *Chaotic and Fractal Dynamics: An Introduction for Applied Scientists and Engineers*. New York: Wiley.
15. J. GUCKENHEIMER and P. HOLMES 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer-Verlag.
16. S. WIGGINS 1990 *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. New York: Springer-Verlag.

APPENDIX: NOMENCLATURE

c_1 linear damping coefficient, 1/s
c_2 clearance bearing damping coefficient, 1/s
e imbalance eccentricity, m
k_1 linear stiffness coefficient, $1/s^2$
k_2 clearance bearing stiffness coefficient, $1/s^2$
q_1 rotor co-ordinate in the horizontal direction, m
q_2 rotor co-ordinate in the vertical direction, m
t time, s
Δ nominal radial gap of clearance bearing, m
ω rotor operating speed, rad/s