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ASYMPTOTIC STATES OF A BILINEAR HYSTERETIC OSCILLATOR IN A FULLY DISSIPATIVE PHASE SPACE

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1. INTRODUCTION

A bilinear hysteretic model is used to study the response of various elastoplastic structures. For fundamental analysis and understanding, a single-degree-of-freedom model often suffices. The idealization in the form of a bilinear response curve results in simpler equations that are easier to analyse. Yet, the results obtained are known to be of direct practical value. Much of the literature on studies of elastoplastic structures is based on this simple model.

There have been numerous studies [1–6] on the forced response of elastoplastic structures using a bilinear hysteretic model. Many of these studies have concentrated on finding periodic responses, resonant frequencies and maximum amplitude motions. There are also several studies [7–9] on chaotic motion of such systems. For many practical applications, however, the understanding of the transient response of the system is equally important. In an earlier paper [10], Pratap *et al*. reported the analysis of free oscillations of an elastoplastic oscillator with kinematic hardening as a parameter and showed that there was a unique limit cycle as an asymptotic state of the system for a range of values of the hardening parameter. A recent paper [6], apparently unaware of the results of reference [10], reports some of those results as a part of the findings. As pointed out in reference [10], the analysis was carried out without considering any viscous damping. The solutions were obtained by piecing together analytical solutions on various branches of the bilinear hysteretic curve. This construction was motivated by Shaw and Holmes's work on piecewise linear systems [11]. In the present work, the same problem as in reference [10] but is considered, but viscous damping is included which makes the phase space fully dissipative. In the earlier work, dissipation was present in finite bands (during the plastic phase of the motion only) and therefore, the phase space was only partially dissipative. In the presence of viscous damping, the authors show that the unique limit cycle reported in the earlier work disappears but other unique asymptotic states emerge that were non-existent before.

The model considered (shown in Figure 1), consists of two massless rigid links of length l each, a concentrated mass m, an elastoplastic torsional spring, and a rotational damper (to model viscous damping).

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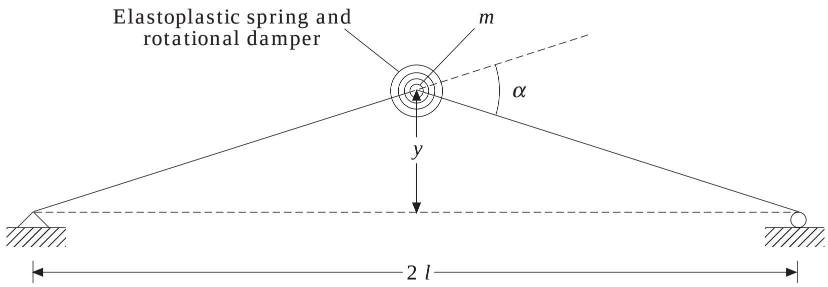

Figure 1. Model.

Figure 2 shows the characteristics of the spring. Moment M is plotted against ξ, normalized displacement, as defined in equation (2). The spring has spring constant k during elastic motion and $\eta^2 k$ during plastic motion. η^2 is the kinematic hardening coefficient, and is between 0 and 1. For $\eta^2 = 0$, the spring is perfectly plastic, and for $\eta^2 = 1$, there is no plastic region and the spring is perfectly elastic. The spring is modelled as having kinematic strain hardening, so that the difference in moment from one extreme of elastic motion to the other (C to D) is always the same. The rotational damper provides for damping proportional to the rate of change of the angle α; the constant of this proportionality is b.

Since the system has only one degree of freedom, the equation of motion is a single order differential equation. This would indicate that there are two state variables, position (either linear or angular displacement) and velocity (again, either linear or angular). However, the constants in the equation of motion for each elastic region of motion depend on the state of the system at the end of the previous plastic region. Therefore, three variables are necessary to fully describe

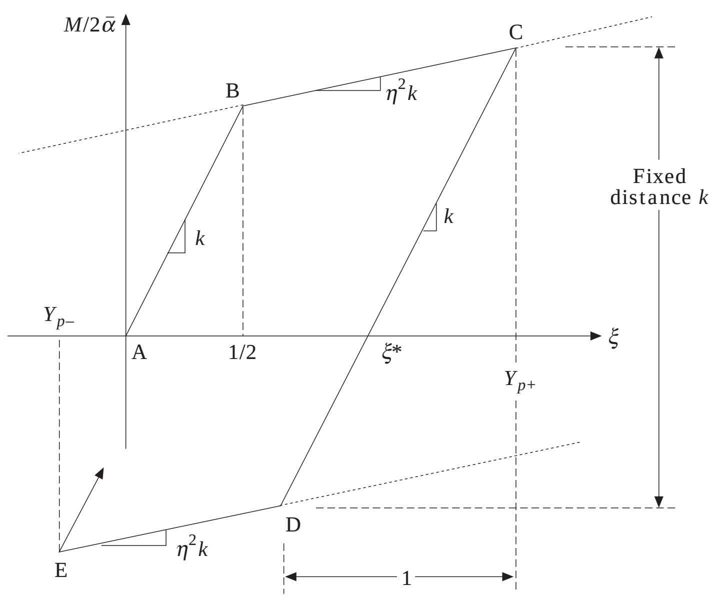

Figure 2. Spring characteristic.

the state of the system: position, velocity, and a third variable to describe the amount of plastic offset in the system (and thus the equilibrium for elastic oscillations). It is possible to construct a model that would include isotropic hardening as well, but that would require five state variables [6].

2. EQUATIONS OF MOTION

The non-dimensionalized equation of motion for the system is

$$\ddot{\xi} + 2\omega\zeta\dot{\xi} + \beta^2\omega^2\xi = c\omega^2, \tag{1}$$

where

$$\xi = q/l\bar{\alpha}, \qquad \omega = \sqrt{4k/ml^2}, \qquad \zeta = b/l\sqrt{km} \tag{2–4}$$

and where

$$\beta = 1 \quad \text{and} \quad c = \xi^* \text{ in the elastic region} \tag{5}$$

and

$$\beta = \eta \quad \text{and} \quad c = -\tfrac{1}{2}(1 - \eta^2)\,\mathrm{sgn}\,(\dot{\xi}) \text{ in the plastic region.} \tag{6}$$

The relationship between the non-dimensionalized rotation v and displacement ξ follows from the small angle approximation:

$$v = \alpha/\bar{\alpha} = 2\xi. \tag{7}$$

The three variables necessary to describe the state of the system are ξ, $\dot{\xi}$, and ξ^*, where $\xi^* = (\xi - \frac{1}{2}\mathrm{sgn}\,(\dot{\xi}))(1 - \eta^2)$ in the plastic region, and remains constant in the elastic region, where it is the equilibrium of the elastic oscillations. Note that c, although constant in each region of motion, depends in each elastic motion upon the final ξ^* of the preceding plastic motion, and so can be considered a dynamic variable. If one were to write equation (1) in state space form, either c or ξ^* would need to be included as a state variable [6]. In the present analysis, however, inclusion of c as a state variable is not required.

3. SOLUTION THROUGH MAPS

Finding the solution of the equation of motion is straightforward, but because there are both ξ and $\dot{\xi}$ terms in the equation, the solution is a transcendental function and therefore time t cannot be solved for explicitly. An iterative process, based on Newton's method, is used to determine the times at which the behavior changes from plastic to elastic or vice versa. The system switches from plastic motion in one direction to elastic motion in the other whenever velocity, $\dot{\xi}$, reaches zero. It then switches from elastic motion to plastic motion in the same direction after ξ changes by a unit amount, i.e., v changes by 2 (this is due to the kinematic strain hardening assumption and the fact the ξ is normalized so that the maximum difference between one extreme of elastic motion and the other is unity). Once the time of a behavior change is found, ξ and $\dot{\xi}$ at that time become initial conditions for the next motion, and, if the ensuing motion is elastic, then ξ^* at the time of the change becomes the equilibrium of the new elastic motion.

The maps shown in Figures 3–5 are the result of iterative simulations of the system's behavior. Y_{p+} and Y_{p-} are the extreme values of motion; the values of ξ at which $\dot{\xi}$ is zero and the behavior changes from plastic motion in one direction to elastic motion in the reverse direction. The map allows prediction of Y_{p-} given Y_{p+}, then use of that value of Y_{p-} to predict the next Y_{p+}, and so on until the system no longer undergoes plastic deformation. It is then possible to find ξ^*, which, since any plastic motion is finished, is the permanent plastic deformation. The size of the elastic region is always the same, so the value of ξ^* when the system switches from elastic to plastic behavior is either $(Y_{p+} - \frac{1}{2})$ or $(Y_{p-} + \frac{1}{2})$.

Although the map in Figure 6 is discussed in reference [10] in detail, the map is reproduced here and its essential features discussed briefly as it is central to our discussion of current results. Each map in Figures 3–6 has two curves: an upper curve which shows the next value of Y_{p-} given Y_{p+}, and a lower curve which shows the next value of Y_{p+} given Y_{p-}. The map is read by taking an initial known value of Y_{p+} or Y_{p-} and finding consecutive Y_p's by following a stairlike path between the two curves, upwards and to the left, until the line of slope 1 passing through (0·5, −0·5) is reached. Without damping ($\zeta = 0$, see Figure 6), the two curves meet at (0·5, −0·5), so any intersection with this line will be at that point or outside the curves. When damping is introduced, the two curves move away from each other along this line, so that the stair-path could end either outside the curves or between them as shown in Figure 7.

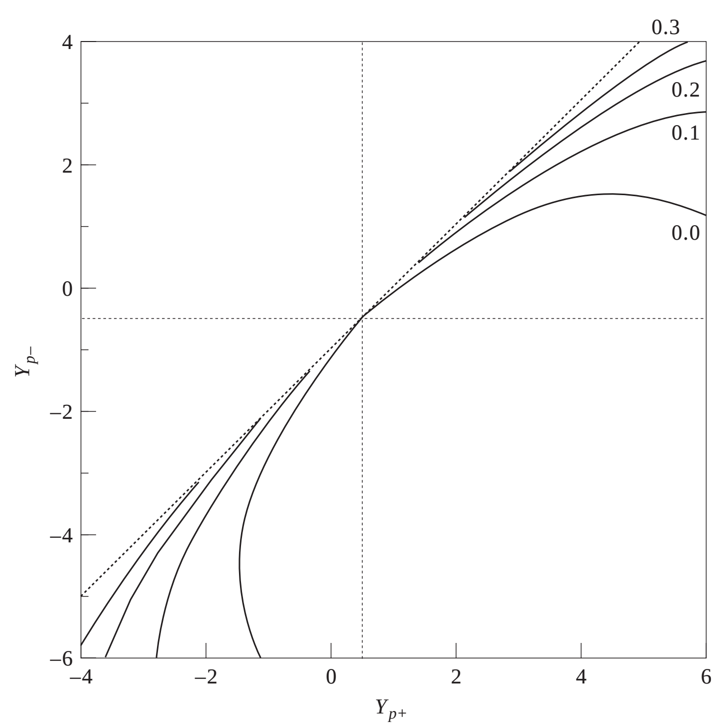

Figure 3. Map for $\eta^2 = 0{\cdot}1$, and various values of ζ.

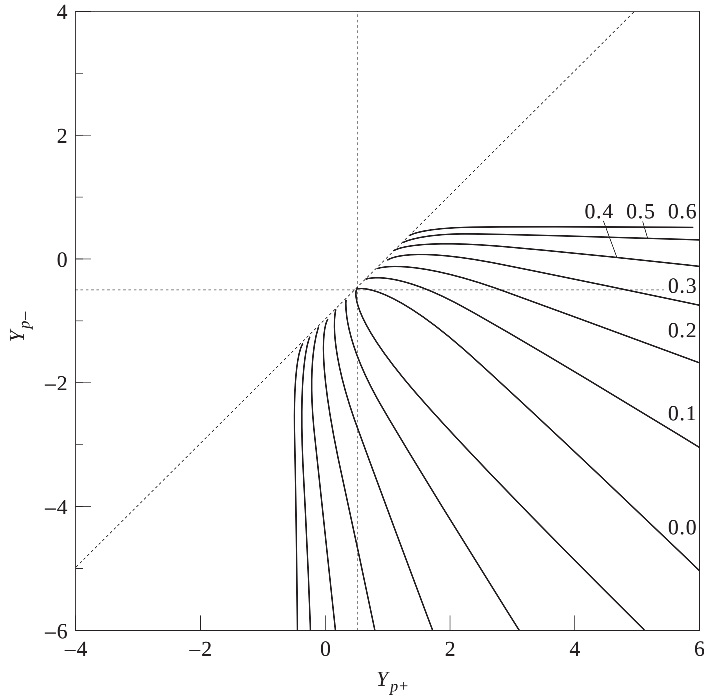

Figure 4. Map for $\eta^2 = 0{\cdot}5$, and various values of ζ.

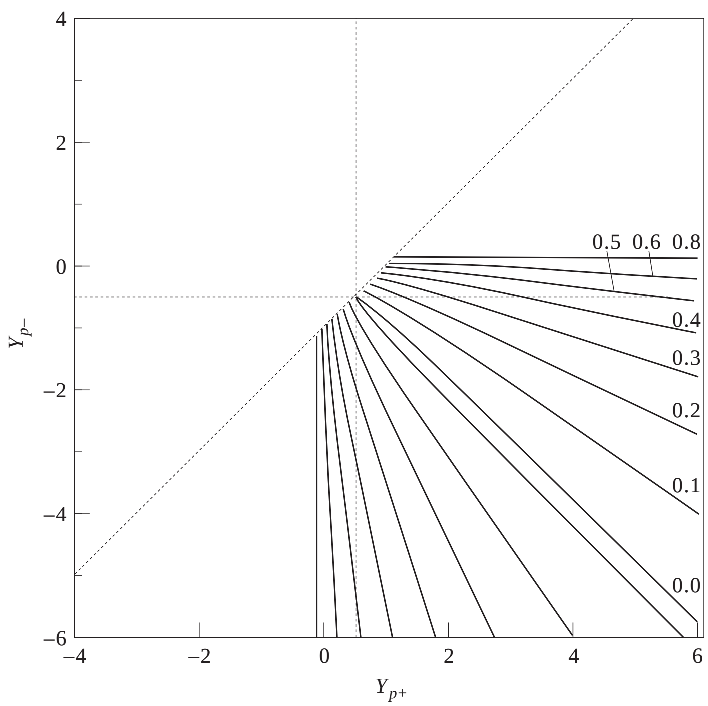

Figure 5. Map for $\eta^2 = 0{\cdot}8$, and various values of ζ.

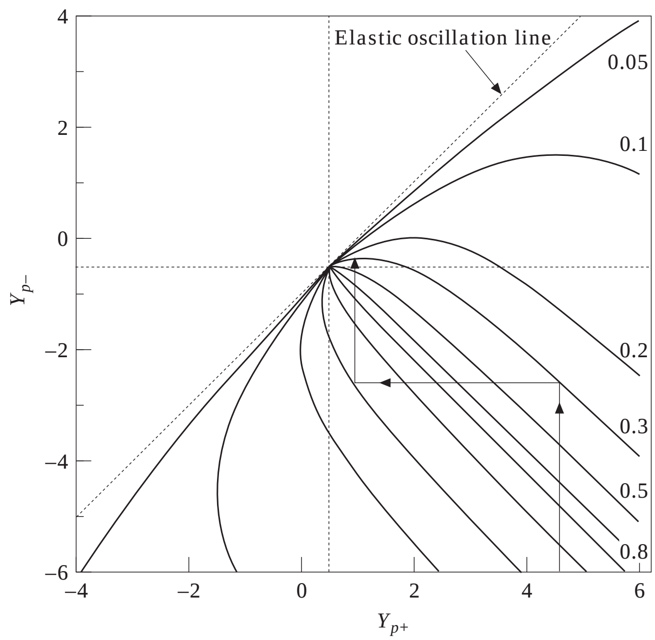

Figure 6. Map for $\zeta = 0$, and various values of η^2. A stair-like path is used to determine the asymptotic state of the oscillator.

The line of slope 1 passing through (0·5, −0·5) will be called the *elastic oscillation line*. On this line, Y_{p+} and Y_{p-} are 1 unit apart, which is the amplitude of undamped elastic motion. Once the stair-path of an undamped beam reaches this line, the beam will continue to undergo elastic oscillations between the Y_{p+} and Y_{p-} values given by the intersection of the path with this line, but no further plastic motion will occur. Since damped elastic oscillations will have amplitudes $\leqslant 1$ (assuming $\zeta > 0$), no further plastic motion will occur for a damped beam as well. Once its stair-path reaches this line, a damped beam will undergo decaying elastic oscillations about the ξ^* value determined by the most recent Y_p. Thus, the elastic oscillation line is the end of all stair paths, and marks the point at which no further plastic motion can occur. The amount of permanent plastic offset, ξ^*_{final}, is found by adding $\frac{1}{2}$ to the Y_{p-} value of the intersection (or substracting $\frac{1}{2}$ from the Y_{p+} value). ξ^*_{final} is zero at (0·5, −0·5) and increases as the distance from this point increases.

Figure 6 shows maps for various values of η^2, without any damping ($\zeta = 0$). As Pratap *et al*. [10] have discussed, $\eta^2 = 0{\cdot}5$ is a critical value. If $\eta^2 > 0{\cdot}5$, all paths lead to (0·5, −0·5). No matter how much energy the vibration starts with, there is no net plastic deformation once the system reaches steady state (ξ^*_{final} is always zero). For $\eta^2 < 0{\cdot}5$, the curves have finite regions to the left of $Y_{p+} = 0{\cdot}5$ and above $Y_{p-} = -0{\cdot}5$. These areas, which result in permanent plastic offset ($\xi^*_{final} \neq 0$), are called *elastic trapping regions*, and are discussed in greater detail in the next section.

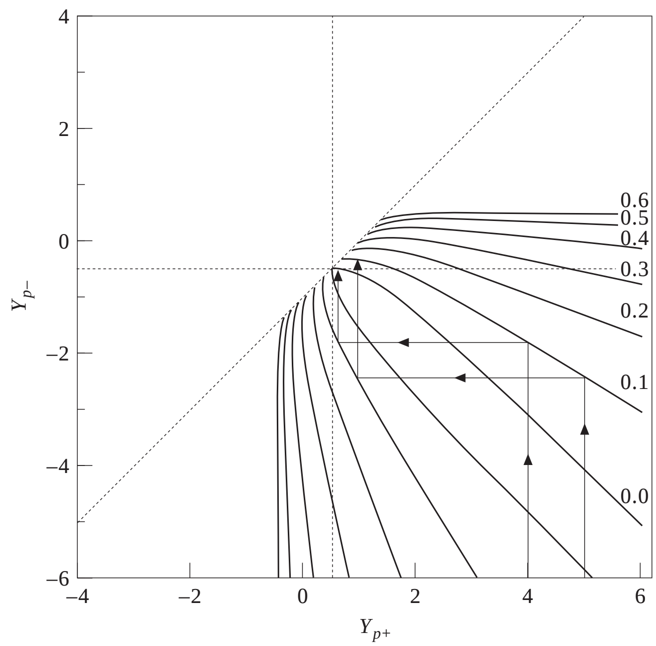

Figure 7. Iterative map for $\eta^2 = 0{\cdot}5$. Due to the presence of damping, the two curves comprising the map move away from each other creating a gap along the elastic oscillation line. Here, two stair-paths are shown, one ending on one of the curves, the other ending in the gap between the two curves.

Figures 3–5 show maps for various amounts of damping, ζ, for three values of η^2 (0·1, 0·5, and 0·8). As damping increases for a constant η^2, the curves spread further apart and a gap opens up at their upper end, along the elastic oscillation line. The portion of this line between the endpoints of the curves is called a *damping trapping region*, and is discussed in greater detail in the next section.

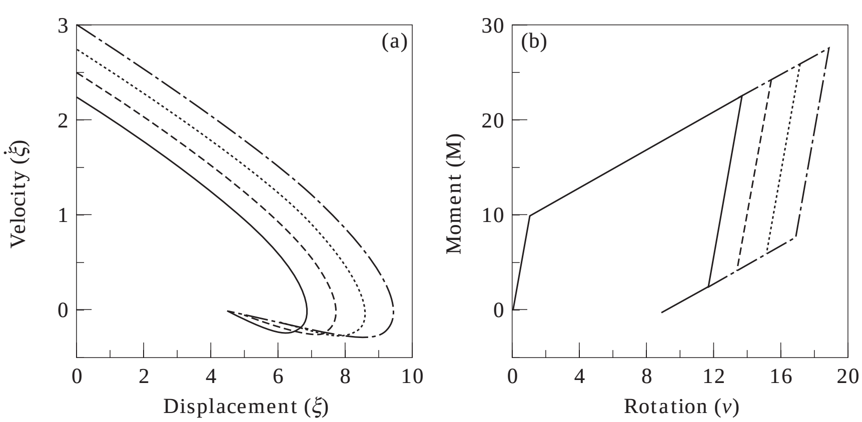

Figure 8. The unique attractor for $\eta^2 = 0{\cdot}5$ ($\zeta = \eta$). (a) The attractor in the phase, (b) the attractor in the M-v space.

4. ASYMPTOTIC STATES

The asymptotic states of the oscillator can be classified according to the various distinct regions of the map that *trap* the system.

4.1 *Elastic Trapping Region*

When the system reaches an extreme of motion in one direction, after which it does not have enough energy left to reach the next onset of plastic behavior, remaining oscillations will be confined (trapped) in the elastic region only. Pratap *et al*. [10] have discussed *elastic trapping regions* for the undamped case. These are the regions of the curves on the maps in Figure 6 which allow the stair-path to end up on the elastic oscillation line outside the curves. This is one type of asymptotic behavior of the oscillator. The asymptotic motion is purely elastic about a permanent plastic offset that is easily found from the map. As damping is increased, this elastic region grows as shown in Figures 3–5. This is because energy is now dissipated by damping as well as plastic deformation, so there is less energy available at the end of each plastic motion. Therefore, the likelihood of further plastic motion decreases and more and more intial conditions lead to eventual elastic oscillations after a particular value of Y_{p+} or Y_{p-}. Thus the effect of damping on this particular asymptotic state is to increase its basin of attraction. It is also worth noting that the maximum possible permanent offset, determined by the maxima of the curves, also increases as damping is increased for any given η^2. This effect is more dramatic for $\eta^2 \geqslant 0{\cdot}5$ for which the elastic region is born essentially due to the existence of damping. For completeness, one also mentions that because of damping, all oscillations will eventually die out and the asymptotic state of the oscillator will be the permanent plastic offset ξ^*, that can be found from the map as discussed above.

4.2 *Damping Trapping Region*

The open line segments on the elastic oscillation line between the ends of the two curves of the solution map present a different type of trapping region. A stair-path terminating on one of these segments implies, once again, purely elastic motion of the system. Approach of this asymptotic state, however, is different from the previous one in that the system is prevented from yielding again in plastic flow due to damping alone. In other words, at the last transition from plastic to elastic motion, the system had enough energy to reach the next plastic yield had there been no damping. Therefore, this region is called the *damping trapping region*. Admittedly, this distinction is somewhat artificial, but it helps in characterizing the different asymptotic states that come into existence due to the introduction of damping.

4.3. *The Unique Attractor*

Without any damping, both curves on any Y_{p+} versus Y_{p-} map eventually reach a slope of -1 as $|Y_p|$ increases. As damping is increased, the curves not only spread apart, their slopes also diverge. The two curves eventually become horizontal and

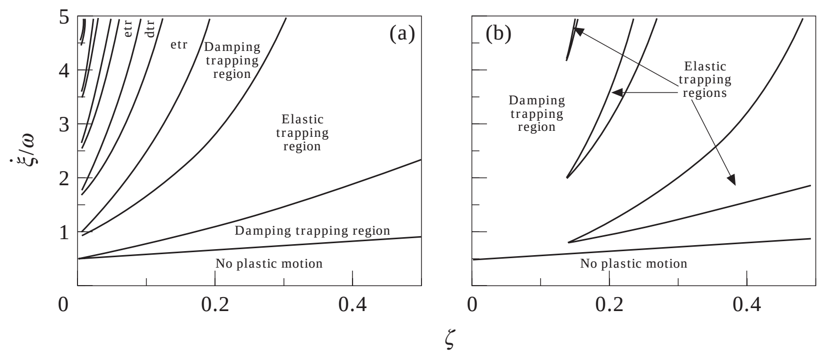

Figure 9. $\dot{\xi}/\omega$ versus ζ for (a) $\eta^2 = 0{\cdot}4$ and (b) 0·6.

vertical as $\zeta \to \eta$. This implies that for any given η^2, there is a unique value* of Y_{p-} and Y_{p+} which the oscillator settles to for all sufficiently large values† of Y_{p+} and Y_{p-} respectively. This asymptotic state corresponds to a unique value of ξ^*. As shown in Figure 8, this state corresponds to zero torque in the spring.

From the equations of motion, one can see that the frequency of oscillation in the plastic region is $\omega\sqrt{\eta^2 - \zeta^2}$. Thus, $\zeta = \eta$ is a critical value above which there are no oscillations. This also implies that the damping is just enough so that the final maximum yield occurs when there is no energy stored in the spring, so that no subsequent oscillations of either the plastic or elastic variety occur. Since, for a given value of hardening parameter, η^2, this asymptotic state is unique the authors call it a unique attractor. As one can see from the location of the limiting curves in the map corresponding to the critical value of ζ, the horizontal and vertical lines move towards the limiting values of $|Y_P| = 0{\cdot}5$ as η^2 increases and will eventually lead to $\xi^* = 0$ for $\eta^2 = 1$ which makes perfect sense since $\eta^2 = 1$ represents the case of a purely elastic oscillator.

4.4. *Basins of the Asymptotic States*

Given a set of initial conditions, it is easy to compute the value of the first Y_P by solving for the condition $|\dot{\xi}| = 0$. Once the first Y_P is determined, one can use the maps discussed above and determine the asymptotic fate of the solution trajectory. Thus, it is possible to construct regions in the initial condition space that lead to similar asymptotic states. What is more interesting, however, is to find the change in these regions with varying ζ, the emphasis here being the effect of damping on the system response. In Figure 9 are shown the basins for different asymptotic states with varying ζ. For constructing the basins of attraction, only velocities have been used as initial conditions. This, however, implies no loss of generality as an initial displacement can always be converted into an equivalent initial velocity based on energy calculations. The two plots shown in Figure 9 indicate how the elastic trapping region expands and the damping trapping region shrinks as η^2 increases.

*Technically, there are two values, one for Y_{p+} and one for Y_{p-}. This, however, is expected due to reflection symmetry in the map.

†This is just to ensure that the stair-paths ending on the elastic oscillation line between the open ends of the two curves are excluded.

5. CONCLUSIONS

The asymptotic states of an elastoplastic oscillator are studied in this paper. The oscillator is assumed to exhibit kinematic hardening. Apart from plastic deformation, viscous damping is considered as another energy dissipation mechanism. The dynamics of the oscillator are studied for various degrees of kinematic hardening and viscous damping. The asymptotic states of the oscillator are obtained and classified by constructing maps that determine the fate of unforced motions. It is shown that presence of viscous damping destroys the unique limit cycle reported in reference [10] for $\eta^2 \geqslant 0{\cdot}5$. It is also shown that there exists a *critical damping* for any value of kinematic hardening, given by $\zeta = \eta$, for which the oscillator has a unique asymptotic state. The variation in the basin of attraction of the various asymptotic states with the amount of viscous damping are also presented and it is shown that the damping trapping regions shrink and the elastic trapping regions grow as the kinematic hardening parameter, η^2 increases.

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