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DIFFERENTIAL QUADRATURE METHOD: APPLICATION TO INITIAL-BOUNDARY-VALUE PROBLEMS

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Two non-linear dynamical systems have been considered. In both cases, the governing equation of motion is reduced to two second-order non-linear non-autonomous ordinary differential equations using the differential quadrature method with a careful distribution of sampling points. To check the numerical results, a comparison with those obtained using the Galerkin approach is proposed.

1. INTRODUCTION

The reduction of continuous non-linear systems to single-degree-of-freedom oscillators can be performed by using various equivalent approaches, as for example the averaging methods or the Ritz–Galerkin techniques [1]. The resulting system can be investigated deeply in the non-linear regime, and harmonic and non-harmonic response can be obtained. Nevertheless, the drastic reduction to a single degree of freedom leads to neglect of some qualitative phenomena, as for example the internal resonances, which can significantly change the system behaviour. Consequently, a number of papers have been devoted to the influence of higher modes on both regular and chaotic responses [2].

In this paper, the reduction of non-linear boundary-initial-value problems to a system of coupled non-linear ordinary differential equations is carried out by using the generalized differential quadrature (GDQ) method [3]. The method can be conveniently applied to the purpose, by substituting the four boundary conditions into the governing equations [4], so that the number of sampling points reduced by four represents the degrees of freedom of the discretized system.

Because of the complexities involved in the analysis of multi-degree-of-freedom systems, attention will be restricted to two-degree-of-freedom systems. So, caution is required in the choice of the sampling points, since their number must be limited to six. Actually, the GDQ method uses the Lagrange interpolated polynomial as test functions and the roots of shifted Legendre polynomial as grid co-ordinates (a brief overview of the method is given in the next section), and this choice is shown to produce inaccurate results, even in the linear regime, at least if the number of sampling points is *a priori* limited. Consequently, a new, more satisfactory distribution is proposed, which is able to reduce the discretization errors.

Finally, the strongly non-linear oscillations of the two-degree-of-freedom systems obtained are studied with the modified Lindstedt–Poincarè (MLP) method, using an automatic code [5].

This perturbation approach, which was recently developed by Cheung and coworkers [6], seems to be the most natural choice for a strongly non-linear, undamped system where no restriction on the non-linearity can be introduced.

As a guideline, two prototype structural models are used, i.e., the simply supported beam resting on a non-linear Winkler soil and the slender clamped–hinged beam in which the axis shortening is taken into account. In both cases, the numerical results have been compared with the results obtained by using the usual GDQ grid co-ordinates, and with the results given by the classical Galerkin approach.

For a different application of the differential quadrature method to the vibration analysis of a geometrically non-linear beam, see reference [7].

2. METHODS OF ANALYSIS

The basis idea of the differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the function values at all discrete points in the domain of that variable. In terms of dimensionless variables, one has that, at a point $\zeta = \zeta_i$, the rth-order derivative of a function $w(\zeta)$, defined in the domain (0, 1) with N discrete grid points, is given by:

$$\left[\frac{\mathrm{d}^r w}{\mathrm{d}\zeta^r}\right]_{\zeta=\zeta_i} = \sum_{j=1}^{N} A_{ij}^{(r)} w_j \quad i = 1,2, \ldots, N, \tag{1}$$

where $A_{ij}^{(r)}$ are the weighting coefficients of the rth-order derivative.

The weighting coefficients are determined by substituting approximating functions to the originary function $w(\zeta)$ in equation (1). In the GDQ method [3, 4], these test functions are assumed to be the Lagrange interpolation polynomials.

The off-diagonal terms of the weighting coefficient matrix of the first-order derivative turns out to be:

$$A_{ij}^{(1)} = \frac{\prod\limits_{\substack{v=1 \\ v \neq i}}^{N} (\zeta_i - \zeta_v)}{(\zeta_i - \zeta_j) \prod\limits_{\substack{v=1 \\ v \neq j}}^{N} (\zeta_j - \zeta_v)} \quad i, j = 1, 2, \ldots, N \quad j \neq i. \tag{2}$$

The off-diagonal terms of the weighting coefficient matrix of the higher-order derivative are obtained through the recurrence relationship:

$$A_{ij}^{(r)} = r\left[A_{ii}^{(r-1)}A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{(\zeta_i - \zeta_j)}\right] \quad i, j = 1, 2, \ldots, N \quad j \neq i, \tag{3}$$

where $2 \leqslant r \leqslant (N-1)$.

The diagonal terms of the weighting coefficient matrix are given by:

$$A_{ii}^{(r)} = -\sum_{\substack{v=1 \\ v \neq i}}^{N} A_{iv}^{(r)} \quad i = 1, 2, \ldots, N, \tag{4}$$

where $1 \leqslant r \leqslant (N-1)$.

Assuming the Lagrange interpolation polynomials as test functions, there is no restriction in the choice of the grid co-ordinates. So, in order to have more accurate solutions, it is possible to generate the sampling points as follows:

$$\zeta_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)}{(N-1)}\pi\right] \quad i = 1, 2, \ldots, N. \tag{5}$$

In order to overcome the problem of the δ-points [8], Shu and Du [4] support the GDQ method with a direct substitution of the boundary conditions into the governing equation. In the present work, the Shu and Du approach is used, but the grid co-ordinates are not given by equation (5).

It is well-known that the distribution of sampling points must be symmetric and it must result in $\zeta_1 = 0$ and $\zeta_N = 1$. So, it will be sufficient to determine $(N-2)/2$ (two, for $N = 6$) points after the first, since the others are automatically fixed by the relation that defines the symmetry of the distribution:

$$\zeta_i + \zeta_{(N+1-i)} = 1.$$

By induction, the following rule has been deduced:

$$\zeta_i = \left(\frac{i-1}{N-1}\right)^{Nb_i/i\sqrt{i}}, \tag{6}$$

where b_i are unknown coefficients to be fixed.

As it has been said, for the symmetry, only b_2 and b_3 need to be fixed.

This distribution has been checked with a linear analysis: the comparison term between numerical and exact results has been given by the critical value of the axial force. After many numerical simulations, it has been possible to see that the third co-ordinate plays an important role in obtaining good results. In fact, for a certain value of b_3, the non-linear results are not influenced very much by varying b_2, whereas the linear results are the same. This is true even if the second point is near to the third. Finally, it has been noted that the results are in good agreement for values of b_3 close to $N/5 = 1{\cdot}2$. For $b_3 = 1{\cdot}2$ and $b_2 = 1$, for example, the following distribution of sampling points is obtained:

$$\{0, \quad 0{\cdot}033, \quad 0{\cdot}281, \quad 0{\cdot}719, \quad 0{\cdot}966, \quad 1\} \tag{7}$$

This is the distribution which is used for the final non-linear results.

3. THE FIRST MODEL: A SIMPLY SUPPORTED BEAM ON A NON-LINEAR WINKLER SOIL

Consider a simply supported beam with span L, Young modulus E, moment of inertia I, mass per unit length m and cross-sectional area A, which rests on an "hardening" non-linear elastic foundation and which is subjected to a compressive load P and to an exciting transverse force $F(z, t) = F(z)\cos\bar{\omega}t$. The foundation is supposed to be defined by the following load–displacement relationship: $q(z) = k_1 v(z) + k_3 v(z)^3$, where $q(z)$ is the force per unit length, k_1 is the linear Winkler foundation stiffness and $k_3 > 0$ is the "hardening" non-linear elastic foundation stiffness.

If the beam is considered to be slender, the equation of motion can be written as:

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial z^4} + P\frac{\partial^2 v}{\partial z^2} + k_1 v + k_3 v^3 = F(z)\cos\bar{\omega}t. \tag{8}$$

Equation (8) can be conveniently written in terms of dimensionless variables as:

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial \zeta^4} + \sigma\frac{\partial^2 w}{\partial \zeta^2} + \theta_1 w + \theta_3 w^3 = f(\zeta)\cos\omega\tau, \tag{9}$$

where

$$w = \frac{v}{L}, \quad \zeta = \frac{z}{L}, \quad \tau = \sqrt{\frac{EI}{m}}\frac{t}{L^2}, \quad \omega = \sqrt{\frac{m}{EI}}L^2\bar{\omega},$$

$$\sigma = \frac{PL^2}{EI}, \quad \theta_1 = \frac{k_1 L^4}{EI}, \quad \theta_3 = \frac{k_3 L^6}{EI}, \quad f(\zeta) = \frac{F(\zeta)L^3}{EI}.$$

In the Galerkin method, the deflection is approximated by:

$$w(\zeta, \tau) = \sum_{m=1}^{M} u_m(\tau)\phi_m(\zeta), \tag{10}$$

where u_m are generalized co-ordinates and $\phi_m = \sin m\pi\zeta$ are the characteristic modal functions of a simply supported beam subjected to a compressive load.

If two co-ordinates are employed, equation (10) can be inserted into equation (9), the result multiplied by ϕ_n and integrated over the span. Then, the orthogonality property of the modal functions ϕ_n, allows one to write the following equations:

$$\ddot{u}_i + \omega_{i0}^2 u_i + \theta_3\left(\tfrac{3}{4}u_i^3 + \tfrac{1}{2}u_i u_k^2\right) = f_i\cos\omega\tau \quad i, k = 1, 2 \quad k \neq i, \tag{11}$$

where the dot denotes differentiation with respect to τ, and

$$\omega_{i0}^2 = i^2\pi^2(i^2\pi^2 - \sigma) + \theta_1, \quad f_i = 2\int_0^1 f(\zeta)\phi_i \mathrm{d}\zeta$$

are the natural frequencies squared and the excitation amplitudes, respectively.

The differential quadrature analog of equation (9) may be written, using the quadrature rules in the ζ co-ordinate only, as

$$\ddot{w}_i + \sum_{j=1}^{N} L_{ij} w_j + \theta_1 w_i + \theta_3 w_i^3 = f_i \cos\omega\tau \quad i = 1, 2, \ldots, N, \tag{12}$$

where

$$L_{ij} = A_{ij}^{(4)} + \sigma A_{ij}^{(2)}, \quad f_i = f(\zeta_i)$$

and N is the number of the sampling points.

It may be noted that the derivative is total, since $w_i = w_i(\tau)$ at a sampling point $\zeta = \zeta_i$. The boundary conditions are:

$$w_1 = 0 \quad w_1'' = 0,$$

$$w_N = 0 \quad w_N'' = 0,$$

which can be immediately written as:

$$\sum_{j=2}^{N-1} A_{1j}^{(2)} w_j = 0, \quad \sum_{j=2}^{N-1} A_{Nj}^{(2)} w_j = 0. \tag{13, 14}$$

The first two (geometric) boundary conditions, in $\zeta = 0$ and in $\zeta = 1$, have already been imposed by changing the summation limits. The other two (natural) boundary conditions, now expressed by equations (13) and (14), can be used in order to obtain w_2 and $w_{(N-1)}$:

$$w_2 = -\frac{1}{D}\sum_{j=3}^{N-2} E_j w_j, \qquad w_{(N-1)} = -\frac{1}{G}\sum_{j=3}^{N-2} H_j w_j, \tag{15, 16}$$

where

$$D = A_{N2}^{(2)} - \frac{A_{N(N-1)}^{(2)}}{A_{1(N-1)}^{(2)}} A_{12}^{(2)}, \qquad E_j = A_{Nj}^{(2)} - \frac{A_{N(N-1)}^{(2)}}{A_{1(N-1)}^{(2)}} A_{1j}^{(2)},$$

$$G = A_{N(N-1)}^{(2)} - \frac{A_{N2}^{(2)}}{A_{12}^{(2)}} A_{1(N-1)}^{(2)}, \qquad H_j = A_{Nj}^{(2)} - \frac{A_{N2}^{(2)}}{A_{12}^{(2)}} A_{1j}^{(2)}.$$

Finally, w_2 and $w_{(N-1)}$ can be substituted into the equations system (12), giving:

$$\ddot{w}_i + \sum_{j=3}^{N-2} R_{ij} w_j + \theta_3 w_i^3 = f_i \cos\omega\tau \quad i = 3, \ldots, (N-2), \tag{17}$$

where

$$R_{ij} = L_{ij} - \frac{E_j}{D} L_{i2} - \frac{H_j}{G} L_{i(N-1)} + \theta_1 \delta_{ij} \tag{18}$$

and δ_{ij} is the Kronecker operator.

Choosing $N = 6$, one obtains a set of two non-linear ordinary differential equations, coupled in the linear part. As already stated, this sytem will be studied by means of the MLP perturbation method, so that it is necessary to reduce it to the following normalized form

$$\ddot{u}_i + \omega_{i0}^2 u_i + (k_{i1} u_i^3 + k_{i2} u_i^2 u_j + k_{i3} u_i u_j^2 + k_{i4} u_j^3) = f_i \cos \omega\tau \quad i, j = 1, 2 \quad j \neq i \tag{19}$$

and this is possible through a change of reference.

The direction cosines of the new rotated axes are the eigenvectors of the matrix **R**, whose elements are defined by equation (18).

Let **V** be the matrix which has columns that are the eigenvectors of **R**, **u** the new co-ordinates vector, **w** the old co-ordinates vector.

From the relation $\mathbf{u} = \mathbf{V}\mathbf{w}$, one obtains:

$$\begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = \frac{1}{C} \begin{pmatrix} -V_{22} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{20}$$

where $C = -\det(V)$ and $\bar{w}_i = w_{(i+2)}$.

After some algebra, it is possible to write the non-linearity coefficients as follows:

$$k_{11} = -\frac{\theta_3}{C^3} V_{22}^3, \qquad k_{12} = 3\frac{\theta_3}{C^3} V_{22}^2 V_{12}, \qquad k_{13} = -3\frac{\theta_3}{C^3} V_{22} V_{12}^2,$$

$$k_{14} = \frac{\theta_3}{C^3} V_{12}^3,$$

$$k_{21} = -\frac{\theta_3}{C^3} V_{11}^3, \qquad k_{22} = 3\frac{\theta_3}{C^3} V_{11}^2 V_{21}, \qquad k_{23} = -3\frac{\theta_3}{C^3} V_{11} V_{21}^2,$$

$$k_{24} = \frac{\theta_3}{C^3} V_{21}^3.$$

It is obvious that the eigenvalues of **R** are the squares of the natural frequencies ω_{i0} of the normalized system.

Now, let σ_{crit} be the critical value of the dimensionless compressive axial load for the associated linear system

The exact value of σ_{crit} is:

$$\sigma_{crit}^{ex} = \pi^2 \left(1 + \frac{\theta_1}{\pi^4} \right).$$

If $\theta_1 = 1$, then $\sigma_{crit}^{ex} = 9{\cdot}97092$.

Using the quadrature method with six grid points ($N = 6$) and with the classical grid distribution [cf. equation (5)], the non-dimensional critical load is found to be equal to $\sigma_{crit}^{DQM} = 8{\cdot}94676$, which is more than 10% smaller than the exact value. This error implies that the subsequent non-linear analysis will be almost meaningless.

If seven points are employed ($N = 7$), then the non-dimensional critical load becomes $\sigma_{crit}^{DQM} = 10{\cdot}0691$, which is less than 1% larger than the exact value. Nevertheless, as already pointed out, the introduction of other grid points greatly complicates the non-linear analysis, and should be avoided, as far as possible.

If the proposed grid distribution [cf. equation (6)] is employed, the non-dimensional critical loads reported in Table 1 are obtained, from which it is possible to realize the good improvement of the linear results.

4. THE SECOND MODEL: CLAMPED–HINGED BEAM

Now consider a slender clamped–hinged beam, having the same physical and mechanical characteristics of the preceding beam and subjected to the same harmonic exciting force. In this case, however, the beam is considered to be so slender that the shortening of the beam axis cannot be neglected.

The equation of motion is:

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial z^4} - \frac{EA}{2L}\left(\int_0^L \left(\frac{\partial v}{\partial z}\right)^2 \mathrm{d}z\right)\frac{\partial^2 v}{\partial z^2} = F(z)\cos\bar{\omega}t. \tag{21}$$

This equation in terms of dimensionless variables becomes:

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^4 w}{\partial \zeta^4} - \kappa\left(\int_0^1 \left(\frac{\partial w}{\partial \zeta}\right)^2 \mathrm{d}\zeta\right)\frac{\partial^2 w}{\partial \zeta^2} = F(\zeta)\cos\omega\tau, \tag{22}$$

where $\kappa = AL^2/2I$.

In this second case, the ϕ_m appearing in equation (10) are given by:

$$\phi_m = A_m(\sin\lambda_m\zeta - B_m\sinh\lambda_m\zeta) \quad m = 1, 2,$$

where

$$A_m = \left[\frac{1}{2}(1 - B_m^2) + \frac{1}{4\lambda_m}(B_m^2\sinh 2\lambda_m - \sin 2\lambda_m)\right]^{-1/2}, \quad B_m = \frac{\sin\lambda_m}{\sinh\lambda_m}$$

and the $\lambda_m = \sqrt{\omega_{m0}}$ are the roots of the transcendental equation $\tan\lambda_m = \tanh\lambda_m$.

TABLE 1

Non-dimensional critical load of a simply supported beam on non-linear soil

b_3	1·056	1·15	1·18	1·19	1·194	1·195	1·2
σ_{crit}	9·1896	9·6984	9·8740	9·9358	9·9568	9·9778	9·9993

The two Galerkin's equations are:

$$\ddot{u}_n + \omega_{n0}^2 u_n - \sum_{m,p,q=1}^{2} \alpha_{nmpq} u_m u_p u_q = f_n \cos\omega\tau \quad n = 1, 2, \tag{23}$$

where

$$\alpha_{nmpq} = 2\kappa \int_0^1 \frac{\partial u_m}{\partial \zeta}\frac{\partial u_p}{\partial \zeta}\,\mathrm{d}\zeta \int_0^1 \frac{\partial^2 u_q}{\partial \zeta^2}\,\phi_n\,\mathrm{d}\zeta, \quad f_n = 2\int_0^1 f(\zeta)\phi_n\,\mathrm{d}\zeta$$

and

$$\alpha_{1121} = \alpha_{1211}, \qquad \alpha_{1122} = \alpha_{1212},$$

$$\alpha_{2121} = \alpha_{2211}, \qquad \alpha_{2122} = \alpha_{2212}.$$

The system (23) is in normal form with:

$$k_{11} = -\alpha_{1111}, \qquad k_{12} = -(\alpha_{1112} + \alpha_{1121} + \alpha_{1211}),$$

$$k_{13} = -(\alpha_{1122} + \alpha_{1212} + \alpha_{1221}), \qquad k_{14} = -\alpha_{1222},$$

$$k_{21} = -\alpha_{2222}, \qquad k_{22} = -(\alpha_{2122} + \alpha_{2212} + \alpha_{2221}),$$

$$k_{23} = -(\alpha_{2112} + \alpha_{2121} + \alpha_{2211}), \qquad k_{24} = -\alpha_{2111}.$$

Before applying the quadrature method to equation (21), it is convenient to rewrite the integral at the first member of this equation as:

$$\left[\frac{\partial w}{\partial \zeta}\,w\right]_0^1 - \int_0^1 \frac{\partial^2 w}{\partial \zeta^2}\,w\,\mathrm{d}\zeta.$$

The bracket is null, so using the quadrature rules in the space co-ordinate, the following equations are obtained:

$$\ddot{w}_i + \sum_{j=2}^{N-1} A_{ij}^{(4)} w_j + \kappa \sum_{k,l,m=2}^{N-1} C_k A_{kl}^{(2)} A_{im}^{(2)} w_k w_l w_m = f_i \cos\omega\tau \quad i = 2, \ldots, (N-1), \tag{24}$$

where C_k are the weighting coefficients of the integral.

It is also worth noting that the boundary conditions

$$w_1 = 0, \qquad w_N = 0$$

have been already introduced.

The remaining boundary conditions are written as follows:

$$\sum_{j=2}^{N-1} A_{1j}^{(1)} w_j = 0, \qquad \sum_{j=2}^{N-1} A_{Nj}^{(2)} w_j = 0. \tag{25, 26}$$

TABLE 2

Non-dimensional critical load of a clamped–hinged beam

b_3	1·194	1·2	1·22	1·225	1·23
σ_{crit}	19·5703	19·7401	20·094	20·1624	20·3009

From these two equations, one can obtain w_2 and $w_{(N-1)}$, which have the same form as (15) and (16), with:

$$D = A_{N2}^{(2)} - \frac{A_{N(N-1)}^{(2)}}{A_{1(N-1)}^{(1)}} A_{12}^{(1)}, \qquad E_j = A_{Nj}^{(2)} - \frac{A_{N(N-1)}^{(2)}}{A_{1(N-1)}^{(1)}} A_{1j}^{(1)},$$

$$G = A_{N(N-1)}^{(2)} - \frac{A_{N2}^{(2)}}{A_{12}^{(1)}} A_{1(N-1)}^{(1)}, \qquad H_j = A_{Nj}^{(2)} - \frac{A_{N2}^{(2)}}{A_{12}^{(1)}} A_{1j}^{(1)}.$$

Finally, w_2 and $w_{(N-1)}$ have to be substituted into the equations system (24). The weighting coefficients C_k are derived by the Newton–Cotes integration formulas:

$$C_k = \int_0^1 \prod_{\substack{i=1 \\ i \neq k}}^{N} \frac{\zeta - \zeta_i}{\zeta_k - \zeta_i} \, \mathrm{d}\zeta,$$

using unequally spaced sampling points.

After all the substitutions, for $N = 6$, the final form of the system (24) may be written as:

$$\ddot{w}_i + \sum_{j=3}^{4} S_{ij} w_j + \kappa \sum_{k,l,m=3}^{4} T_{iklm} w_k w_l w_m = f_i \cos \omega\tau \quad i = 3, 4, \tag{27}$$

where

$$S_{ij} = A_{ij}^{(4)} - \frac{E_j}{D} A_{i2}^{(4)} - \frac{H_j}{G} A_{i(N-1)}^{(4)}. \tag{28}$$

The T_{iklm} coefficients have been calculated with a simple *Mathematica* code.

The normalization of the system (27) requires the calculation of the eigenvalues and the eigenvectors of the matrix $\mathbf{S}$ whose elements are defined by equation (28).

The new k_{ij} coefficients are more complex than the preceding ones and they have been calculated with the quoted *Mathematica* code.

Imagine now that the beam examined is subjected to an axial load P.

Let σ be the dimensionless compressive axial load and σ_{crit} the critical value of it for the associated linear system.

The exact value of σ_{crit} is $\sigma_{crit}^{ex} = 20{\cdot}1907$.

Using the quadrature method with the distribution (5), for $N = 6$ results $\sigma_{crit}^{DQM} = 16{\cdot}5861$.

Using the quadrature method with the distribution (6), one obtain the results reported in Table 2.

5. NUMERICAL RESULTS

The solutions obtained by using the sampling points generated by the relation (5) are compared with the results obtained by using the distribution given by (7).

The cases $f = 0, f = 1, f = 100$ have been studied. Since f is constant, operating with the quadrature method results in $f_i = f$, whereas with the Galerkin's procedure it results in $f_i = 2f\int_0^1 \phi_i \, d\zeta$.

Fundamental resonance with $\omega \approx \omega_{10}$ has been considered.

Figures 1 and 2 show the relationship frequency–initial amplitude of the first mode (A_{10}) for the examined cases and $f = 0$.

The curves obtained for $f > 0$ are not reported, because only for $f = 100$ and for small initial amplitudes A_{10}, the discrepancies become noticeable.

5.1. FIRST MODEL

Assume $\sigma = 0{\cdot}1$ and $\theta_1 = \theta_3 = 1$. The matrix $\mathbf{R}$ reflects the system symmetry with the peculiarity of having equal diagonal elements. It will be:

$$\mathbf{R} = \begin{pmatrix} 516{\cdot}2777 & -429{\cdot}6952 \\ -429{\cdot}6952 & 516{\cdot}2777 \end{pmatrix}$$

if the distribution (5) is adopted and:

$$\mathbf{R} = \begin{pmatrix} 601{\cdot}7479 & -503{\cdot}9223 \\ -503{\cdot}9223 & 601{\cdot}7479 \end{pmatrix}$$

if the distribution (7) is chosen.

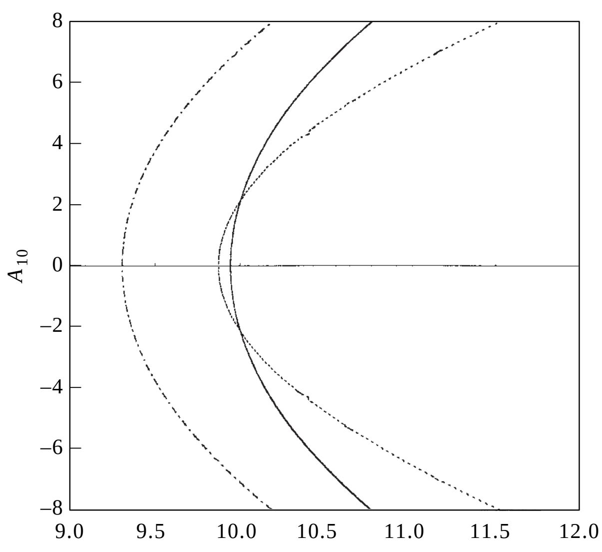

Figure 1. Free vibration response ω–A_{10} for the simply supported beam examined. — ·—·, Distribution (5); —— distribution (7); ---- Galerkin method.

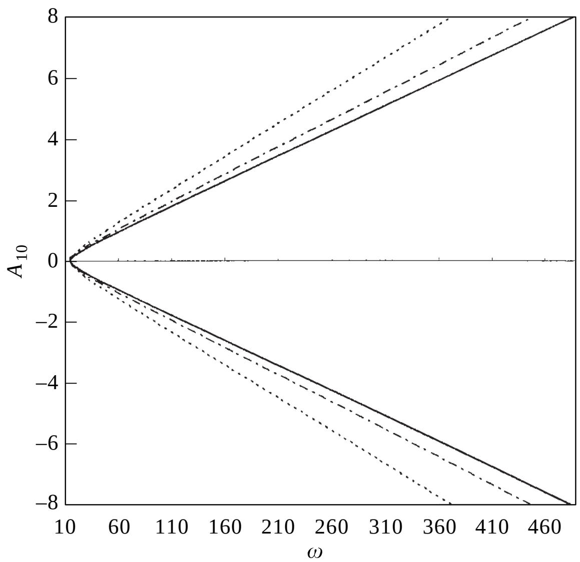

Figure 2. Free vibration response ω–A_{10} for the clamped–hinged beam examined. — ·—·, Distribution (5); —— distribution (7); ---- Galerkin method.

The natural frequencies are: for the distribution (5), $\omega_{10} = 9{\cdot}305$ and $\omega_{20} = 30{\cdot}7568$; for the distribution (7), $\omega_{10} = 9{\cdot}9411$ and $\omega_{20} = 33{\cdot}2666$. As one can see, $\omega_{20} \approx 3\omega_{10}$, which is the condition of internal resonance.

The non-linear coefficients are, for both cases:

$$k_{11} = 0{\cdot}35355, \qquad k_{12} = 1{\cdot}06066, \qquad k_{13} = k_{12}, \qquad k_{14} = k_{11},$$

$$k_{21} = k_{11}, \qquad k_{22} = -k_{12}, \qquad k_{23} = k_{12}, \qquad k_{24} = -k_{11},$$

There is a σ value (σ_0) close to 22·8 for which $R_{ii} = 0$.

If $\sigma < \sigma_0$, the results are $R_{ii} > 0$ and $R_{ij} < 0$; if $\sigma > \sigma_0$, then $R_{ii} < 0$ and $R_{ij} < 0$, but only if $\sigma \leqslant \sigma_{crit}$ the frequencies are real. In particular, if $\sigma = \sigma_{crit}$ all the **R** elements are equal in absolute value so that one frequency is null. If $\sigma > \sigma_{crit}$ one frequency becomes imaginary.

Finally, as expected, as σ is increased, the frequency values reduce since all the **R** elements in absolute value reduce.

The application of Galerkin's method produces the following results:

$$\omega_{10} = 9{\cdot}8703, \qquad \omega_{20} = 39{\cdot}4411.$$

5.2. SECOND MODEL

It is assumed that $\kappa = 12$.

The quadrature method with the distribution (5) leads to the following results:

$$\mathbf{S} = \begin{pmatrix} 1309{\cdot}5985 & -752{\cdot}1546 \\ -478{\cdot}8968 & 536{\cdot}9786 \end{pmatrix}$$

$$\omega_{10} = 14{\cdot}4754, \qquad \omega_{20} = 40{\cdot}4603,$$

$$k_{11} = 3847{\cdot}14, \qquad k_{12} = 4473{\cdot}144, \qquad k_{13} = 2764{\cdot}44, \qquad k_{14} = 8228{\cdot}388,$$

$$k_{21} = 56{\cdot}352, \qquad k_{22} = -1398{\cdot}684, \qquad k_{23} = -1365{\cdot}696, \qquad k_{24} = -3582{\cdot}516.$$

The quadrature method with the distribution (7) gives:

$$\mathbf{S} = \begin{pmatrix} 1604{\cdot}0068 & -766{\cdot}903 \\ -747{\cdot}4094 & 667{\cdot}4618 \end{pmatrix}$$

$$\omega_{10} = 15{\cdot}6693, \qquad \omega_{20} = 45{\cdot}0105,$$

$$k_{11} = 5083{\cdot}89, \qquad k_{12} = 5903{\cdot}25, \qquad k_{13} = 3449{\cdot}23, \qquad k_{14} = 1157{\cdot}99,$$

$$k_{21} = -220{\cdot}258, \qquad k_{22} = -2312{\cdot}26, \qquad k_{23} = -3532{\cdot}11, \qquad k_{24} = -5738{\cdot}42.$$

The results obtained by the application of the Galerkin's method are:

$$\omega_{10} = 15{\cdot}4182, \qquad \omega_{20} = 49{\cdot}8238,$$

$$k_{11} = 3172{\cdot}632, \qquad k_{12} = 3539{\cdot}4696, \qquad k_{13} = 12\,684{\cdot}6096, \qquad k_{14} = 4390{\cdot}776,$$

$$k_{21} = 43\,940{\cdot}88, \qquad k_{22} = 13\,172{\cdot}328, \qquad k_{23} = k_{13}, \qquad k_{24} = 1179{\cdot}8232.$$

6. CONCLUSIONS

In this paper a convenient approach is proposed, in order to study the non-linear dynamics of continuous systems taking into account the internal resonances and other phenomena which are neglected in the usual reduction to a single-degree-of-freedom oscillator. In particular, attention is drawn to two-degree-of-freedom systems. So, the initial-boundary value problem is first discretized by using the generalized differential quadrature method, with an optimized choice of the sampling points. In this way, the non-linear partial differential equation reduces to a set of two non-linear ordinary differential equations. The problem, so governed by two coupled equations, is then investigated in detail by using a modified version of the Lindstedt–Poincarè method, in which the non-linearity needs not to be small.

As an example, two simple structural models have been investigated by using three different methods, and the numerical results show that the proposed approach behaves quite satisfactorily.

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