Journal of Sound and Vibration (1998) **218**(4), 659–696
Article No. 981854

COMPOSITE ELEMENT METHOD FOR VIBRATION ANALYSIS OF STRUCTURE, PART II: C^1 ELEMENT (BEAM)

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(*Received* 20 *May* 1997, *and in final form* 29 *June* 1998)

This paper is the second in a series of two devoted to a detailed study of Composite Element Method for vibration analysis of structures. The first paper focused on the principle and C^0 element of the Composite Element Method. The present one concentrates on developing the beam element of the Composite Element Method. The related contents cover the expression of displacement field, the stiffness matrix and the consistent mass matrix of beam element, and transformation matrix. Especially, the detailed numerical verifications for the beam element of Composite Element Method are presented, which involve the *h*-version and the *c*-version. Also, some applications to the vibration analyses of lathe, automobile and frame are given in detail.

1. INTRODUCTION

As has been shown in Part I of this series of papers, the Composite Element Method is a new approach combining the conventional FEM and the classical theory, with the goal of utilizing the advantages of both FEM and classical theory, i.e., the former's versatility and the latter's closed analytical solution. This paper addresses the derivation of the beam element of Composite Element Method, covering the formulation of stiffness and mass matrices, the *h*-version, the *c*-version, the superconvergence, and the applications to the vibration analysis of lathe, automobile and frame.

2. DISPLACEMENT FIELD

Consider a bending beam as shown in Figure 1 where the local x-axis is taken in the axial direction of the element with origin at a corner (or local node) 1. Assume that the rotary inertia and shear deformation can be neglected. We construct the displacement field function $W(x)$ as:

$$W(x) = W_{FEM}(x) + W_{CT}(x) \tag{1}$$

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● Derivation of $W_{FEM}(x)$ by FEM

Since there are four nodal displacements v_1, θ_1, v_2, θ_2, we assume a cubic displacement model for $W_{FEM}(x)$, which can be expressed as [1]

$$\begin{aligned} W_{FEM}(\xi) &= v_1(1-3\xi^2+2\xi^3)+\theta_1 l(\xi-2\xi^2+\xi^3)+v_2(3\xi^2-2\xi^3)+\theta_2 l(\xi^3-\xi^2) \\ &= \mathbf{N}(\xi)\mathbf{q} \end{aligned} \tag{2}$$

where

$$\mathbf{N}(\xi) = [(1-3\xi^2+2\xi^3) \quad l(\xi-2\xi^2+\xi^3) \quad (3\xi^2-2\xi^3) \quad l(\xi^3-\xi^2)] \tag{3}$$

$$\mathbf{q} = [v_1 \ \theta_1 \ v_2 \ \theta_2]^{\mathrm{T}}$$

$$\xi = \frac{x}{l}. \tag{4}$$

● Derivation of $W_{CT}(x)$ by classical theory

We know that the Composite Element requires an analytical solution under the coupling boundary conditions. Here, for the beam element of Composite Element Method, these coupling boundary conditions should be

$$\begin{aligned} &W_r(x)|_{x=0} = 0, \qquad W_r(x)|_{x=l} = 0 \\ &W_r'(x)|_{x=0} = 0, \qquad W_r'(x)|_{x=l} = 0 \qquad r = 1, 2, 3, \ldots \end{aligned} \tag{5}$$

where $W_r(x)$ is the permissible displacement functions of deflection. Actually, this is the case of the clamped-clamped beam [see Figure 1(b)]. The corresponding solution under the condition (5) can yield the characteristic equation for natural frequency

$$\cos \lambda_r^* \cdot \cosh \lambda_r^* = 1, \qquad r = 1, 2, 3, \ldots \tag{6}$$

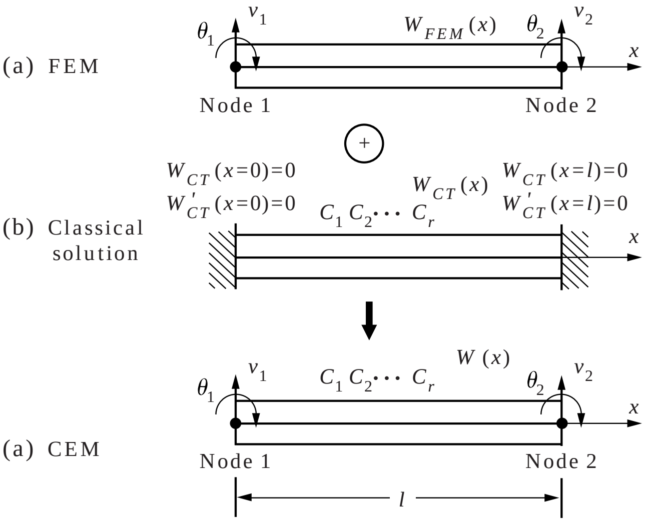


Figure 1. Constructing of CEM for bending beam.

and the eigenfunctions (natural mode shapes) are found to be

$$
\begin{aligned}
W_r(x) &= c_r \cdot \mathscr{F}_r(\lambda_r^*, x) \\
&= c_r\left[\sin \lambda_r^* \frac{x}{l} - \sinh \lambda_r^* \frac{x}{l} - \frac{\sin \lambda_r^* - \sinh \lambda_r^*}{\cos \lambda_r^* - \cosh \lambda_r^*}\left(\cos \lambda_r^* \frac{x}{l} - \cosh \lambda_r^* \frac{x}{l}\right)\right] \quad (7)
\end{aligned}
$$

where c_r are a set of constants. Finally, the solution of dynamic problem, $w(x, t)$, is given as

$$
\begin{aligned}
w(x, t) &= W_r(x) \cdot G_r(t) \\
&= c_r \mathscr{F}_r(\lambda_r^*, x) \cdot G_r(t) \\
&= c_r\left[\sin \lambda_r^* \frac{x}{l} - \sinh \lambda_r^* \frac{x}{l} - \frac{\sin \lambda_r^* - \sinh \lambda_r^*}{\cos \lambda_r^* - \cosh \lambda_r^*}\left(\cos \lambda_r^* \frac{x}{l} - \cosh \lambda_r^* \frac{x}{l}\right)\right] \\
&\quad \cdot \sin \omega_r^* t \qquad (8)
\end{aligned}
$$

where

$$
\omega_r^{*2} = \frac{EI}{\rho l^4} \lambda_r^{*4}, \qquad r = 1, 2, 3, \ldots . \qquad (9)
$$

Note that $W_r(x)$ are a set of natural mode functions, which will be combined or embedded into the displacement field of the bending beam element in CEM together with the interpolation polynomial function of the conventional FEM.

So, as to the $W_{CT}(x)$ of equation (1), we take

$$
\begin{aligned}
W_{CT}(x) &= \sum_{r=1}^{n} c_r \mathscr{F}_r(\lambda_r^*, x) \\
&= \phi(x)\mathbf{c} \qquad (10)
\end{aligned}
$$

where

$$
\phi(\xi) = [\mathscr{F}_1(\lambda_1^*, \xi) \;\; \mathscr{F}_2(\lambda_2^*, \xi) \;\; \cdots \;\; \mathscr{F}_r(\lambda_r^*, \xi)] \qquad (11)
$$

$$
\mathbf{c} = [c_1 \;\; c_2 \;\; \cdots \;\; c_r]^{\mathrm{T}} \qquad (12)
$$

$$
\mathscr{F}_i(\lambda_i^*, x) = \sin \lambda_i^* \frac{x}{l} - \sinh \lambda_i^* \frac{x}{l} - \left(\frac{\sin \lambda_i^* - \sinh \lambda_i^*}{\cos \lambda_i^* - \cosh \lambda_i^*}\right)\left(\cos \lambda_i^* \frac{x}{l} - \cosh \lambda_i^* \frac{x}{l}\right)
$$

$$
i = 1, 2, 3, \ldots \qquad (13)
$$

and the λ_i^* satisfies equation (6).

- **Combination of $W_{FEM}(x)$ and $W_{CT}(x)$**

According to equation (1), we combine $W_{FEM}(x)$ and $W_{CT}(x)$ into $W(x)$, i.e.,

$$\begin{aligned} W(\xi) &= W_{FEM}(x) + W_{CT}(x) \\ &= v_1(1-3\xi^2+2\xi^3) + \theta_1 l(\xi-2\xi^2+\xi^3) + v_2(3\xi^2-2\xi^3) + \theta_2 l(\xi^3-\xi^2) \\ &\quad + c_1\mathscr{F}_1(\lambda_1^*,\xi) + c_2\mathscr{F}_2(\lambda_2^*,\xi) + \cdots + c_r\mathscr{F}_r(\lambda_r^*,\xi) \\ &= \mathbf{N}(\xi)\mathbf{q} + \phi(\xi)\mathbf{c} \\ &= \mathbf{S}(\xi)\cdot\delta \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathbf{S}(\xi) &= [\mathbf{N}(\xi)\;\; \phi(\xi)] \\ &= [(1-3\xi^2+2\xi^3) \quad l(\xi-2\xi^2+\xi^3) \quad (3\xi^2-2\xi^3) \quad l(\xi^3-\xi^2) \\ &\qquad \mathscr{F}_1(\lambda_1^*,\xi) \quad \mathscr{F}_2(\lambda_2^*,\xi)\cdots\mathscr{F}_r(\lambda_r^*,\xi)] \end{aligned} \tag{15}$$

$$\begin{aligned} \delta &= [\mathbf{q}^{\mathrm{T}}\;\; \mathbf{c}^{\mathrm{T}}] \\ &= [v_1\;\; \theta_1\;\; v_2\;\; \theta_2\;\; c_1\;\; c_2\;\; \cdots\;\; c_r]^{\mathrm{T}} \end{aligned} \tag{16}$$

and the generalized shape function matrix and the coordinates (or DOF) of CEM respectively. From the beading theory of beam, the axial strain induced in the element is given by

$$\varepsilon = -y\frac{\partial^2 W(x)}{\partial x^2} \tag{17}$$

where y is the distance from the neural axis. From equation (14) the strain–displacement relation can be expressed as

$$\begin{aligned} \varepsilon &= -y\frac{\partial^2 W(x)}{\partial x^2} \\ &= -\frac{y}{l^2}\frac{\partial^2 \mathbf{S}(\xi)}{\partial \xi^2}\cdot\delta = \mathbf{B}(\xi)\cdot\delta \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathbf{B}(\xi) = -y\Bigg[&\frac{1}{l^2}(12\xi-6) \quad \frac{l}{l}(6\xi-4) \quad -\frac{1}{l^2}(12\xi-6) \quad \frac{l}{l}(6\xi-2) \\ &\mathscr{F}_1''(\lambda_1^*,\xi)\;\; \mathscr{F}_2''(\lambda_2^*,\xi)\;\; \cdots\;\; \mathscr{F}_r''(\lambda_r^*,\xi)\Bigg] \end{aligned} \tag{19}$$

$$\begin{aligned} \mathscr{F}_r''(\lambda_r^*,\xi) = -\frac{\lambda_r^{*2}}{l^2}\Bigg[&\sin\lambda_r^*\xi + \sinh\lambda_r^*\xi \\ &-\left(\frac{\sin\lambda_r^* - \sinh\lambda_r^*}{\cos\lambda_r^* - \cosh\lambda_r^*}\right)(\cos\lambda_r^*\xi + \cosh\lambda_r^*)\Bigg] \qquad r = 1, 2, 3, \ldots \end{aligned} \tag{20}$$

and the λ_r^* satisfies equation (6).

3. STIFFNESS MATRIX AND CONSISTENT MASS MATRIX

When the matrix **S** of shape function and the matrix **B** of strain–displacement relation of composit element are available, we can derive the stiffness matrix and the consistent mass matrix by the following expressions [2]

$$\mathbf{k}^e = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}V \tag{21}$$

$$\mathbf{m}^e = \int_V \rho\mathbf{S}^{\mathrm{T}}\mathbf{S}\,\mathrm{d}V \tag{22}$$

where **D** is the elastic matrix, it will be equal to E (Young's modulus) in the case of planar beam element in the case of pure bending, and the superscript e denotes for each element. We calculate the stiffness matrix $\mathbf{k}^e$ of the element according to the expression (21) as

$$\begin{aligned} \mathbf{k}^e &= \int_V \mathbf{B}^{e\mathrm{T}}\mathbf{D}^e\mathbf{B}^e\,\mathrm{d}V \\ &= E\int_A\int_0^l \mathbf{B}^{e\mathrm{T}}\mathbf{B}^e\,\mathrm{d}x\,\mathrm{d}A \\ &= \frac{EI}{l^3}\cdot \begin{array}{cc} \begin{array}{ccccc} v_1 & \theta_1 & v_2 & \theta_2 & c \end{array} & \\ \left[\begin{array}{cccc|c} 12 & & & & \\ 6l & 4l^2 & & sym. & \\ -12l & -6l & 12 & & 0 \\ 6l & 2l^2 & -6l & 4l^2 & \\ \hline & & 0 & & \mathbf{k}_{cc} \end{array}\right] & \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ c \end{array} \end{array} \end{aligned} \tag{23}$$

where $I = \int_A \bar{y}^2\,\mathrm{d}A$ and $\mathbf{k}_{cc}$ is given by

$$\mathbf{k}_{cc} = \begin{array}{cc} \begin{array}{ccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & \cdots & c_r \end{array} & \\ \left[\begin{array}{ccccccc} 1{\cdot}035936\lambda_1^{*4} & & & & & & \\ & 0{\cdot}998447\lambda_2^{*4} & & & 0 & & \\ & & 1{\cdot}000067\lambda_3^{*4} & & & & \\ & & & 0{\cdot}9999971\lambda_4^{*4} & & & \\ & & 0 & & 1{\cdot}0\lambda_5^{*4} & & \\ & & & & & \ddots & \\ & & & & & & 1{\cdot}0\lambda_r^{*4} \end{array}\right] & \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_r \end{array} \end{array} \tag{24}$$

in which $\lambda_1^*, \ldots, \lambda_r^*$ are the constants, i.e.,

$$\begin{aligned}
\lambda_1^* &= 4{\cdot}730041\\
\lambda_2^* &= 7{\cdot}853205\\
\lambda_3^* &= 10{\cdot}995608\\
\lambda_4^* &= 14{\cdot}137165\\
&\vdots\\
\lambda_r^* &= (r + 0{\cdot}5)\pi, \qquad r \geqslant 4.
\end{aligned} \tag{25}$$

Similarly, from the expression (22) the consistent mass matrix $\mathbf{m}^e$ of the element is given as

$$\begin{aligned}
\mathbf{m}^e &= \int_V \rho \mathbf{S}^{e\mathrm{T}} \mathbf{S}^e \, \mathrm{d}V\\
&= \rho A \int_0^l \mathbf{S}^{e\mathrm{T}} \mathbf{S}^e \, \mathrm{d}x\\
&= \rho A l \cdot \begin{array}{c} \\ \left[\begin{array}{ccccc} v_1 & \theta_1 & v_2 & \theta_2 & c \\ \dfrac{13}{35} & & & & \\ \dfrac{11}{210} l & \dfrac{1}{105} l^2 & & sym. & \\ \dfrac{9}{70} & \dfrac{13}{420} l & \dfrac{13}{35} & & sym. \\ -\dfrac{13}{420} l & -\dfrac{1}{140} l^2 & -\dfrac{11}{210} l & \dfrac{1}{105} l^2 & \\ & & \mathbf{m}_{qc} & & \mathbf{m}_{cc} \end{array} \right] \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ c \end{array}
\end{aligned} \tag{26}$$

where $\mathbf{m}_{cc}$ is given as

$$\mathbf{m}_{cc} = \left[\begin{array}{ccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & \cdots & c_r \\ 1{\cdot}035936 & & & & & & \\ & 0{\cdot}998447 & & & & 0 & \\ & & 1{\cdot}000067 & & & & \\ & & & 0{\cdot}9999971 & & & \\ & & 0 & & 1{\cdot}0 & & \\ & & & & & \ddots & \\ & & & & & & 1{\cdot}0 \end{array} \right] \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_r \end{array} \tag{27}$$

and $\mathbf{m}_{qc}$ is

$$\mathbf{m}_{qc} = \begin{array}{c} \\ \left[\begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \\ 0{\cdot}42282930 & 0{\cdot}09098435l & 0{\cdot}42282930 & -0{\cdot}09098435l \\ 0{\cdot}25467310 & 0{\cdot}03240400l & -0{\cdot}25467310 & 0{\cdot}03240400l \\ 0{\cdot}18189080 & 0{\cdot}01654269l & 0{\cdot}18189080 & -0{\cdot}01654269l \\ 0{\cdot}14147110 & 0{\cdot}01000702l & -0{\cdot}14147110 & 0{\cdot}01000702l \\ 0{\cdot}11574900 & 0{\cdot}00669892l & 0{\cdot}11574900 & -0{\cdot}00669892l \\ \vdots & \vdots & \vdots & \vdots \end{array}\right] \end{array} \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \end{array} \qquad (28)$$

Note that the generalized coordinate δ^e is composed of two parts: the nodal coordinate $\mathbf{q}$ (or nodal DOF) and the c-coordinate (or c-DOF) $\mathbf{c}$:

$$\delta^e = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad | \quad c_1 \quad c_2 \quad \cdots \quad c_r]^{\mathrm{T}}. \qquad (29)$$

Stiffness and mass matrices of the planar beam element in CEM possess same properties as those of the conventional FEM [3], i.e.,

(1) Both the stiffness matrix and the mass matrix in the Composite Element Method are symmetric.

(2) The stiffness matrix in the Composite Element Method is positive semi-definite. Also, after elimination of rigid body motion, a stiffness matrix will be positive definite.

(3) The diagonal elements of both stiffness matrix and mass matrix are always positive.

4. SPATIAL BEAM ELEMENT AND COORDINATE TRANSFORMATION

4.1. SPATIAL BEAM ELEMENT

A spatial beam element is a straight beam of uniform cross section which is capable of resisting axial forces, bending moments about the two principal axes in the plane of its cross section and twisting moment about its centroidal axis. The corresponding displacement degrees of freedom are shown in Figure 2.

If the local axes (xyz system) are chosen to coincide with the principal axes of the cross section, it is easier to construct the mass and stiffness matrices. According to the engineering theory of bending and torsion of beams, the axial displacements q_1 and q_4 depend only on the axial forces, and the torsional displacements θ_1 and θ_4 depend only on the torsional moments. However, for arbitrary choice of the xyz coordinate system, the bending displacements in the xy plane, namely, q_2, θ_2, q_5 and θ_5 depend not only on the bending forces acting in that plane (i.e., shear forces acting in the y-direction and the bending moments acting in the xz plane), but also on the bending forces acting in the plane xz. On the other hand, if the xy, and xz planes coincide with the principle axes of the cross section, the bending displacements and forces in the two planes can be considered to be independent

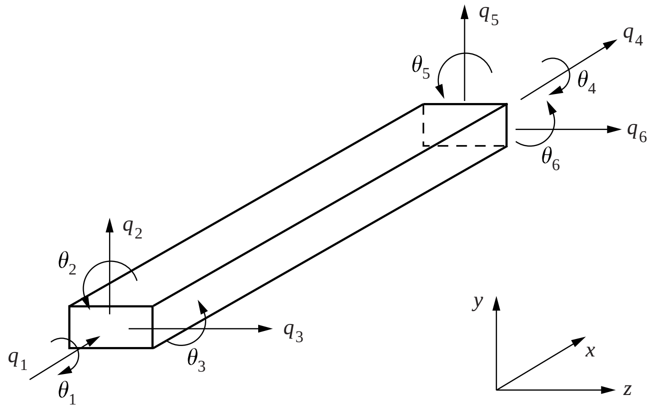


Figure 2. A spatial beam element.

of each other. In this section we shall choose the local xyz coordinate system to coincide with the principle axes of the cross section with the x-axis representing the centroidal axis of the spatial beam element. Thus the displacements can be separated into four groups each of which can be considered independently of others.

We will first utilize the mass and stiffness matrices corresponding to different independent sets of displacements, which have been derived in the above, and then obtain the total mass and stiffness matrices of the element by superposition.

4.2. COORDINATE TRANSFORMATION

As we know, the element characteristics are calculated in the local coordinate systems suitably oriented for minimizing computational effort. However, the local coordinate system may be different for different elements. In such a case, before the element equations can be assembled, it is necessary to transform the element equations derived in local coordinate systems so that all the elemental equations are referred to a common global coordinate system.

In order to find the stiffness matrix and the mass matrix of the bar element of the Composite Element Method in the global coordinate system, we need to search the transformation matrix. Let a transformation matrix $\mathbf{T}^e$ exist between the local and the global coordinate systems such that

$$\delta^e = \mathbf{T}^e \overline{\delta}^e \tag{30}$$

where $\overline{\delta}^e$ is the generalized coordinates in the global coordinate system. The stiffness matrix $\mathbf{K}^e$ and the mass matrix $\mathbf{M}^e$ of the element corresponding to the global coordinate system are given as [4]

$$\mathbf{K}^e = \mathbf{T}^{e\mathrm{T}} \mathbf{k}^e \mathbf{T}^e \tag{31}$$

$$\mathbf{M}^e = \mathbf{T}^{e\mathrm{T}} \mathbf{m}^e \mathbf{T}^e \tag{32}$$

The main properties of transformation can be found as:

(1) The transformation is carried out only for nodal coordinate, not for c-coordinate. The reason for this is that the c-coordinate is always defined in the

local coordinate system in a closed form, contributes only to the internal displacement field of the element and does not therefore influence its edge displacements.

(2) Since the transformation matrix $\mathbf{T}^e$ is the matrix of direction cosines relating the two coordinate systems, it is orthogonal.

5. NUMERICAL VERIFICATION

5.1. A FREE-CLAMPED BEAM

Now, we give a validation for Composite Element Method, using a free-clamped beam as an example. The contents include: discretization of 1, 2 as well as 4 elements, effect of the number of c-DOF.

Consider the bending vibration of a free-clamped beam as shown in Figure 3(a). L is the length of beam, ρ, E are the mass density and Young's modulus respectively. Now, we idealize this beam into 1 element, 2 elements, and 4 elements, and then apply CEM to calculate the natural frequencies.

Let

$$\lambda_i^4 = \frac{\rho A L^4}{EI}\,\omega_i^2, \qquad i = 1, 2, \ldots \tag{33}$$

where ω_i is the natural frequencies. We present the results below.

5.1.1. *Discretization of* 1 *element*

If we take total beam as 1 beam element, then consider several calculating schemes wherein 1c-DOF, 4c-DOF, 6c-DOF, 10c-DOF and 16c-DOF are chosen. Various order of eigenvalues λ_i resulting from the above calculating schemes are presented in Table 1, in comparison with the exact solutions.

From the results shown in Table 1, we can see that the resultant eigenvalues from λ_1 to λ_n within the scope of the c-DOF number in each scheme will be very close to the exact solution (the maximum relative error <0·05%). For example,

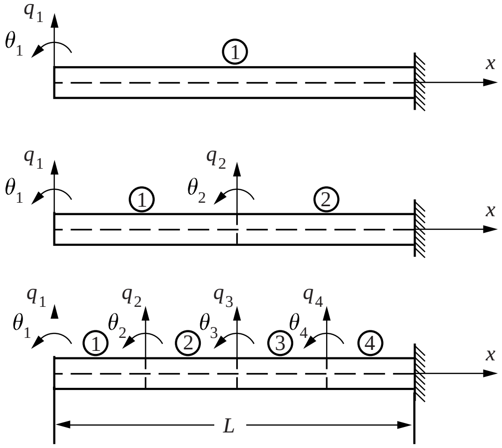


Figure 3. A free-clamped beam and its discretization.

in the scheme of CEM ($1 \times 16c$), i.e., using one composite element with $16c$-DOF, the total-DOF is 18 (i.e. c-DOF plus nodal DOF), the resultant eigenvalues from λ_1 to λ_{16} are very close to the exact solution, and the maximum relative error (i.e., λ_{16}) only reaches 0·03575%.

5.1.2. *Discretization of* 2 *elements*

Now, we idealize this free-clamped beam into 2 elements [shown in Figure 3(b)]. Consider several calculating schemes wherein FEM, $1c$-DOF, $2c$-DOF are chosen. Various orders of eigenvalues λ_i in various schemes are presented in Table 2, which are compared with the exact solution.

The results of Table 2 also show that the calculated eigenvalues from λ_1 to λ_n within the scope of c-DOF number in each scheme approximate the exact solution very well (the maximum relative error $<$0·5%). For instance, in the scheme of CEM ($2 \times 2c$), i.e., using two composite elements with $2c$-DOF, the total-DOF is 8 (i.e., c-DOF plus nodal DOF), the resulted eigenvalues from λ_1 to λ_4 are very close to the exact solution, and the maximum relative error (i.e., λ_4) only reaches 0·25692%.

Obviously, a comparison of Tables 1 and 2 shows that, in the case of an equal amount of computational efforts, the results of the scheme with 2 elements discretization are not better than those with 1 element discretization.

5.1.3. *Discretization of* 4 *elements*

Here, we idealize this free-clamped beam into 4 elements shown in Figure 3(c), and also consider several calculating schemes: FEM, $1c$-DOF, $2c$-DOF. Various orders of eigenvalues λ_i in various schemes are presented in Table 3, which are compared with the exact solution.

The results of Table 3 also show that the calculated eigenvalues from λ_1 to λ_n within the scope of the c-DOF number in each scheme approximate the exact solution very well (the maximum relative error $<$0·5%). For instance, in the scheme of CEM ($4 \times 2c$), i.e., using four composite elements with $2c$-DOF, the total-DOF is 16 (i.e., c-DOF plus nodal DOF), the resulting eigenvalues from λ_1 to λ_8 are very closed to the exact solution, and the maximum relative error (i.e., λ_8) only reaches 0·44593%.

Obviously, a comparison of Tables 1, 2 and 3 shows that, in the case of an equal amount of effort, the results of the scheme with 4 elements discretization are not better than those with 1 or 2 element discretization.

5.1.4. *Effect of number of c-DOF*

If we fix the number of total DOF as 12, then consider several schemes: CEM ($1 \times 10c$) (i.e., total c-DOF is 10), CEM ($3 \times 2c$) (i.e., total c-DOF is 6), CEM ($4 \times 1c$) (i.e., total c-DOF is 4), FEM (i.e., total c-DOF is 0). The purpose of doing so is to investigate the effect of the number of c-DOF on eigenvalue with the computational effort remaining the same. The detailed results and comparisons are presented in Table 4.

From Table 4, we find that the accuracy achieved by CEM is superior to that by the conventional FEM. Moreover, the scheme with more c-DOF is superior

TABLE 1

λ_i of various schemes in case of 1 element discretization

Order	Exact	CEM ($1 \times 1c$)* c-DOF:1 Total-DOF:3	CEM ($1 \times 4c$)* c-DOF:4 Total-DOF:6	CEM ($1 \times 6c$)* c-DOF:6 Total-DOF:8	CEM ($1 \times 10c$)* c-DOF:10 Total-DOF:12	CEM ($1 \times 16c$)* c-DOF:16 Total-DOF:18
λ_1	1·875104	1·875429	1·875109	1·875105	1·875104	1·875104
λ_2	4·694091	4·703879	4·694419	4·694165	4·694100	4·694092
λ_3	7·854757		7·857543	7·855485	7·854857	7·854769
λ_4	10·99554		11·00451	10·99836	10·99599	10·99560
λ_5	14·13717		14·15447	14·14405	14·13845	14·13736
λ_6	17·27876			17·29133	17·28159	17·27923
λ_7	20·42035			20·43861	20·42555	20·42132
λ_8	23·56195				23·57028	23·56367
λ_9	26·70354				26·71556	26·70635
λ_{10}	29·84513				29·86092	29·84937
λ_{11}	32·98672				33·00567	32·99274
λ_{12}	36·12832					36·13641
λ_{13}	39·26991					39·28030
λ_{14}	42·41150					42·42432
λ_{15}	45·55309					45·56833
λ_{16}	48·69469					48·71210
λ_{17}	51·83628					51·85543

* Note: the symbol CEM ($1 \times 1c$) of Table 1 means using one composite element with $1c$-DOF, CEM ($1 \times 4c$) means using one composite element with $4c$-DOF, and so on.

TABLE 2

λ_i of various schemes in case of 2 elements discretization

Order	Exact	FEM (2e)* *c*-DOF:0 Total-DOF:4	CEM (2 × 1*c*) *c*-DOF:2 Total-DOF:6	CEM (2 × 2*c*) *c*-DOF:4 Total-DOF:8
λ_1	1·875104	1·875557	1·875111	1·875107
λ_2	4·694091	4·713966	4·697333	4·694305
λ_3	7·854757	8·669320	7·870405	7·855822
λ_4	10·99554	14·76957	11·11274	11·02379
λ_5	14·13717			14·18547
λ_6	17·27876			17·51730
λ_7	20·42035			24·94674
λ_8	23·56195			33·88184

* Note: the symbol FEM (2e) of Table 2 denotes using 2 beam elements of the conventional FEM, CEM (2 × 2*c*) means using 2 composite element with 2*c*-DOF each.

to that with less *c*-DOF. Note that the above comparison is based on the same computational effort used (i.e., total-DOF of each scheme is 12). If we compare the errors of each scheme with the exact solution, e.g., as to λ_8, the relative error of the CEM (1 × 10*c*) scheme is 0·03535%, and the error of the FEM (6e) scheme already reaches 7·6085%; as to λ_6, the relative error of the CEM (1 × 10*c*) scheme is 0·01638%, that of the CEM (3 × 2*c*) scheme is 0·3732%, and that of the FEM (6e) scheme already reaches 0·83009%; as to λ_4, the relative error of the CEM (1 × 10*c*) scheme is 0·0040925%, that of the CEM (3 × 2*c*) scheme is 0·00864%, that of the CEM (4 × 1*c*) scheme is 0·1489%, and that of the FEM (6e) scheme already reaches 0·32177%; as to λ_1, the relative error of the CEM (1 × 10*c*) scheme is 0·0%, that of the CEM (3 × 2*c*) scheme is 0·0%, the relative error by CEM (4 × 1*c*) scheme is 0·0%, and that of the FEM (6e) scheme already reaches 0·000907%.

TABLE 3

λ_i of various schemes in case of 4 elements discretization

Order	Exact	FEM (4e)* *c*-DOF:0 Total-DOF:8	CEM (4 × 1*c*) *c*-DOF:4 Total-DOF:12	CEM (4 × 2*c*) *c*-DOF:8 Total-DOF:16
λ_1	1·875104	1·875133	1·875104	1·875104
λ_2	4·694091	4·696825	4·694152	4·694110
λ_3	7·854757	7·885103	7·856252	7·854947
λ_4	10·99554	11·07509	11·01191	10·99633
λ_5	14·13717	15·10421	14·15534	14·13819
λ_6	17·27876	19·14127	17·40013	17·28335
λ_7	20·42035	24·10067	20·73256	20·44178
λ_8	23·56195	30·87079	23·93568	23·66702

* Note: the symbol FEM (4e) of Table 3 denotes using 4 beam elements of the conventional FEM, CEM (4 × 1*c*) means using 4 composite element with 1*c*-DOF each, and so on.

5.1.5. *Brief remarks*

We briefly summarize some features of the *c*-DOF for the dynamic analysis of a bending beam, according to the above numerical results.

- The ability of the *c*-DOF to improve accuracy is greatly superior to that of the traditional node-DOF of FEM.
- The eigenvalues achieved by the *c*-DOF approximate very well the exact solution within the scape of *c*-DOF number (e.g., from Table 4, in the case of the same computational efforts used, the CEM ($1 \times 10c$) scheme brings about only the relative error of 0·03535% for λ_8, but the FEM (6e) scheme produces that of 7·6085%)
- Increasing the number of the *c*-DOF and decreasing the number of element (i.e., node-DOF) will efficiently improve the accuracy for dynamic analysis of structure. The numerical examples also show that an increase of *c*-DOF can lead to a superconvergence
- In the case of multi-discretization elements, the scheme with well-allocated *c*-DOF for each element is superior to other schemes.

5.2. COMPARISONS BETWEEN CEM AND FEM

We will compare the CEM and FEM regarding their *h*-version and *c*-version. Below all comparisons are symbolized by the computational effort (i.e., total-DOF).

5.2.1. *h-version*

The *h*-version of CEM is completely similar to that of FEM, i.e., improving the accuracy by refining the element mesh. In the cases of using 1*c*-DOF CEM and 2*c*-DOF CEM, we present the detailed results of the *h*-version in Tables 5 and 6 respectively.

TABLE 4

λ_i of various schemes in case of total-DOF = 12

Order	Exact	CEM ($1 \times 10c$) *c*-DOF:10 Total-DOF:12	CEM ($3 \times 2c$) *c*-DOF:6 Total-DOF:12	CEM ($4 \times 1c$) *c*-DOF:4 Total-DOF:12	FEM (6e) *c*-DOF:0 Total-DOF:12
λ_1	1·875104	1·875104	1·875104	1·875104	1·875087
λ_2	4·694091	4·694100	4·694145	4·694152	4·694673
λ_3	7·854757	7·854857	7·855258	7·856252	7·861944
λ_4	10·99554	10·99599	10·99649	11·01191	11·03092
λ_5	14·13717	14·13845	14·14494	14·15534	14·24302
λ_6	17·27876	17·28159	17·34325	17·40013	17·44219
λ_7	20·42035	20·42555	20·48156	20·73256	21·63393
λ_8	23·56195	23·57028	23·90490	23·93568	25·35466

- **1 *c element***

In the case of using the 1c-DOF CEM, for a free-clamped beam shown in Figure 3, we consider the following schemes: when total-DOF is assigned as 6, two 1c-DOF elements (i.e., CEM ($2 \times 1c$)) and 3 FEM elements (i.e. FEM (3e)) are used respectively; when total-DOF is assigned as 12, four 1c-DOF elements (i.e., CEM ($4 \times 1c$)) and 6 FEM elements (i.e., FEM (6e)) are used respectively; when total-DOF is assigned as 18, six 1c-DOF elements (i.e., CEM ($6 \times 1c$)) and 9 FEM elements (i.e., FEM (9e)) are used respectively. All results are listed in Table 5. Relative errors are shown in Figures 4 and 5.

- **2 *c element***

In the case of using the 2c-DOF CEM, for a free-clamped beam shown in Figure 3, we consider the following schemes: when total-DOF is assigned as 8, two 2c-DOF elements (i.e., CEM ($2 \times 2c$)) and 4 FEM elements (i.e. FEM (4e)) are used respectively; when total-DOF is assigned as 12, three 2c-DOF elements (i.e., CEM ($3 \times 2c$)) and 6 FEM elements (i.e., FEM (6e)) are used respectively; when total-DOF is assigned as 20, five 2c-DOF elements (i.e., CEM ($5 \times 2c$)) and 10 FEM elements (i.e., FEM (10e)) are used respectively. All results are listed in Table 6. The relative errors are shown in Figures 4 and 5.

5.2.2. *c-version*

The c-version of CEM is to increase c-DOF terms when choosing the trial function of displacement field in order to improve the accuracy. Previously, many numerical results of increasing c-DOF have shown the high efficiency and good approximation on eigenvalues, especially for higher-order eigenvalues (see Tables 1–4). Now, also for a free-clamped beam shown in Figure 3, we present a more detailed comparison of the c-version of CEM with the conventional FEM. The numerical results listed in Table 7 and the relative error curves shown in Figures 4 and 5 indicate that by the c-version of CEM we can obtain a superconvergence for eigenvalues of structure, especially for higher-order eigenvalues. For instance, as to λ_1, the result of the c-version of CEM (total-DOF = 6) will nearly correspond to that of FEM (total-DOF = 18); as to λ_4, the result of the c-version of CEM (total-DOF = 6) will be better than that of FEM (total-DOF = 12); as to higher-order eigenvalue λ_{16}, the relative error of c-version (total-DOF = 18) is only 0·03576%, but the relative error by using FEM (total-DOF = 18) already reaches 18·2475%.

5.2.3. *Comments*

From the detailed numerical results above, we can sum up some features of the h-version and c-version of beam element in the Composite Element Method as follows.

(1) The convergence of the h-version and c-version of the beam element is obviously superior to that of conventional FEM. With less computation effort, both the h-version and c-version of CEM can approximate the desired solution. Usually, against the same computational effort, the error of CEM is one order of magnitude less than that of the conventional FEM.

TABLE 5

λ_i of various schemes by h-version using 1c element

		Total-DOF:6		Total-DOF:12		Total-DOF:18	
Order	Exact	CEM ($2 \times 1c$) c-DOF:2	FEM (3e)	CEM ($4 \times 1c$) c-DOF:4	FEM (6e)	CEM ($6 \times 1c$) c-DOF:6	FEM (9e)
λ_1	1·875104	1·875111	1·875199	1·875103	1·875087	1·875080	1·875140
λ_2	4·694091	4·697333	4·701793	4·694152	4·694673	4·694101	4·694209
λ_3	7·854757	7·870405	7·903542	7·856252	7·861944	7·854922	7·856264
λ_4	10·99554	11·11274	11·86048	11·01191	11·03092	10·99687	11·00339
λ_8	23·56195			23·93568	25·35466	23·66916	23·81524
λ_{10}	29·84513					30·30250	31·46236

TABLE 6

λ_i of various schemes by h-version using 2c element

		Total-DOF:8		Total-DOF:12		Total-DOF:20	
Order	Exact	CEM ($2 \times 2c$) c-DOF:4	FEM (4e)	CEM ($3 \times 2c$) c-DOF:6	FEM (6e)	CEM ($5 \times 2c$) c-DOF:10	FEM (10e)
λ_1	1·875104	1·875107	1·875133	1·875104	1·875087	1·875104	1·875085
λ_2	4·694091	4·694305	4·696825	4·694145	4·694673	4·694096	4·694164
λ_3	7·854757	7·855822	7·855103	7·855258	7·861944	7·854846	7·855755
λ_4	10·99554	11·02379	11·07509	10·99649	11·03092	10·99591	11·00078
λ_8	23·56195			23·90490	25·35466	23·57403	23·75331
λ_{10}	29·84513					29·99291	34·74266

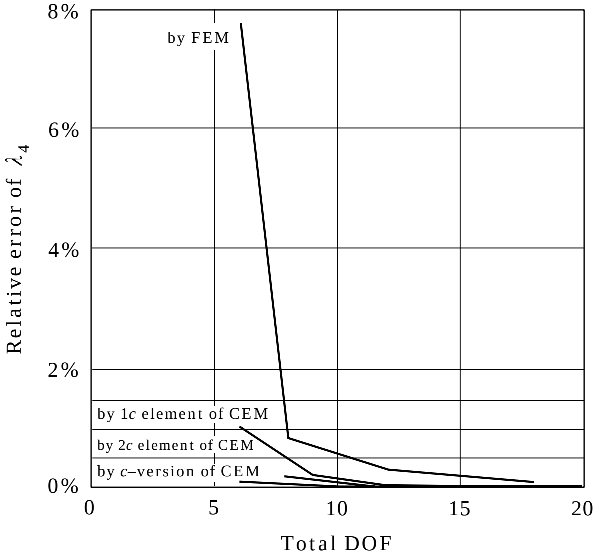


Figure 4. The relative errors of the 4th-order eigenvalue for a free-clamped beam.

(2) For low order of eigenvalues, both the *h*-version and *c*-version of the beam element can arrive at a superconvergence, although the *c*-version of CEM is superior to the *h*-version of CEM. It means that, with only few *c*-DOF, the CEM can obtain highly-accurate results. For higher-order of eigenvalues, only the *c*-version of CEM can continue to arrive at a superconvergence.

6. APPLICATIONS

6.1. VIBRATION ANALYSIS OF A LATHE

As an example, consider a simple lathe shown in Figure 6 [5].

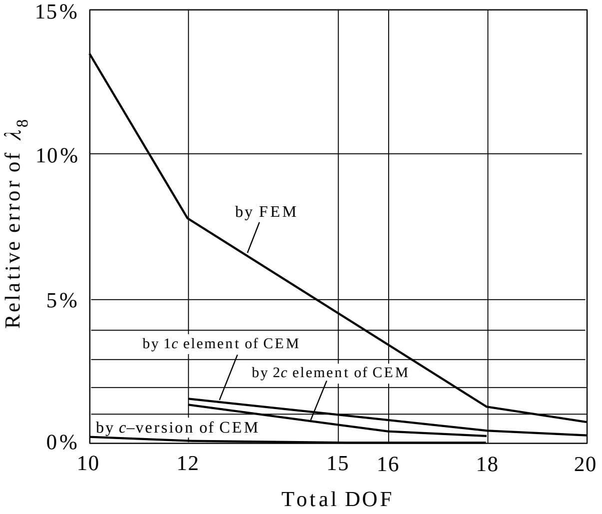


Figure 5. The relative errors of the 8th-order eigenvalue for a free-clamped beam.

TABLE 7

λ_i *of various schemes by c-version*

		Total-DOF:6		Total-DOF:12		Total-DOF:18	
Order	Exact	CEM (1 × 4c) c-DOF:4	FEM (3e)	CEM (1 × 10c) c-DOF:10	FEM (6e)	CEM (1 × 16c) c-DOF:16	FEM (9e)
λ_1	1·875104	1·875109	1·875199	1·875104	1·875087	1·875104	1·875140
λ_2	4·694091	4·694419	4·701793	4·694100	4·694673	4·694092	4·694209
λ_3	7·854757	7·857543	7·903542	7·854857	7·861944	7·854769	7·856264
λ_4	10·99554	11·00451	11·86048	10·99599	11·03092	10·99560	11·00339
λ_8	23·56195			23·57028	25·35466	23·56367	23·81524
λ_{10}	29·84513			29·86092	34·47107	29·84937	31·46236

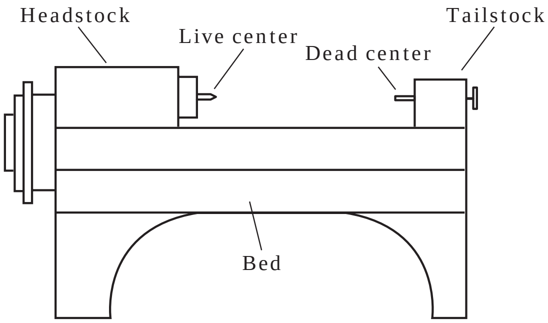
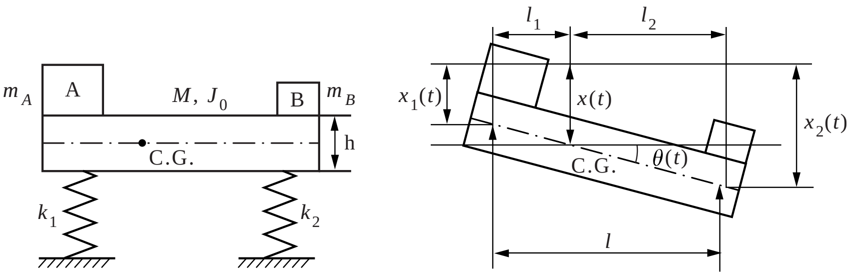


Figure 6. Components of a lathe.



(a) (b)

Figure 7. Rigid model of a lathe.

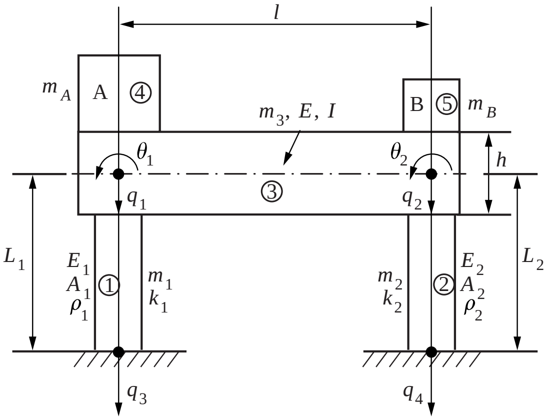


Figure 8. CEM model of a lathe.

6.1.1. *Rigid model*

For a simplified vibration analysis, the lathe bed can be considered as a rigid body having mass and inertia, and the headstock and tailstock can each be replaced by lumped masses. The bed can be assumed to be supported on springs at the ends. Thus the final model will be a rigid body of total mass m and mass moment of inertia J_0 and its C.G., resting on springs of stiffnesses k_1 and k_2, as shown in Figure 7(a).

We assume that the parameters of the lathe in Figure 7 are chosen as: the mass of headstock $m_A = 1000$ kg, the mass of tailstock $m_B = 500$ kg, the mass of bed $m_3 = 2500$ kg, the length of lathe $l = 2{\cdot}4$ m, the height of bed $h = 0{\cdot}4$ m, the support springs $k_1 = 1 \times 10^7$ N/m, $k_2 = 8 \times 10^6$ N/m. So one can calculate the position of C.G. point as: $l_1 = 0{\cdot}8$ m, $l_2 = 1{\cdot}6$ m, the total mass $m = 4000$ kg, the total mass moment of inertia $J_0 = 3553{\cdot}33$ kg $\cdot$ m^2.

For this system with two degree of freedom shown in Figure 7, any of the following sets of co-ordinates may be used to describe the motion:

(a) Deflections $x(t)$ of the C.G. and rotation $\theta(t)$.
(b) Deflection $x_1(t)$ and $x_2(t)$ of the two ends of the lathe AB.

● Equations of motion using $x(t)$ and $\theta(t)$

From the free-body diagram shown in Figure 7(b), with the positive values of the motion variables as indicated, the force equilibrium equation in the vertical direction can be written as

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \tag{34}$$

and the moment equation about the C.G. can be expressed as

$$J_0\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2. \tag{35}$$

Equations (34) and (35) can be rearranged and written in the matrix form as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1l_1 - k_2l_2) \\ -(k_1l_1 - k_2l_2) & (k_1l_1^2 + k_2l_2^2) \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{36}$$

From equation (36), one get the natural circular frequencies: $\omega_1 = 63{\cdot}56171$ rad/s, $\omega_2 = 89{\cdot}58034$ rad/s.

● Equations of motion using $x_1(t)$ and $x_2(x)$

Consider the transformation relation between (x, θ) and (x_1, x_2) [see Figure 7(b)].

$$x = \left(\frac{x_2 - x_1}{l_1 + l_2}\right)l_1 + x_1 \tag{37}$$

$$\theta = \frac{x_2 - x_1}{l_1 + l_2} \tag{38}$$

Table 8

Natural frequencies of a lathe by CEM (considering the effects of different masses of supports)

Case	ω_1 (rad/s)	ω_2 (rad/s)	· · ·	· · ·	
Rigid model	0·6356170E + 02	0·8958034E + 02			
$m_1 = 0$ kg, $m_2 = 0$ kg	0·67042500E + 02 0·44014340E + 05	0·88497730E + 02	0·25634080E + 04	0·80921490E + 04	0·24115390E + 05
$m_1 = 100$ kg, $m_2 = 100$ kg	0·66493650E + 02 0·19885570E + 04	0·87195230E + 02 0·25638380E + 04	0·89522130E + 03 0·80835970E + 04	0·99905130E + 03 0·24093830E + 05	0·17790290E + 04 0·43978030E + 05
$m_1 = 200$ kg, $m_2 = 200$ kg	0·65954190E + 02 0·14086130E + 04	0·85936240E + 02 0·25563700E + 04	0·63798470E + 03 0·80746430E + 04	0·71061770E + 03 0·24073180E + 05	0·12609010E + 04 0·43948040E + 05
$m_1 = 300$ kg, $m_2 = 300$ kg	0·65423930E + 02 0·11518990E + 04	0·84719310E + 02 0·25503090E + 04	0·52485360E + 03 0·80658890E + 04	0·58360590E + 03 0·24052440E + 05	0·10316680E + 04 0·43921460E + 05
$m_1 = 500$ kg, $m_2 = 300$ kg	0·64866520E + 02 0·10317240E + 04	0·83874820E + 02 0·25467930E + 04	0·45680150E + 03 0·80614030E + 04	0·52516480E + 03 0·24041340E + 05	0·89487460E + 03 0·43908440E + 05

Table 9

Natural frequencies of a lathe by CEM (considering the effects of different stiffness of lathe bed)

E (N/m^2)	ω_1 (rad/s)	ω_2 (rad/s)	· · ·	· · ·	
Rigid model	0·63561710E + 02	0·89580340E + 02			
$E = 2{\cdot}1 \times 10^{10}$	0·64684360E + 02 0·90421300E + 03	0·83845150E + 02 0·10400810E + 04	0·45488550E + 03 0·25514440E + 04	0·52140030E + 03 0·76027570E + 04	0·80147160E + 03 0·13883880E + 05
$E = 2{\cdot}1 \times 10^{11}$	0·64866520E + 02 0·10317240E + 04	0·83874820E + 02 0·25467930E + 04	0·45680150E + 03 0·80614030E + 04	0·52516480E + 03 0·24041340E + 05	0·89487460E + 03 0·43908440E + 05
$E = 2{\cdot}1 \times 10^{12}$	0·64885570E + 02 0·10322970E + 04	0·83872970E + 02 0·80391520E + 04	0·45693010E + 03 0·25490360E + 05	0·52540100E + 03 0·76030670E + 05	0·89513820E + 03 0·13884860E + 06
$E = 2{\cdot}1 \times 10^{13}$	0·64882060E + 02 0·10323450E + 04	0·83782220E + 02 0·25417710E + 05	0·45694170E + 03 0·80607070E + 05	0·52542330E + 03 0·24040690E + 06	0·89516150E + 03 0·43908580E + 06

i.e.,

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \frac{1}{(l_1 + l_2)} \begin{bmatrix} l_2 & l_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{39}$$

where **T** is the transformation matrix, i.e.,

$$\mathbf{T} = \frac{1}{(l_1 + l_2)} \begin{bmatrix} l_2 & l_1 \\ -1 & 1 \end{bmatrix}.$$

So, equation (39) can be transformed by **T**

$$\frac{1}{(l_1 + l_2)} \begin{bmatrix} l_2^2 m + J_0 & l_1 l_2 m - J_0 \\ l_1 l_2 m - J_0 & l_1^2 m + J_0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{40}$$

From equation (40), one also get the natural circular frequencies: $\omega_1 = 63{\cdot}56171$ rad/s, $\omega_2 = 89{\cdot}58034$ rad/s.

6.1.2. *Composite Element Model*

An accurate model of this machine tool would involve the consideration of the lathe bed as an elastic beam with lumped masses attached to it as shown in Figure 8.

Now we apply the Composite Element Method to deal with it. For each element, first we write the stiffness and mass matrices as follows:

For bar element (1) (i.e., left support spring), we take a CEM bar element with $2c$-DOF CEM, i.e.,

$$\mathbf{k}^{(1)} = k_1 \cdot \begin{array}{c} \begin{array}{cccc} q_1 & q_3 & c_{11} & c_{12} \end{array} \\ \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{\pi^2}{2} & 0 \\ 0 & 0 & 0 & \frac{4\pi^2}{2} \end{array}\right] \end{array} \begin{array}{c} \\ q_1 \\ q_3 \\ c_{11} \\ c_{12} \end{array} \tag{41}$$

$$\mathbf{m}^{(1)} = m_1 \cdot \begin{array}{c} \begin{array}{cccc} q_1 & q_3 & c_{11} & c_{12} \end{array} \\ \left[\begin{array}{cc|cc} \frac{1}{3} & \frac{1}{6} & \frac{1}{\pi} & \frac{1}{2\pi} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{\pi} & -\frac{1}{2\pi} \\ \hline \frac{1}{\pi} & \frac{1}{\pi} & \frac{1}{2} & 0 \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & \frac{1}{2} \end{array}\right] \end{array} \begin{array}{c} \\ q_1 \\ q_3 \\ c_{11} \\ c_{12} \end{array} \tag{42}$$

Table 10

Natural frequencies of a lathe by CEM (considering several simplified models)

Case	ω_1 (rad/s)	ω_2 (rad/s)	. . .	. . .	
Rigid model	0·63561710E + 02	0·89580340E + 02			
Case 1	0·64925860E + 02	0·44831220E + 03	0·51570890E + 03	0·88738960E + 03	0·10196570E + 04
	0·12217690E + 04				
Case 2	0·64886710E + 02	0·83877330E + 02	0·45693960E + 03	0·52541850E + 03	0·89515470E + 03
	0·10323350E + 04	0·10435490E + 05			
Case 3	0·67081870E + 02	0·12136260E + 04			
Case 4	0·67042500E + 02	0·88497730E + 02	0·25634080E + 04	0·80921490E + 04	0·24115390E + 05
	0·44014340E + 05				

where $k_1 = (E_1 A_1)/L_1$ is the stiffness coefficient, $m_1 = \rho_1 A_1 L_1$ is the mass of bar element. Note that only two c-DOF is taken for this bar element.

For bar element (2) (i.e., right support spring), we take a CEM bar element with $2c$-DOF CEM, i.e.,

$$\mathbf{k}^{(2)} = k_2 \cdot \begin{array}{cc} \begin{array}{cccc} q_2 & q_4 & c_{21} & c_{22} \end{array} & \\ \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & \dfrac{\pi^2}{2} & 0 \\ 0 & 0 & 0 & \dfrac{4\pi^2}{2} \end{array}\right] & \begin{array}{c} q_2 \\ q_4 \\ c_{21} \\ c_{22} \end{array} \end{array} \qquad (43)$$

$$\mathbf{m}^{(2)} = m_2 \cdot \begin{array}{cc} \begin{array}{cccc} q_2 & q_4 & c_{21} & c_{22} \end{array} & \\ \left[\begin{array}{cc|cc} \dfrac{1}{3} & \dfrac{1}{6} & \dfrac{1}{\pi} & \dfrac{1}{2\pi} \\ \dfrac{1}{6} & \dfrac{1}{3} & \dfrac{1}{\pi} & -\dfrac{1}{2\pi} \\ \hline \dfrac{1}{\pi} & \dfrac{1}{\pi} & \dfrac{1}{2} & 0 \\ \dfrac{1}{2\pi} & -\dfrac{1}{2\pi} & 0 & \dfrac{1}{2} \end{array}\right] & \begin{array}{c} q_2 \\ q_4 \\ c_{21} \\ c_{22} \end{array} \end{array} \qquad (44)$$

where $k_2 = (E_2 A_2)/L_2$ is the stiffness coefficient, $m_2 = \rho_2 A_2 L_2$ is the mass of bar element.

For the element (3) (i.e., lathe body), we use a CEM beam element with $2c$-DOF, i.e.,

$$\mathbf{k}^{(3)} = \frac{EI}{l^3} \cdot \begin{array}{cc} \begin{array}{cccccc} q_1 & \theta_1 & q_2 & \theta_2 & c_{31} & c_{32} \end{array} & \\ \left[\begin{array}{cccc|cc} 12 & & & & 0 & 0 \\ 6l & 4l^2 & & sym. & 0 & 0 \\ -12 & -6l & 12 & & 0 & 0 \\ 6l & 2l^2 & -6l & 4l^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1{\cdot}035936\lambda_1^{*4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}998447\lambda_2^{*4} \end{array}\right] & \begin{array}{c} q_1 \\ \theta_1 \\ q_2 \\ \theta_2 \\ c_{31} \\ c_{32} \end{array} \end{array} \qquad (45)$$

where $\lambda_1^* = 4{\cdot}730041$, $\lambda_2^* = 7{\cdot}853205$.

$$\mathbf{m}^{(3)} = m_3 \cdot$$

$$\begin{array}{c}
\begin{array}{cccccc} q_1 & \theta_1 & q_2 & \theta_2 & c_{31} & c_{32} \end{array} \\
\left[\begin{array}{cccc|cc}
\frac{13}{35} & & & & & \\
\frac{11}{210}l & \frac{1}{105}l^2 & & sym. & sym. & \\
\frac{9}{70} & \frac{13}{420}l & \frac{13}{35} & & & \\
-\frac{13}{420}l & -\frac{1}{140}l^2 & -\frac{11}{210}l & \frac{1}{105}l^2 & & \\
\hline
0\cdot42282930 & 0\cdot09098435l & 0\cdot42282930 & -0\cdot09098435l & 1\cdot035936 & 0 \\
0\cdot25467310 & 0\cdot03240400l & -0\cdot25467310 & 0\cdot03240400l & 0 & 0\cdot998447
\end{array}\right]
\end{array} \tag{46}$$

For the element (4) (i.e., headstock), it is considered as a rigid lump, i.e.,

$$\mathbf{k}^{(4)} = [0] \tag{47}$$

$$\mathbf{m}^{(4)} = [m_A] \tag{48}$$

For the element (5) (i.e., tailstock), also it is considered as a rigid lump, i.e..,

$$\mathbf{k}^{(5)} = [0] \tag{49}$$

$$\mathbf{m}^{(5)} = [m_B] \tag{50}$$

The assembled mass and stiffness matrices are given by

$$\mathbf{K} = \sum_{e=1}^{5} \mathbf{k}^{(e)} \tag{51}$$

$$\mathbf{M} = \sum_{e=1}^{5} \mathbf{m}^{(e)}. \tag{52}$$

Since the bottoms of the bar elements 1 and 2 are fixed, i.e., $q_3 = q_4 = 0$, and these two degree of freedoms have to be eliminated from the global stiffness and mass matrices. The parameters of lathe bed (as a beam element) are assumed as: the modulus $E = 2{\cdot}1 \times 10^{11}$ N/m^2, the area moment of inertia of cross section $I = 0{\cdot}005333$ m^4. Others take the same parameters as the rigid model. Now we analyze the effects of different masses of supports on the natural circular frequencies of the lathe by the CEM model proposed above. The results are presented in Table 8, which are compared with those of the rigid model.

From Table 8, one can summarize that the masses of supports will give an obvious impact on vibration of the total lathe, especially on a higher order of vibration. The CEM is a good means to analyze these influences with less computational effort.

If fix the support masses as $m_1 = 500$ kg, $m_2 = 300$ kg, we study the effects of stiffness of the lathe bed (i.e., take different Young's modulus) by the CEM model proposed above. The results are presented in Table 9, which are compared with those of the rigid model.

From Table 9, one can find that the stiffness of the lathe bed will make less impact on the lower order of vibration, but an obvious impact on the higher order of vibration.

In order to investigate the ability of CEM, we discuss several simplified below.

6.1.3. *Case* 1

Consider the supports with masses and take the lathe bed as a rigid body. So, we let $\theta_1 = \theta_2 = c_{31} = c_{32} = 0$. One has the following expressions of the mass and stiffness matrices from the globe mass and stiffness matrices.

$$\mathbf{K} = \begin{array}{cc} \begin{matrix} q_1 & q_2 & c_{11} & c_{12} & c_{21} & c_{22} \end{matrix} & \\ \begin{bmatrix} k_1 + \dfrac{12EI}{l^3} & -\dfrac{12EI}{l^3} & & & 0 & \\ -\dfrac{12EI}{l^3} & k_2 + \dfrac{12EI}{l^3} & & & & \\ & & \dfrac{\pi^2}{2} k_1 & & & \\ & & & \dfrac{4\pi^2}{2} k_1 & & 0 \\ & 0 & & & \dfrac{\pi^2}{2} k_2 & \\ & & 0 & & & \dfrac{4\pi^2}{2} k_2 \end{bmatrix} & \begin{matrix} q_1 \\ q_2 \\ c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{matrix} \end{array} \tag{53}$$

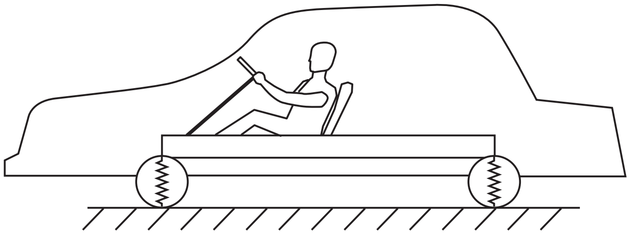


Figure 9. Components of an automobile.

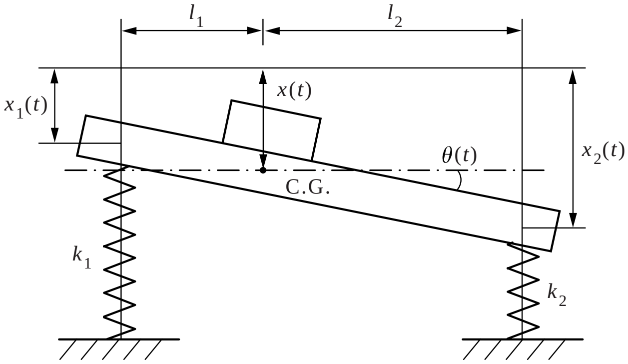


Figure 10. Rigid model of an automobile.

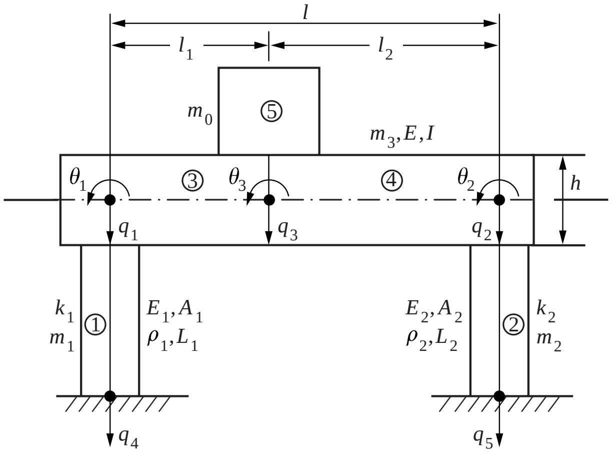


Figure 11. CEM model of an automobile.

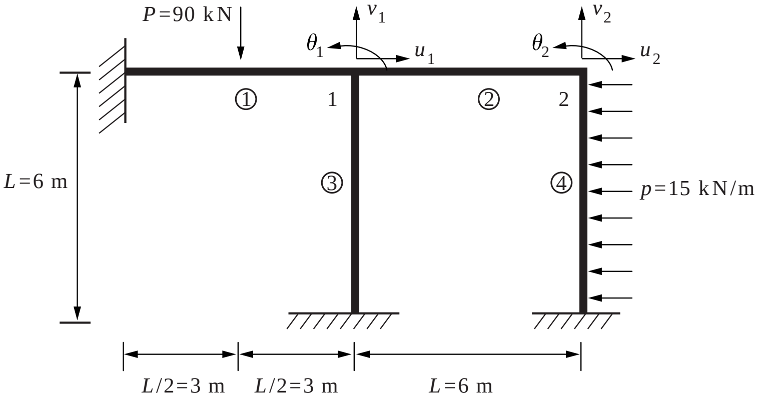


Figure 12. Vibration analysis of a frame by CEM.

$$
\mathbf{M} = \left[\begin{array}{cc|cccc}
m_A + \frac{1}{3} m_1 + \frac{13}{35} m_3 & & & & sym. & \\
\frac{9}{70} m_3 & m_B + \frac{1}{3} m_2 + \frac{13}{35} m_3 & & & & \\
\hline
\frac{1}{\pi} m_1 & 0 & \frac{1}{2} m_1 & & & \\
-\frac{1}{2\pi} m_1 & 0 & & \frac{1}{2} m_1 & & 0 \\
0 & \frac{1}{\pi} m_2 & & & \frac{1}{2} m_2 & \\
0 & -\frac{1}{2\pi} m_2 & & 0 & & \frac{1}{2} m_2
\end{array}\right]
\begin{array}{c} q_1 \\ q_2 \\ c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{array}
\tag{54}
$$

6.1.4. *Case* 2

Also consider the supports with masses and take the lathe bed as a rigid body. But we let $\theta_1 = \theta_2 \neq 0$, and $c_{31} = c_{32} = 0$. This simplified case can more reasonably reflect the property of the rigid body of the lathe bed.

6.1.5. *Case* 3

Consider the lathe bed as a rigid body and neglect the masses of the two support bars. So, we let $\theta_1 = \theta_2 = c_{31} = c_{32} = 0$, $c_{11} = c_{12} = c_{21} = c_{22} = 0$, $m_1 = m_2 = 0$. One has the following mass and stiffness matrices

$$
\mathbf{K} = \begin{array}{c}
\begin{array}{cc} q_1 & \qquad q_2 \end{array} \\
\left[\begin{array}{cc}
k_1 + \frac{12EI}{l^3} & -\frac{12EI}{l^3} \\
-\frac{12EI}{l^3} & k_2 + \frac{12EI}{l^3}
\end{array}\right]
\end{array}
\begin{array}{c} \\ q_1 \\ q_2 \end{array}
\tag{55}
$$

$$
\mathbf{M} = \begin{array}{c}
\begin{array}{cc} q_1 & \qquad q_2 \end{array} \\
\left[\begin{array}{cc}
m_A + \frac{13}{35} m_3 & \frac{9}{70} m_3 \\
\frac{9}{70} m_3 & m_B + \frac{13}{35} m_3
\end{array}\right]
\end{array}
\begin{array}{c} \\ q_1 \\ q_2 \end{array}
\tag{56}
$$

Consider the lathe bed as an elastic beam and neglect the masses of the two support bars. So, we let $c_{11} = c_{12} = c_{21} = c_{22} = 0$, as well as $m_1 = m_2 = 0$. One has the following mass and stiffness matrices.

$$\boldsymbol{K} = \frac{EI}{l^3}\cdot$$

$$\begin{array}{cc}
\begin{array}{cccccc} q_1 & \theta_1 & q_2 & \theta_2 & c_{31} & c_{32} \end{array} & \\
\left[\begin{array}{cccccc}
\dfrac{k_1 l^3}{EI} + 12 & & & & 0 & 0 \\
6l & 4l^2 & & sym. & 0 & 0 \\
-12 & -6l & \dfrac{k_2 l^3}{EI} + 12 & & 0 & 0 \\
6l & 2l^2 & -6l & 4l^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1{\cdot}035936\lambda_1^{*4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0{\cdot}998447\lambda_2^{*4}
\end{array}\right]
&
\begin{array}{c} q_1 \\ \theta_1 \\ q_2 \\ \theta_2 \\ c_{31} \\ c_{32} \end{array}
\end{array} \tag{57}$$

where $\lambda_1^* = 4{\cdot}730041$, $\lambda_2^* = 7{\cdot}853205$.

$$\mathbf{M} = m_3\cdot$$

$$\begin{array}{c}
\begin{array}{cccccc} q_1 & \theta_1 & q_2 & \theta_2 & c_{11} & c_{12} \end{array} \\
\left[\begin{array}{cccccc}
\dfrac{m_A}{m_3} + \dfrac{13}{35} & & & & & \\
\dfrac{11}{210}l & \dfrac{1}{105}l^2 & & sym. & sym. & \\
\dfrac{9}{70} & \dfrac{13}{420}l & \dfrac{m_B}{m_3} + \dfrac{13}{35} & & & \\
-\dfrac{13}{420}l & -\dfrac{1}{140}l^2 & -\dfrac{11}{210}l & \dfrac{1}{105}l^2 & & \\
0{\cdot}42282930 & 0{\cdot}09098435l & 0{\cdot}42282930 & -0{\cdot}09098435l & 1{\cdot}035936 & \\
0{\cdot}25467310 & 0{\cdot}03240400l & -0{\cdot}25467310 & 0{\cdot}03240400l & 0 & 0{\cdot}998447
\end{array}\right]
\end{array} \tag{58}$$

Similarly, the parameters of the lathe bed (as a beam element) are: $E = 2{\cdot}1 \times 10^{11}$ N/m^2, $I = 0{\cdot}005333$ m^4, the parameters of supports are: $m_1 = 500$ kg, $m_2 = 300$ kg, $k_1 = 1 \times 10^7$ N/m, $k_2 = 8 \times 10^6$ N/m. Other parameters are the same as rigid model. Now we calculate the natural frequencies of the lathe by the four simplified models of CEM which are proposed above. The results are presented in Table 10, which are compared with those of the rigid model.

From Table 10, we can conclude:

1. Since we have assumed that $\theta_1 = \theta_2 \neq 0$ in model 2 (case 2), the results are fully close to those of the rigid model.
2. In simplified model 4 (case 4), we have assumed that $m_1 = m_2 = 0$. The results also agree with those of the rigid model.
3. Simplified models 1 and 3 give poor results. It is very likely that due to the fact that the assumption $\theta_1 = \theta_2 = 0$ is not appropriate.

6.2. VIBRATION ANALYSIS OF AN AUTOMOBILE

As an example, consider the automobile shown in Figure 9 [5].

6.2.1. *Rigid model*

For a simplified vibration analysis, the automobile body and driver can be considered as a rigid body having mass and inertia, and it can be assumed to be supported on springs at the ends. Thus the final model will be a rigid body of total mass m and mass moment of inertia J_0 and its C.G., resting on springs of stiffnesses k_1 and k_2, as shown in Figure 10.

We assume that the parameters in Figure 10 are: the mass of driver $m_0 = 80$ kg, the mass of automobile body $m_3 = 1200$ kg, the length of automobile $l = 3{\cdot}5$ m, the height of frame beam $h = 0{\cdot}2$ m, the stiffness of support wheels $k_1 = k_2 = 2 \times 10^5$ N/m, the position of C.G. point $l_1 = 1{\cdot}5$ m, $l_2 = 2$ m. So one can calculate the total mass $m = 1280$ kg, the total mass moment of inertia $J_0 = 1304$ kg $\cdot$ m^2.

Similar to the last example, we use the $[x(t), \theta(t)]$ coordinate system and the $[x_1(t), x_2(t)]$ coordinate system to describe the equation of motion.

- **Equations of motion using $x(t)$ and $\theta(t)$**

Similar to the lathe problem, the equation of motion can be written as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{59}$$

From equation (59), one obtains the natural circular frequencies: $\omega_1 = 17{\cdot}41713$ rad/s, $\omega_2 = 31{\cdot}1084$ rad/s.

- **Equations of motion using $x_1(t)$ and $x_2(x)$**

Using the transformation relationship $\mathbf{T}$ between (x, θ) and (x_1, x_2), i.e.,

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \frac{1}{(l_1 + l_2)} \begin{bmatrix} l_2 & l_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{60}$$

we have

$$\frac{1}{(l_1 + l_2)^2} \begin{bmatrix} l_2^2 m + J_0 & l_1 l_2 m - J_0 \\ l_1 l_2 m - J_0 & l_1^2 m + J_0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{61}$$

From equation (61), one obtains the natural circular frequencies: $\omega_1 = 17{\cdot}41713$ rad/s, $\omega_2 = 31{\cdot}1084$ rad/s.

6.2.2. *Composite element model*

An accurate model of this automobile should consider the automobile body as an elastic beam with the lumped mass (driver) and the mass-coupled spring elements attached to it as shown in Figure 11. Now we apply the CEM to analyze it. First of all, for each element we write the stiffness and mass matrices as follows:

For element (1) (i.e., the left wheel, we use a CEM bar element with $2c$-DOF CEM), one has

$$\mathbf{k}^{(1)} = k_1 \cdot \begin{array}{c} \begin{array}{cccc} q_1 & q_4 & c_{11} & c_{12} \end{array} \\ \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{\pi^2}{2} & 0 \\ 0 & 0 & 0 & \frac{4\pi^2}{2} \end{array}\right] \begin{array}{c} q_1 \\ q_4 \\ c_{11} \\ c_{12} \end{array} \end{array} \tag{62}$$

$$\mathbf{m}^{(1)} = m_1 \cdot \begin{array}{c} \begin{array}{cccc} q_1 & q_4 & c_{11} & c_{12} \end{array} \\ \left[\begin{array}{cc|cc} \frac{1}{3} & \frac{1}{6} & \frac{1}{\pi} & \frac{1}{2\pi} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{\pi} & -\frac{1}{2\pi} \\ \hline \frac{1}{\pi} & \frac{1}{\pi} & \frac{1}{2} & 0 \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & \frac{1}{2} \end{array}\right] \begin{array}{c} q_1 \\ q_4 \\ c_{11} \\ c_{12} \end{array} \end{array} \tag{63}$$

where $k_1 = (E_1 A_1)/L_1$ is the stiffness coefficient, $m_1 = \rho_1 A_1 L_1$ is the mass of bar element. Note that we take only two c-DOF is taken for this bar element.

Table 11

Natural frequencies of an automobile by CEM (considering the effects of different masses of wheels and driver)

Case	ω_1 (rad/s)	ω_2 (rad/s)	...	...	
Rigid model	0·17417130E + 02	0·31108400E + 02			
$m_0 = 80$ kg	0·17666970E + 02	0·31564880E + 02	0·11240210E + 04	0·31707920E + 04	0·61679760E + 04
$m_1 = m_2 = 0$ kg	0·10100520E + 05	0·16542480E + 05	0·31480200E + 05	0·47886630E + 05	0·78623740E + 05
$m_0 = 80$ kg	0·17530440E + 02	0·30800190E + 02	0·25742860E + 03	0·26029870E + 03	0·51292930E + 03
$m_1 = m_2 = 30$ kg	0·51488350E + 03	0·11179390E + 04	0·31471150E + 04	0·61176740E + 04	0·10021010E + 05
	0·16400550E + 05	0·31105920E + 05	0·46728700E + 05	0·75773760E + 05	
$m_0 = 80$ kg	0·17396240E + 02	0·30076540E + 02	0·18288430E + 03	0·18668170E + 03	0·36318100E + 03
$m_1 = m_2 = 60$ kg	0·36545020E + 03	0·11091880E + 04	0·31236140E + 04	0·60696980E + 04	0·99465790E + 04
	0·16271110E + 05	0·30762870E + 05	0·45741330E + 05	0·73511840E + 05	
$m_0 = 120$ kg	0·17137280E + 02	0·30054640E + 02	0·18282680E + 03	0·18667320E + 03	0·36314020E + 03
$m_1 = m_2 = 60$ kg	0·36544540E + 03	0·10911260E + 04	0·31042750E + 04	0·59965000E + 04	0·98175590E + 04
	0·16134800E + 05	0·30543490E + 05	0·45719980E + 05	0·73195130E + 05	
$m_0 = 200$ kg	0·16651340E + 02	0·30016260E + 02	0·18272100E + 03	0·18665840E + 03	0·36306480E + 03
$m_1 = m_2 = 60$ kg	0·36543700E + 03	0·10601710E + 04	0·30722870E + 04	0·58807070E + 04	0·96346710E + 04
	0·15953680E + 05	0·30260540E + 05	0·45713640E + 05	0·72786510E + 05	

TABLE 12

Natural frequencies of an automobile by CEM (considering the effects of different stiffnesses of automobile body)

E (N/m^2)	ω_1 (rad/s)	ω_2 (rad/s)	. . .	. . .	
Rigid model	0·17417130E + 02	0·31108400E + 02			
$E = 2{\cdot}1 \times 10^{10}$	0·17316540E + 02	0·30057430E + 02	0·18048950E + 03	0·18643350E + 03	0·33540810E + 03
	0·36488960E + 03	0·38624920E + 03	0·99080920E + 03	0·19209460E + 04	0·31462720E + 04
	0·51459520E + 04	0·97283420E + 04	0·14469630E + 05	0·23250610E + 05	
$E = 2{\cdot}1 \times 10^{11}$	0·17396240E + 02	0·30076540E + 02	0·18288430E + 03	0·18668170E + 03	0·36318100E + 03
	0·36545020E + 03	0·11091880E + 04	0·31236140E + 04	0·60696980E + 04	0·99465790E + 04
	0·16271110E + 05	0·30762870E + 05	0·45741330E + 05	0·73511840E + 05	
$E = 2{\cdot}1 \times 10^{12}$	0·17397450E + 02	0·30097970E + 02	0·18306700E + 03	0·18670590E + 03	0·36358110E + 03
	0·36549970E + 03	0·34989120E + 04	0·98748820E + 04	0·19192520E + 05	0·31452750E + 05
	0·51452880E + 05	0·97265480E + 05	0·14459710E + 06	0·23248150E + 06	
$E = 2{\cdot}1 \times 10^{13}$	0·17405690E + 02	0·30215600E + 02	0·18308480E + 03	0·18670950E + 03	0·36361730E + 03
	0·36550470E + 03	0·11061950E + 05	0·31226120E + 05	0·60691750E + 05	0·99462200E + 05
	0·16270510E + 06	0·30758030E + 06	0·73426810E + 06	0·45733800E + 06	

Table 13

Natural circular frequencies of an automobile by CEM (considering several simplified models)

Case	ω_1 (rad/s)	ω_2 (rad/s)	. . .	. . .	
Rigid model	0·17417130E + 02	0·31108400E + 02			
Case 1	0·17405000E + 02	0·18125360E + 03	0·18305520E + 03	0·36248310E + 03	0·36355620E + 03
	0·10734790E + 04	0·22105650E + 04			
Case 2	0·17676370E + 02	0·10821920E + 04	0·22312500E + 04		
Case 3	0·17666970E + 02	0·31564880E + 02	0·11240210E + 04	0·31707920E + 04	0·61679760E + 04
	0·10100520E + 05	0·16542480E + 05	0·31480200E + 05	0·47886630E + 05	0·78623740E + 05

Table 14

Natural circular frequencies of a 4-beam fram by CEM

	Element (DOF)	ω_1 (rad/s)	ω_2 (rad/s)	. . .	. . .
CEM	4 (22)	0·11794540E + 02	0·12301100E + 02	0·15841500E + 02	0·20125440E + 02
		0·21708130E + 02	0·25305280E + 02	0·31505780E + 02	0·39506160E + 02
		0·47215750E + 02	0·55810070E + 02	0·60401240E + 02	0·61077850E + 02
		0·73408120E + 02	0·79044710E + 02	0·87110100E + 02	0·99372490E + 02
		0·12780200E + 03	0·12954950E + 03	0·13554450E + 03	0·15016910E + 03
		0·17765660E + 03	0·22132360E + 03		

For element (2) (i.e., the right wheel, we also use a CEM bar element with $2c$-DOF CEM) one has

$$\mathbf{k}^{(2)} = k_2 \cdot \begin{array}{cc} \begin{array}{cccc} q_2 & q_5 & c_{21} & c_{22} \end{array} & \\ \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{\pi^2}{2} & 0 \\ 0 & 0 & 0 & \frac{4\pi^2}{2} \end{array}\right] & \begin{array}{c} q_2 \\ q_5 \\ c_{21} \\ c_{22} \end{array} \end{array} \tag{64}$$

$$\mathbf{m}^{(2)} = m_2 \cdot \begin{array}{cc} \begin{array}{cccc} q_2 & q_5 & c_{21} & c_{22} \end{array} & \\ \left[\begin{array}{cc|cc} \frac{1}{3} & \frac{1}{6} & \frac{1}{\pi} & \frac{1}{2\pi} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{\pi} & -\frac{1}{2\pi} \\ \hline \frac{1}{\pi} & \frac{1}{\pi} & \frac{1}{2} & 0 \\ \frac{1}{2\pi} & -\frac{1}{2\pi} & 0 & \frac{1}{2} \end{array}\right] & \begin{array}{c} q_2 \\ q_5 \\ c_{21} \\ c_{22} \end{array} \end{array} \tag{65}$$

where $k_2 = (E_2A_2)/L_2$ is the stiffness coefficient, $m_2 = \rho_2A_2L_2$ is the mass of bar element.

For the element (3) (i.e., the left part of the automobile body), we can take a CEM beam element with $2c$-DOF, i.e.,

$$\mathbf{k}^{(3)} = \frac{EI}{l_1^3} \cdot \begin{array}{cc} \begin{array}{cccccc} q_1 & \theta_1 & q_3 & \theta_3 & c_{31} & c_{32} \end{array} & \\ \left[\begin{array}{cccc|cc} 12 & & & & 0 & 0 \\ 6l_1 & 4l_1^2 & & sym. & 0 & 0 \\ -12 & -6l_1 & 12 & & 0 & 0 \\ 6l_1 & 2l_1^2 & -6l_1 & 4l_1^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1{\cdot}035936\lambda_1^{*4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}998447\lambda_2^{*4} \end{array}\right] & \begin{array}{c} q_1 \\ \theta_1 \\ q_3 \\ \theta_3 \\ c_{31} \\ c_{32} \end{array} \end{array} \tag{66}$$

where $\lambda_1^* = 4{\cdot}730041$, $\lambda_2^* = 7{\cdot}853205$.

$$\mathbf{m}^{(3)} = \frac{m_3 l_1}{l} \cdot \begin{array}{c} \begin{array}{cccc|cc} q_1 & \theta_1 & q_3 & \theta_3 & c_{31} & c_{32} \end{array} \\ \left[\begin{array}{cccc|cc} \frac{13}{35} & & & & & \\ \frac{11}{210} l_1 & \frac{1}{105} l_1^2 & & sym. & sym. & \\ \frac{9}{70} & \frac{13}{420} l_1 & \frac{13}{35} & & & \\ -\frac{13}{420} l_1 & -\frac{1}{140} l_1^2 & -\frac{11}{210} l_1 & \frac{1}{105} l_1^2 & & \\ \hline 0{\cdot}42282930 & 0{\cdot}09098435 l_1 & 0{\cdot}42282930 & -0{\cdot}09098435 l_1 & 1{\cdot}035936 & 0 \\ 0{\cdot}25467310 & 0{\cdot}03240400 l_1 & -0{\cdot}25467310 & 0{\cdot}03240400 l_1 & 0 & 0{\cdot}998447 \end{array}\right] \end{array} \qquad (67)$$

where m_3 is the total mass of automobile body.

For the element (4) (the right part of automobile body), we take a CEM beam element with $2c$-DOF, i.e.,

$$\mathbf{k}^{(4)} = \frac{EI}{l_2^3} \cdot \begin{array}{cccc|ccc} q_3 & \theta_3 & q_2 & \theta_2 & c_{41} & c_{42} & \\ 12 & & & & 0 & 0 & q_3 \\ 6l_2 & 4l_2^2 & & sym. & 0 & 0 & \theta_3 \\ -12 & -6l_2 & 12 & & 0 & 0 & q_2 \\ 6l_2 & 2l_2^2 & -6l_2 & 4l_2^2 & 0 & 0 & \theta_2 \\ \hline 0 & 0 & 0 & 0 & 1{\cdot}035936\lambda_1^{*4} & 0 & c_{41} \\ 0 & 0 & 0 & 0 & 0 & 0{\cdot}998447\lambda_2^{*4} & c_{42} \end{array} \tag{68}$$

$$\mathbf{m}^{(4)} = \frac{m_3 l_2}{l} \cdot \begin{array}{cccc|cc} q_3 & \theta_3 & q_2 & \theta_2 & c_{41} & c_{42} \\ \frac{13}{35} & & & & & \\ \frac{11}{210} l_2 & \frac{1}{105} l_2^2 & & sym. & sym. & \\ \frac{9}{70} & \frac{13}{420} l_2 & \frac{13}{35} & & & \\ -\frac{13}{420} l_2 & -\frac{1}{140} l_2^2 & -\frac{11}{210} l_2 & \frac{1}{105} l_2^2 & & \\ \hline 0{\cdot}42282930 & 0{\cdot}09098435 l_2 & 0{\cdot}42282930 & -0{\cdot}09098435 l_2 & 1{\cdot}035936 & 0 \\ 0{\cdot}25467310 & 0{\cdot}03240400 l_2 & -0{\cdot}25467310 & 0{\cdot}03240400 l_2 & 0 & 0{\cdot}998447 \end{array} \tag{69}$$

For the element (5) (i.e., the driver), it is considered as a rigid lump, i.e.,

$$\mathbf{k}^{(5)} = [0] \tag{70}$$

$$\mathbf{m}^{(5)} = [m_0] \tag{71}$$

The assembled mass and stiffness matrices are given as

$$\mathbf{K} = \sum_{e=1}^{5} \mathbf{k}^{(e)} \tag{72}$$

$$\mathbf{M} = \sum_{e=1}^{5} \mathbf{m}^{(e)}. \tag{73}$$

Since the bottoms of two bar elements are fixed, one has $q_4 = q_5 = 0$, and these two degrees of freedom have to be eliminated from the global stiffness and mass matrices. The parameters of the automobile body (as a beam element) are: $E = 2{\cdot}1 \times 10^{11}$ N/m^2, the area moment of inertia of cross section $I = 0{\cdot}0006667$ m^4. Other parameters are the same as the rigid model. Now we analyze the effects of different masses of support wheels and driver on the natural circular frequencies of the lathe by the CEM model proposed above. The results are presented in Table 11, which are compared with those of the rigid model.

From Table 11, one can summarize that the masses of wheels and drivers will give an obvious impact on vibration properties of the total automobile, especially on the higher order of vibration.

The masses of wheels and drivers are fixed as $m_1 = m_2 = 60$ kg, $m_0 = 80$ kg. We study the effects of stiffness of the automobile body (i.e., consider different Young's modulus) on the vibration properties of the total automobile by the CEM model proposed above. The results are presented in Table 12, which are compared with those of the rigid model.

From Table 12, one can find that the stiffness of the automobile body will make less impact on the lower order of vibration of the total automobile, but an obvious impact on the higher order of vibration.

Below we discuss several simplified cases.

6.2.3. *Case* 1

Consider the automobile body as a rigid body. So, we take $\theta_1 = \theta_2 = \theta_3 = c_{31} = c_{32} = c_{41} = c_{42} = 0$. One can obtain the corresponding expressions of the mass and stiffness matrices from the globe mass and stiffness matrices.

6.2.4. *Case* 2

Consider the automobile body as a rigid body, and neglect the masses of two support springs. So, we take $\theta_1 = \theta_2 = \theta_3 = c_{31} = c_{32} = c_{41} = c_{42} = 0$, $c_{11} = c_{12} = c_{21} = c_{22} = 0$, $m_1 = m_2 = 0$. One can obtain the corresponding expressions of the mass and stiffness matrices from the globe mass and stiffness matrices.

6.2.5. *Case* 3

Consider the automobile body as a elastic beam, and neglect the masses of two support bars. So, we take $c_{11} = c_{12} = c_{21} = c_{22} = 0$, as well as $m_1 = m_2 = 0$. Also one can obtain the corresponding mass and stiffness matrices.

From Table 13, we can find that the simplified model 3 (case 3) gives a good result to the rigid model since we have neglected the masses of support wheels (i.e., $m_1 = m_2 = 0$), but the simplified models 1 and 2 give poor appropriate results since we have let $\theta_1 = \theta_2 = 0$ which is not suitable for the rigid case.

6.3. VIBRATION ANALYSIS OF A FRAME

Consider a frame made of 4 beams shown in Figure 12, and find the natural frequencies. The related data are: $L = 6$ m, cross-sectional area $A = 0{\cdot}1$ m^2, area moment of inertia of the cross section $I = 1 \times 10^{-2}$ m^4, density $\rho = 7800$ kg/m^3, Young's modulus $E = 10^3$ MPa. We idealize the frame into 4 beam elements by the CEM. The resultant natural frequencies are listed in Table 14.

ACKNOWLEDGMENT

This Project 59775017 is supported by National Natural Science Foundation of China. Also, the work involved in the paper is supported by Alexander von Humboldt Foundation of Germany and Doctoral Dissertation Foundation of Tsinghua University.

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