



LETTERS TO THE EDITOR



HYDRODYNAMIC MODES IN PIPES WITH SUPERIMPOSED UNIFORM MEAN FLOW AND VISCO-THERMAL EFFECTS

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1. INTRODUCTION

Some recent papers have dealt with the propagation of sound in narrow pipes carrying mean flow. This new interest has come especially from the objective of modelling automobile catalytic converters [1–5].

In a general sheared mean flow, the problem of acoustic propagation in circular pipes has no analytical solution. With account taken of viscothermal dissipation, solutions have been determined when a parabolic mean flow profile occurs, by variational or numerical methods [3–5]. In a different way, Dokumaci has studied the problem of propagation in circular pipes with superimposed uniform mean flow in reference [1] and generalized this study to rectangular pipes in reference [2]. The uniform mean flow profile leads to an analytical eigenequation for the wavenumber (equation (13) in reference [1]). In reference [2], it was shown that the results on attenuation are not significantly different when uniform or parabolic mean flow profiles are used. Then with the simplified Dokumaci approach it is possible to begin the study of the effect of a non-turbulent flow on the acoustic dissipation.

Without any dissipation, fluctuations of both vorticity and entropy are only convected by the flow. Those fluctuations often have little influence on the pressure fluctuations but generate velocity fluctuations. So, to a linearized approximation acoustic fluctuations on the one hand and vorticity and entropy fluctuations on the other hand are decoupled (the splitting theorem [6]). With dissipation, it is natural to investigate fluctuations convected by the mean flow as well. In reference [2], Dokumaci stated that his attempts to extract numerically hydrodynamic modes from his analytical eigenequation had been abortive. In this paper, those modes are found by writing equation (13) in reference [1] in a slightly different form from which it is more convenient to extract hydrodynamic modes.

In the first section of this letter, the same dispersion relation as the one found by Dokumaci is given, but in a slightly different form. This latter will be found to be more suitable for the numerical search for the hydrodynamic modes. Some notations differ from the original derivation [1] to make the interpretation of hydrodynamic modes easier. Then, the approximate reduced wave number of the hydrodynamic modes are found when the Mach number is small and a physical interpretation of these modes is given.

2. DERIVATION OF THE EIGENEQUATION

In this section, the basic linear equation is presented which governs the propagation of axisymmetric fluctuations in a tube of radius a , carrying a uniform mean flow of velocity U_0 , in presence of viscosity and thermal diffusion. The fluctuating variables used here are pressure p , axial velocity u , radial velocity v , density ρ and temperature T . The parameters which specify the properties of the fluid are the ambient value of density ρ_0 , the temperature T_0 , the kinematic viscosity ν , the coefficient of thermal conductivity κ , the specific heat coefficient at constant pressure C_p , the specific heat coefficients ratio γ and the universal gas constant R .

It is assumed that the pressure is uniform on any section and that any axial gradient is small compared to the radial gradient of the same quantity (low reduced frequency assumptions; cf. reference [7]). Thus, the simplified equations are the following: the Navier–Stokes equations,

$$Du/Dt = (1/\rho_0)\partial p/\partial x + \nu\Delta_{\perp}u, \quad \partial p/\partial r = 0; \quad (1, 2)$$

the continuity equation,

$$D\rho/Dt + \rho_0(\partial u/\partial x + (1/r)\partial rv/\partial r) = 0; \quad (3)$$

the energy equation

$$\rho_0 C_p DT/Dt = Dp/Dt + \kappa\Delta_{\perp}T; \quad (4)$$

the state equation for a perfect gas,

$$\rho = p/RT_0 - \rho_0 T/T_0. \quad (5)$$

Here the substantive derivative is $D/Dt = \partial/\partial t + U_0\partial/\partial x$ and the transverse Laplacian is $\Delta_{\perp} = 1/r\partial(r\partial/\partial r)/\partial r$.

Associated with these equations are the boundary conditions (at $r = 0$, $v = 0$ and u and T have finite values and at $r = a$, $u = v = 0$, $T = 0$). This set of equations is the basis for the calculation of the fluctuating approximate fields outside the sources.

For an harmonic motion ($\partial/\partial t \equiv i\omega$, ω denotes the radian frequency), solutions are sought in the general form $q(x, r, t) = \hat{q}(r) e^{i(\omega t - kx)}$ for $q = u, v, T$. Equation (2) leads to $\hat{p}(r) = C$. By using equations (1) and (4), the axial velocity and the temperature can be written as

$$\hat{u}(r) = (1/\rho_0 c_0)(AJ_0(\beta r) + [K/(1 - KM)]C), \quad \hat{T}(r) = (1/\rho_0 C_p)(BJ_0(\sigma\beta r) + C), \quad (6, 7)$$

where c_0 is the speed of sound, $K = k/(\omega/c_0)$ the reduced wave number, $M = U_0/c_0$ the Mach number, $\beta^2 = -i(1 - KM)\omega/\nu$, $\sigma^2 = \rho_0\nu C_p/\kappa$ the Prantld number and J_m the Bessel function of order m .

Imposing the boundary conditions $u = 0$ and $T = 0$ at $r = a$ in equations (6) and (7), gives two equations for the three constants (A, B, C). A third equation is obtained by calculating v from equation (3) with the help of equation (5) and

imposing the boundary condition $v = 0$ at $r = a$. Eventually, this leads to the following system for (A, B, C) :

$$AJ_0(\beta a) + [K/(1 - KM)]C = 0, \quad BJ_0(\sigma\beta a) + C = 0, \quad (8a, b)$$

$$\frac{J_1(\beta a)}{\beta a} A + (\gamma - 1) \frac{1 - KM}{K} \frac{J_1(\sigma\beta a)}{\sigma\beta a} B + \frac{1 - KM}{2K} \left(\left(\frac{K}{1 - KM} \right)^2 - 1 \right) C = 0. \quad (8c)$$

The determinant of the system must vanish to have a non-trivial zero solution. This gives the eigenequation for the wavenumber K :

$$\gamma J_0(\beta a) J_0(\sigma\beta a) + (\gamma - 1) J_0(\beta a) J_2(\sigma\beta a) + \left(\frac{K}{1 - MK} \right)^2 J_0(\sigma\beta a) J_2(\beta a) = 0. \quad (9)$$

This equation was initially given by Dokumaci [1] in the form of equation (9) divided by $\gamma J_0(\beta a) J_0(\sigma\beta a)$. The form (9) is more appropriate to the calculation of hydrodynamic modes in the sense that the singularities of this equation in Dokumaci's form ($J_0(\sigma\beta a) = 0$ and $J_0(\beta a) = 0$) are very close to the exact hydrodynamic solutions.

3. SOLUTIONS OF THE EIGENEQUATION

Two kinds of solutions of equation (9) can be found.

(a) The first kind of solution corresponds to travelling in both axial directions. The phase velocity of these modes, relatively to the mean flow, is close to c_0 when $S_h = a\sqrt{\omega/\nu}$ is high. So, these two solutions could be called "true sound" or *acoustic modes* even if the phase velocity decreases quickly when S_h tends towards zero. For high shear numbers, $S_h \gg 1$, the propagation constants are given by [8] $K^\pm = \pm K_0/(1 \pm K_0 M)$ where K_0 is the classical Kirchhoff solution

$$K_0 = 1 + (1 - i)\alpha_0 \quad \text{with} \quad \alpha_0 = (1/\sqrt{2}S_h)(1 + (\gamma - 1)/\sigma).$$

For $M = 0$, the solution of equation (9) is simply

$$K^2 = -[\gamma J_0(\beta a) J_0(\sigma\beta a) + (\gamma - 1) J_0(\beta a) J_2(\sigma\beta a)]/J_0(\sigma\beta a) J_2(\beta a),$$

with $\beta^2 = -i\omega/\nu$, the classical "low reduced frequency" solution.

(b) The other kind of modes have a phase velocity close to the flow velocity (so they are travelling only in the downstream direction). They are called "pseudo-sound" or *hydrodynamic modes*.

The two kinds of modes are illustrated in Figure 1(a), where the numerical solutions of the dispersion equation are shown. On this figure are plotted the lines where the real part (dashed lines) and the imaginary part (solid lines) of the left side of equation (9) are equal to zero. At very intersection of these two kinds of lines (circles in Figure 1(a)) a solution of equation (9) is found. The modes A^+ and A^- correspond to the classical acoustic modes. The hydrodynamic modes are

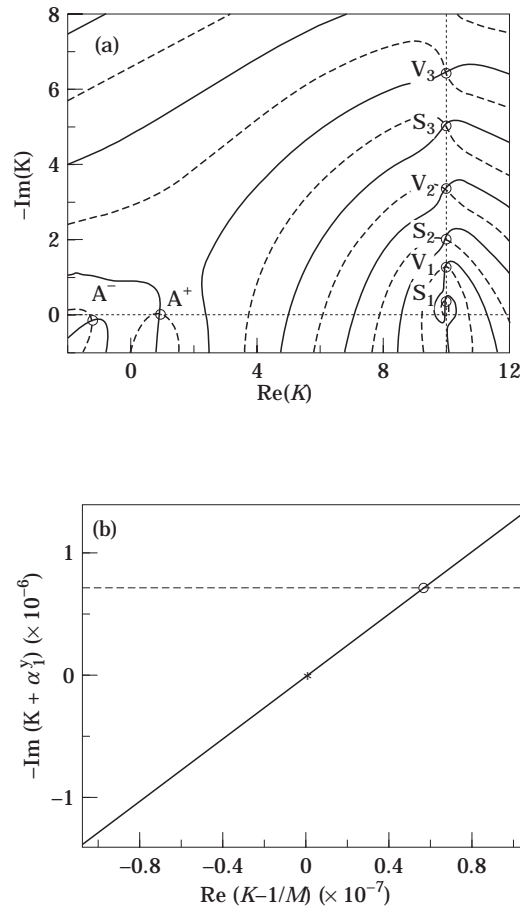


Figure 1. (a) Numerical solutions of equation (9) in the complex plane ($\text{real}(K)$, $-\text{imag}(K)$): solid lines: imaginary part of (9), dashed lines: real part of (9), circle: solutions of equation; (b) detail of (a) showing the hydrodynamic mode S1: ★, approximate solution; ○, more precise solution given by $\varepsilon(\omega a/c_0) = 0.0092$, $S_h = 14.5$, $M = 0.1$, $Pr = 0.71$, standard air condition).

called S_i and V_i because it will be shown that they can be split into two families: entropy and vorticity modes.

Analytically, it is possible to go further to obtain more insights into the hydrodynamic modes when the Mach number is small. The propagation constants of the hydrodynamic modes are sought under the form $K_h = 1/M - i\alpha + \varepsilon$ where α is the attenuation of these modes and ε is a small correction to be determined. When the Mach number is weak, the influence of ε is disregarded (it will be seen to be of the order of M). At this first order, the arguments of the Bessel functions in equation (9) are real: $\beta = \sqrt{\alpha M \omega / \nu}$ and the third term of equation (9) has the order M^{-2} (α will be seen to be of the order M^{-1}). For small Mach numbers, the first two terms of the eigenequation can be neglected and equation (9) simply reduces to

$$J_2(\beta a) \simeq 0 \quad \text{or} \quad J_0(\sigma \beta a) \simeq 0,$$

which are only estimations of the locations of the hydrodynamic modes. The attenuation is then given by $\alpha_n^v = j_{2,n+1}^2 / MS_h^2$ or by $\alpha_n^s = j_{0,n}^2 / M\sigma^2 S_h^2$ where $j_{m,n}$ is the n th zero of the Bessel function of order m with $n \geq 1$. The solution corresponding to $j_{2,1}(\alpha_0^v = 0)$ is excluded since it yields a trivial solution of the problem (all variables equal to zero). A more precise solution can be obtained by taking into account ε in K . For instance, when $J_0(\sigma\beta a) \simeq 0$, the Bessel function could be developed around its zero, $J_0(\sigma\beta a) = -i\varepsilon M\sigma^2 S_h^2 J_1(j_{0,n}) / 2j_{0,n} + o(\varepsilon^2)$ and, by introducing this value in equation (9), ε can be evaluated:

$$\varepsilon = -2i(\gamma - 1)Mj_{0,n}J_0(j_{0,n}/\sigma)J_2(j_{0,n})/\sigma^2 S_h^2 (1 + i(\sigma S_h/j_{0,n})^2)^2 J_2(j_{0,n}/\sigma)J_1(j_{0,n}).$$

Here the indices for ε have been omitted. The correction ε is of the order M and is very small compared to $1/M$ and to $\alpha_n^{v,s}$, because in the depicted case, S_h is much larger than 1. For instance, for the S_1 mode in Figure 1(a) (which is detailed in Figure 1(b)), $1/M = 10$ and $\alpha_1^s = 0.389$ while the correction ε is equal to $5.4 \times 10^{-8} - i 7 \times 10^{-7}$. One can see in Figure 1(b) that the numerical exact solution (at the intersection of the two lines) is very close to the more precise solution given by ε (circle). The same kind of result could be found with the other family of solutions ($J_2(\beta a) \simeq 0$).

Thus, two new families of solutions for the equation (9) have been found. When the Mach number is small, these solutions have wavenumbers approximately given by

$$k_n^v \simeq \omega/U_0 - ij_{2,n}^2 v/U_0 a^2 \quad \text{and} \quad k_n^s \simeq \omega/U_0 - ij_{0,n}^2 v/U_0 \sigma^2 a^2. \quad (10)$$

Remark. it should be mentioned that the assumption of this study ($\partial/\partial r \gg \partial/\partial x$) imposes a new condition due to the solution of the hydrodynamic modes. Actually $|\partial/\partial x| \simeq \omega K/c_0 \simeq \omega/c_0 M$ and $|\partial/\partial r| \simeq \beta\delta \simeq j_{m,n}/a \geq j_{0,1}/a$ ($\delta = 1$ or σ). This yields $\omega/c_0 \ll Mj_{0,1}/a$.

4. INTERPRETATION OF HYDRODYNAMIC MODES

For the first family of hydrodynamic modes ($J_2(\beta a) \simeq 0$ or more precisely $J_2(\beta a) = \mathcal{O}(M^2)$), equations (8a, b) imply that C and B are of the order of MA . In this case $A \gg B$ and $A \gg C$, and this mode corresponds to a normalized velocity $\rho_0 c_0 \hat{u}$ that is very large compared to the normalized temperature $\rho_0 C_p \hat{T}$ and to the pressure \hat{p} . The compressibility effects are also very weak. This mode could be seen as a ‘‘vorticity mode’’. For the second family of hydrodynamic modes ($J_0(\sigma\beta a) \simeq 0$ or more exactly $J_0(\sigma\beta a) = \mathcal{O}(M^2)$), equations (8a, b) imply that C is of the order of MA and $M^2 B$. In this case $B \gg A \gg C$, and this mode corresponds to a very high normalized temperature compared to the normalized velocity, this normalized velocity being high compared to the normalized pressure. This temperature mode with small velocity and pressure fluctuations may be called ‘‘entropy mode’’.

The same kinds of results have been found numerically by Astley and Cummings [4]. The less attenuated hydrodynamic mode they found (2^+) is a ‘‘temperature perturbation’’ and the second hydrodynamic mode (3^+) is a ‘‘velocity perturbation’’.

Like available numerical results for shear flow [4, 5], one finds that the hydrodynamic modes have an attenuation independent of the frequency. An estimation of the ratio of the attenuation of hydrodynamic modes and acoustic modes is given by

$$\alpha_i^s/\alpha_0 = [\sqrt{2}j_{0,1}^2/\sigma(\sigma + \gamma - 1)](1/MS_h).$$

The attenuation of the hydrodynamic modes is much higher than the attenuation of the acoustic modes when the product MS_h is much smaller than 4 (for standard air conditions).

5. CONCLUSION

New modes for the propagation of fluctuations with superimposed uniform flow and viscothermal dissipation have been found. They constitute supplementary solutions of the classical acoustic modes. A simple analysis has been used to show that they can be decomposed into two families: vorticity and entropy modes.

Möhring *et al.* [9] pointed out that, for the uniform profile, the pressure obeys a second order equation while the velocity obeys a third order equation. In the general case of shear flow, both pressure and velocity obey a third order equation. This shows the very peculiar case of a uniform mean flow profile compared to a shear flow one where both pressure and velocity equations are third order. Consequently, further studies are planned in order to include both shear flow and viscothermal dissipation effects.

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