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ON FREE VIBRATION OF A PIEZOELECTRIC COMPOSITE RECTANGULAR PLATE

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1. INTRODUCTION

Vibrations of piezoelectric plates have been studied for a long time. In fact, in as early as 1952, Mindlin [1] derived the two-dimensional approximate theory of thickness and bending vibrations for piezoelectric plates. Dokmeci [2] made a review on the main works of vibrations of piezoelectric crystals before 1980. Since then, the research has been further broadened and deepened [3–5].

The state space method is very powerful in the study of elastic layered structures [6]. Sosa and Castro [7] generalized the method to study the plane problem of piezoelectric layered structures. Lee and Jiang [8] and Chen *et al*. [9] independently derived the three-dimensional static state space formulae for transversely isotropic piezoelectricity and analyzed the bending problem of piezoelectric plates.

This letter presents the non-dimensional state equations for the vibration problem of transversely isotropic piezoelectric body. The direct expression of the corresponding transfer matrix is given, so that the inversion of matrix is avoided to improve the calculation efficiency. The simplified theory corresponding to the three-generalized-variable elastic plate theory is also given. Numerical comparison between these two theories is made by considering the free vibration of a simply supported PZT-4 ceramic plate.

2. THE STATE SPACE METHOD

Assuming the isotropic plane is perpendicular to the z axis, one can write down the constitutive relations of a transversely isotropic piezoelectric body as follows [10]

$$\begin{gathered}\sigma_x = c_{11}u_{,x} + c_{12}v_{,y} + c_{13}w_{,z} + e_{31}\phi_{,z}, \qquad \tau_{xz} = c_{44}(u_{,z} + w_{,x}) + e_{15}\phi_{,x},\\ D_x = e_{15}(u_{,z} + w_{,x}) - \varepsilon_{11}\phi_{,x}, \qquad \sigma_y = c_{12}u_{,x} + c_{11}v_{,y} + c_{13}w_{,z} + e_{31}\phi_{,z},\\ \tau_{yz} = c_{44}(v_{,z} + w_{,y}) + e_{15}\phi_{,y}, \qquad D_y = e_{15}(v_{,z} + w_{,y}) - \varepsilon_{11}\phi_{,y},\\ \sigma_z = c_{13}u_{,x} + c_{13}v_{,y} + c_{33}w_{,z} + e_{33}\phi_{,z}, \qquad \tau_{xy} = \tfrac{1}{2}(c_{11} - c_{12})(u_{,y} + v_{,x}),\\ D_z = e_{31}(u_{,y} + v_{,x}) + e_{33}w_{,z} - \varepsilon_{33}\phi_{,z}, \qquad (1)\end{gathered}$$

where u, v and w are components of displacement; σ_i and D_i are components of stress and electric displacement respectively; ϕ is electric potential; c_{ij}, e_{ij} and ε_{ij} are elastic, piezoelectric and dielectric constants respectively. A comma in subscript indicates partial derivatives with respect to the followed variables. The

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governing equations can also be found in reference [10] and they can be non-dimensionalized by introducing the following parameters, in accordance with the geometry of the plate (Figure 1)

$$\begin{cases} U = u/a, \quad V = v/b, \quad W = w/h, \quad D = D_z/e_{33}, \quad \Phi = \phi_{33}/he_{33}\varepsilon, \quad Z = \sigma_z/c_{33}, \\ X = \tau_{xz}/c_{44}, \quad Y = \tau_{yz}/c_{44}, \quad \xi = x/a, \quad \eta = y/b, \quad \zeta = z/h, \quad s_1 = a/b, \quad s_2 = h/b, \\ \tau = v_0 t/h, \quad v_0^2 = c_{44}/\rho, \quad f_1 = c_{11}/c_{44}, \quad f_2 = c_{12}/c_{44}, \quad f_3 = c_{13}/c_{44}, \quad f_4 = c_{33}/c_{44}, \\ f_5 = e_{15}/e_{33}, \quad f_6 = e_{31}/e_{33}, \quad f_7 = \varepsilon_{11}/\varepsilon_{33}, \quad f_8 = e_{33}^2/(\varepsilon_{33}c_{44}) \end{cases} \tag{2}$$

where ρ is density. By generalizing the state space method in elasticity [6], one can establish the following state equation for piezoelectricity:

$$\frac{\partial}{\partial \zeta} A = KA \tag{3}$$

where $A = [U, V, D, Z, X, Y, \Phi, W]^{\mathrm{T}}$ and

$$K = \begin{bmatrix} 0 & K_1 \\ K_2 & 0 \end{bmatrix}$$

with

$$K_1 = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 p_1 & \alpha_3 p_1 \\ 0 & \alpha_4 & \alpha_5 p_2 & \alpha_6 p_2 \\ \alpha_7 p_1 & \alpha_8 p_2 & \alpha_9 p_1^2 + \alpha_{13} p_2^2 & 0 \\ \alpha_{11} p_1 & \alpha_{12} p_2 & 0 & p^2/f_4 \end{bmatrix} \tag{4-1}$$

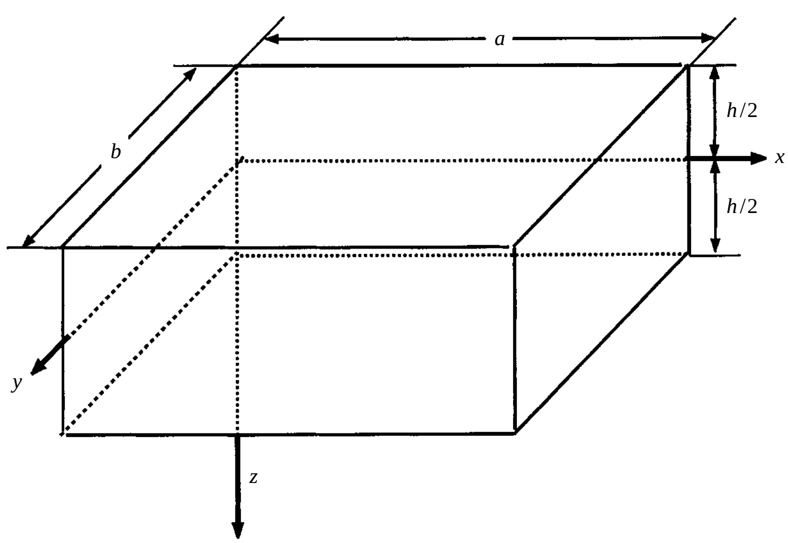

Figure 1. The geometry of a rectangular plate.

$$K_2 = \begin{bmatrix} \beta_1 p^2 + \alpha_{13} p_1^2 + \alpha_{14} p_2^2 & \alpha_{15} p_1 p_2 & \alpha_{16} p_1 & \alpha_{17} p_1 \\ \alpha_{18} p_1 p_2 & \beta_2 p^2 + \alpha_{19} p_1^2 + \alpha_{20} p_2^2 & \alpha_{21} p_2 & \alpha_{22} p_2 \\ \alpha_{23} p_1 & \alpha_{24} p_2 & \alpha_{25} & \alpha_{26} \\ \alpha_{27} p_1 & \alpha_{28} p_2 & \alpha_{29} & \alpha_{30} \end{bmatrix} \tag{4-2}$$

where $p = \partial/\partial\tau$, $p_1 = \partial/\partial\xi$ and $p_2 = \partial/\partial\eta$, and

$$\left\{ \begin{array}{l} \alpha_1 = s_2/s_1, \quad \alpha_2 = -\alpha_1^2 f_5 f_8, \quad \alpha_3 = -\alpha_1^2, \quad \alpha_4 = s_2, \quad \alpha_5 = -\alpha_4^2 f_5 f_8, \quad \alpha_6 = -\alpha_4^2, \\ \alpha_7 = -f_5\alpha_1, \quad \alpha_8 = -f_5\alpha_4, \quad \alpha_9 = -(f_5^2 f_8 + f_7)\alpha_3, \quad \alpha_{10} = -(f_5^2 f_8 + f_7)\alpha_6, \\ \alpha_{11} = -\alpha_1/f_4, \quad \alpha_{12} = -\alpha_4/f_4, \\ \alpha_{13} = \alpha_1(f_3^2 + 2f_6 f_3 f_8 - f_6^2 f_4 f_8 - f_1 f_4 - f_1 f_8)/(f_4 + f_8), \\ \alpha_{14} = -\frac{1}{2}(f_1 - f_2)s_1 s_2, \quad \alpha_{15} = \alpha_{13} + \frac{1}{2}(f_1 - f_2)\alpha_1, \quad \alpha_{16} = f_8\alpha_1\alpha_{23}, \quad \alpha_{17} = f_4\alpha_1\alpha_{27}, \\ \alpha_{18} = \alpha_{15}s_1, \quad \alpha_{19} = \alpha_{14}/s_1^3, \quad \alpha_{20} = \alpha_{13}s_1, \quad \alpha_{21} = \alpha_{16}s_1, \quad \alpha_{22} = \alpha_{17}s_1, \\ \alpha_{23} = \alpha_{24} = (f_4 f_6 - f_3)/(f_4 + f_8), \quad \alpha_{26} = -\alpha_{25} = \alpha_{30} = f_4/(f_4 + f_8), \\ \alpha_{29} = f_8/(f_4 + f_8), \quad \alpha_{27} = \alpha_{28} = -(f_3 + f_6 f_8)/(f_4 + f_8), \\ \beta_1 = s_1/s_2, \quad \beta_2 = 1/s_2 \end{array} \right. \tag{5}$$

Considering the following simply supported conditions,

$$\sigma_x = V = W = \Phi = 0 \quad \text{for} \quad \zeta = 0, 1 \quad \text{and} \quad \sigma_y = U = W = \Phi = 0 \ \text{for} \ \eta = 0, 1 \tag{6}$$

it is assumed that

$$\begin{bmatrix} U \\ V \\ D \\ Z \\ X \\ Y \\ \Phi \\ W \end{bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} U_{mn}(\zeta)\cos(m\pi\xi)\sin(n\pi\eta) \\ V_{mn}(\zeta)\sin(m\pi\zeta)\cos(n\pi\eta) \\ D_{mn}(\zeta)\sin(m\pi\zeta)\sin(n\pi\eta) \\ Z_{mn}(\zeta)\sin(m\pi\xi)\sin(n\pi\eta) \\ X_{mn}(\zeta)\cos(m\pi\xi)\sin(n\pi\eta) \\ Y_{mn}(\zeta)\sin(m\pi\xi)\cos(n\pi\eta) \\ \Phi_{mn}(\zeta)\sin(m\pi\xi)\sin(n\pi\eta) \\ W_{mn}(\zeta)\sin(m\pi\zeta)\sin(n\pi\eta) \end{bmatrix} \mathrm{e}^{i\Omega\tau} \tag{7}$$

where the non-dimensional frequency is defined as $\Omega = \omega h/v_0$ and ω is the circular frequency. Obviously, equation (7) identically satisfies the boundary conditions (6). Substituting it into equation (3) gives

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} A_{mn} = K_{mn} A_{mn} \tag{8}$$

where $A_{mn} = [U_{mn}, V_{mn}, D_{mn}, Z_{mn}, X_{mn}, Y_{mn}, \Phi_{mn}, W_{mn}]^{\mathrm{T}}$ and K_{mn} is an eighth order coefficient matrix whose elements can be readily obtained from equation (4). According to the matrix theory, the solution to equation (8) can be obtained as

$$A_{mn}(\zeta) = \exp[(\zeta + 0{\cdot}5)K_{mn}] \cdot A_{mn}(-0{\cdot}5) \tag{9}$$

where the exponential function matrix $\exp[(\zeta + 0{\cdot}5)K_{mn}]$ is known as the transfer matrix. For the sake of convenience, we will focus our discussion on matrix $\exp(\zeta K_{mn})$ instead of this transfer matrix in the following and the results can be easily applied to it. By virtue of the Hamilton–Cayley theorem, one can obtain

$$\exp(\zeta K_{mn}) = a_1(\zeta)I + \sum_{i=1}^{7} a_{i+1}(\zeta)K_{mn}^{i} \tag{10}$$

where I is an 8×8 unit matrix, a_i $(i = 1, 2, \ldots, 8)$ are relative to the eigenvalues of matrix K_{mn}, of which the eigen equation is

$$\lambda^8 + b_1\lambda^6 + b_2\lambda^4 + b_3\lambda^2 + b_4 = 0 \tag{11}$$

where b_i $(i = 1, 2, \ldots, 4)$ can be expressed by the elements of matrix K_{mn}. Since equation (11) is a quadruplicate algebraic equation about λ^2, its eight eigenvalues can be written as $\lambda_k = -\lambda_{k+4}$ $(k = 1, 2, \ldots, 4)$. When these eigenvalues are distinct, one can get a_i as follows

$$\left\{\begin{aligned}
&a_1 = \sum_{k=1}^{4} h_k\lambda_{k+1}^2\lambda_{k+2}^2\lambda_{k+3}^2, \quad a_2 = \sum_{k=1}^{4} g_k\lambda_{k+1}^2\lambda_{k+2}^2\lambda^2{}_{k+3,}\\
&a_3 = -\sum_{k=1}^{4} h_k(\lambda_{k+1}^2\lambda_{k+2}^2 + \lambda_{k+1}^2\lambda_{k+3}^2 + \lambda_{k+2}^2\lambda_{k+3}^2),\\
&a_4 = -\sum_{k=1}^{4} g_k(\lambda_{k+1}^2\lambda_{k+2}^2 + \lambda_{k+1}^2\lambda_{k+3}^2 + \lambda_{k+2}^2\lambda_{k+3}^2),\\
&a_5 = \sum_{k=1}^{4} h_k(\lambda_{k+1}^2 + \lambda_{k+2}^2 + \lambda_{k+3}^2), \quad a_6 = \sum_{k=1}^{4} g_k(\lambda_{k+1}^2 + \lambda_{k+2}^2 + \lambda_{k+3}^2),\\
&a_7 = -\sum_{k=1}^{4} h_k, \quad a_8 = -\sum_{k=1}^{4} g_k
\end{aligned}\right. \tag{12}$$

and

$$h_k = \frac{(\mathrm{e}^{\lambda_k\zeta} + \mathrm{e}^{-\lambda_k\zeta})}{2d_k}, \qquad g_k = \frac{(\mathrm{e}^{\lambda_k\zeta} - \mathrm{e}^{-\lambda_k\zeta})}{2\lambda_k d_k},$$
$$d_k = (\lambda_{k+1}^2 - \lambda_k^2)(\lambda_{k+2}^2 - \lambda_k^2)(\lambda_{k+3}^2 - \lambda_k^2) \tag{13}$$

where $k = 1, 2, 3, 4$. It is easy to prove that $a_i = \bar{a}_i$ ($i = 1, 2, \ldots, 8$), i.e., all a_i are real. For equal eigenvalues, one shall solve a_i in other manners [11]. For the plane problem, the transfer matrix is 6×6 and its eigen equation is a cubic algebraic equation about λ^2, see reference [7]. By looking into equations (15) and (18) of that paper, one can find that the negative sign was missed in coefficients a_3, a_4. The same error can also be found in reference [12]. Setting $\zeta = 0{\cdot}5$ in equation (9) gives

$$[U_{mn}(0{\cdot}5), \quad V_{mn}(0{\cdot}5), \quad 0, \quad 0, \quad 0, \quad 0, \quad \Phi_{mn}(0{\cdot}5), \quad W_{mn}(0{\cdot}5)]^{\mathrm{T}}$$

$$= \exp(K_{mn})[U_{mn}(-0{\cdot}5),\ V_{mn}(-0{\cdot}5),\ 0,\ 0,\ 0,\ 0,\ \Phi_{mn}(-0{\cdot}5),\ W_{mn}(-0{\cdot}5)]^{\mathrm{T}}. \quad (14)$$

The third to the sixth equations in equations (14) yield

$$H_1[U_{mn}(-0{\cdot}5), \quad V_{mn}(-0{\cdot}5), \quad \Phi_{mn}(-0{\cdot}5), \quad W_{mn}(-0{\cdot}5)]^{\mathrm{T}} = \{0\} \qquad (15)$$

where H is a fourth order square matrix derived from matrix $\exp(K_{mn})$ by eliminating the relevant columns and rows. The vanishing of the coefficient determinant of equation (15) gives the exact frequency equation for every couple of (m, n):

$$\det|H_1| = 0. \qquad (16)$$

3. TWO-DIMENSIONAL PLATE THEORY

For two-dimensional analysis, assuming

$$u = -z\psi_x(x, y), \qquad v = -z\psi_y(x, y), \qquad w = w(x, y), \qquad \phi = g(z)f(x, y) \quad (17)$$

where $g(z)$ denotes the distribution of electric potential along the thickness. Equation (17) corresponds to the three-generalized-variable plate theory if the piezoelectric effect is not considered [13]. Similar to elasticity, we can derive the governing equations as follows:

$$\begin{cases} R\kappa_1\psi_{x,xx} + Rc_{66}\psi_{x,yy} + (R\kappa_2 + Rc_{66})\psi_{y,xy} + c_{44}h(w_{,x} - \psi_x) + e_{15}khf_{,x} - \rho R\psi_{x,tt} = 0 \\ (R\kappa_2 + Rc_{66})\psi_{x,xy} + Rc_{66}\psi_{y,xx} + R\kappa_1\psi_{y,yy} + c_{44}h(w_{,y} - \psi_y) + e_{15}khf_{,y} - \rho R\psi_{y,tt} = 0 \\ c_{44}h\nabla^2 w + e_{15}kh\nabla^2 f - c_{44}h(\psi_{x,x} + \psi_{y,y}) - \rho h w_{,tt} = 0 \\ e_{15}h\nabla^2 w - \varepsilon_{11}kh\nabla^2 f - e_{15}h(\psi_{x,x} + \psi_{y,y}) = 0 \end{cases} \qquad (18)$$

where

$$k = (1/h)\int_{-h/2}^{h/2} g(z)\,\mathrm{d}z, \qquad R = h^3/12, \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

and

$$\kappa_1 = c_{11} - (\alpha c_{13} + \beta e_{31})/\gamma, \qquad \kappa_2 = c_{12} - (\alpha c_{13} + \beta e_{31})/\gamma,$$

$$\alpha = c_{13}\varepsilon_{33} + e_{31}e_{33}, \qquad \beta = c_{13}e_{33} - c_{33}e_{31}, \qquad \gamma = e_{33}^2 + \varepsilon_{33}c_{33}. \qquad (19)$$

For simply supported rectangular plate, analogy to the three-dimensional analysis, by expanding the unknown functions ψ_x, ψ_y, w and f in terms of trigonometric series, a system of four linear algebraic homogeneous equations can be derived from equation (18), which gives the corresponding two-dimensional frequency equation as follows:

$$\det|H_2| = 0. \tag{20}$$

For the sake of simplicity, the elements in H_2 are not given here.

4. NUMERICAL CALCULATIONS

Consider the free vibration of a PZT-4 ceramic plate, whose material constants can be found in reference [14]. Frequency equation (16) corresponds to three-dimensional theory so that for every couple of (m, n), one can get an arbitrary number of the non-dimensional frequency Ω. On the other hand, equation (20) corresponds to the two-dimensional plate theory, which is finally shown to be a cubic algebraic equation about Ω^2, so that at most only three real frequencies can be obtained. Because the smallest frequency is the most important in general engineering consideration, we will thus pay attention to it in what follows. Figures 2 and 3 display the curves of the smallest non-dimensional frequency Ω versus thickness-to-span ratio s_2 for a rectangular plate, whose length-to-width ratio is chosen to be 2. In both figures, the solid lines correspond to three-dimensional theory while the dotted ones to the two-dimensional theory.

It can be seen from the figures that the non-dimensional frequency Ω increases as the thickness-to-span ratio increases. A comparison shows that the non-dimensional Ω calculated by the two-dimensional plate theory is always larger than the corresponding one by three-dimensional theory. It can also be seen that the two-dimensional curve deviates gradually from the corresponding three-dimensional one with the increase of the thickness-to-span ratio. These facts are

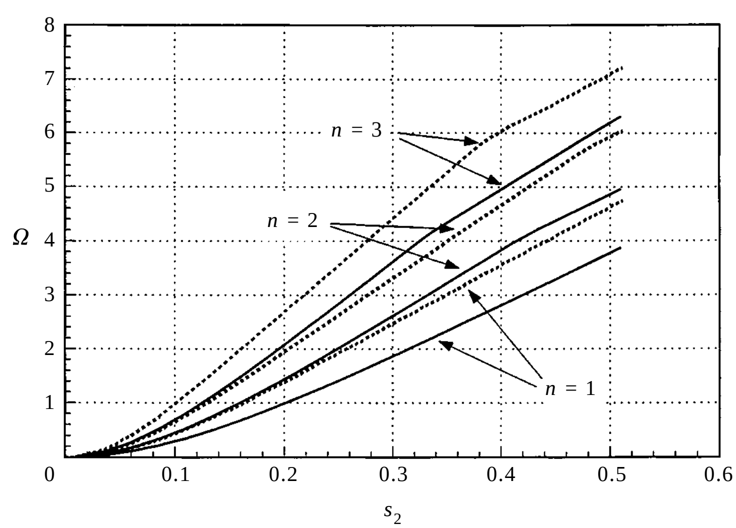

Figure 2. Non-dimensional frequencies for rectangular plates ($s_1 = 0{\cdot}5, m = 1$).

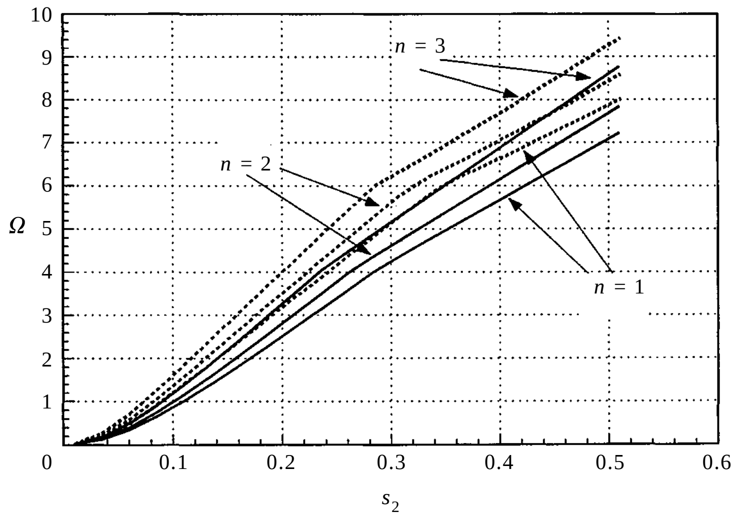

Figure 3. Non-dimensional frequencies for rectangular plates ($s_1 = 0{\cdot}5, m = 2$).

in fact identical with those of the elastic plate. For a square plate there must be no difference between x and y directions. Though numerical results are not presented, it has actually been proved in our calculations.

5. CLOSURE

This letter derived the non-dimensional state equation of a transversely isotropic piezoelectric body. The direct expression of the transfer matrix was derived so that the inversion of matrix is avoided. The free vibration of a simply supported rectangular plate was therefore analyzed. The two-dimensional analysis was also presented. It is worth mentioning here that the state space method can also be applied to analyze bending and stability problems of laminated plates.

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