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ON A GENERAL MODEL FOR DAMPING

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When a structure exhibits a damped dynamic behavior that does not conform to the classical and well-known viscous or hysteretic models, a more general approach must be sought. Several questions may then be raised, like (i) in which theoretical background should one base the investigation?, or (ii) can one apply common modal analysis tools to solve the problem? Recent works have shown that some types of materials and therefore structures demand a more rigorous behavior description and that it seems possible to address the problem by means of the theory of fractional derivatives, leading to a model in terms of general damping parameters. Such an approach reveals itself somewhat complicated to implement in practice and some simplifications are necessary. The authors discuss the use of a generalized damping concept for modeling the dynamic behavior of linear systems and show how this concept allows for a clearer interpretation and explanation of the behavior displayed by the common viscous and hysteretic models.

1. INTRODUCTION

Nowadays, it is known that the behavior of certain types of materials, namely viscoelastic ones, are better described—in what damping characteristics are concerned—by a model that takes into consideration terms that are non-integer powers of the exciting frequency.

The theory that supports such models is based on fractional derivatives. Fractional derivatives are part of the now called fractional calculus which is not a new subject. The idea started in the 17th century when L'Hôpital wrote to Leibnitz asking whether he thought that a non-integer derivative would make any sense. Many other well-known scientists like Euler, Lacroix, Abel, Fourier, Liouville, etc., have devoted some of their interest to the subject, along the years [1]. Since then, the subject has evolved slowly and it is only in the last 20 years that the theory of fractional calculus has become more known, particularly with the advent of the theory of Fractals [2].

The articles by Bagley and Torvik [3, 4] and by Gaul [5–7] have shown the importance that the application of fractional derivatives has in modelling in a more accurate way the damping characteristics and consequently the complex Young

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modulus of some materials, leading to a more precise reconciliation between the model response and the actual behavior of the structure under analysis.

Here, the authors address the problem with the objective of building a general formulation for possible applications in modal analysis and show that the usual damping models can be described as particular cases of a general one.

2. THEORETICAL DEVELOPMENT

The first order time-derivative of the displacement is the velocity, the second order derivative the acceleration, etc. What can be the meaning of the derivative of order 0·5, or −1·7, or even 1·2–i0·3 (the general case of complex order derivatives has been particularly addressed in the works of Campos [8])?

The general definition of a fractional derivative, given by Riemann–Liouville as the inverse operation of fractional integration, is [1, 9]:

$$D^{v}(x(t)) = \frac{\mathrm{d}^{v}x(t)}{\mathrm{d}t^{v}} = \frac{1}{\Gamma(1-v)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\frac{x(\tau)}{(t-\tau)^{v}}\,\mathrm{d}\tau \tag{1}$$

where $0 < v < 1$ and Γ is the Gamma function, defined as $\Gamma(1-v) = \int_0^\infty x^{-v}\,\mathrm{e}^{-x}\,\mathrm{d}x$. When considering harmonic vibrations and harmonic waves in mechanical systems, the lower limit of the integral in (1) is replaced by $-\infty$ [9, 10]:

$$D^{v}(x(t)) = \frac{\mathrm{d}^{v}x(t)}{\mathrm{d}t^{v}} = \frac{1}{\Gamma(1-v)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\frac{x(\tau)}{(t-\tau)^{v}}\,\mathrm{d}\tau. \tag{2}$$

The possible applications to modal analysis, associated with some degree of physical interpretation, are related to some particular and useful properties of definition (2), concerning Laplace and Fourier transforms. For zero initial conditions, the following relationships are verified:

$$\mathscr{L}(D^{v}(x(t))) = s^{v}\mathscr{L}(x(t)) \tag{3}$$

and

$$\mathscr{F}(D^{v}(x(t))) = (\mathrm{i}\omega)^{v}\mathscr{F}(x(t)) \tag{4}$$

Now concentrate on this last expression and take the case of harmonic motion:

$$x(t) = X\,\mathrm{e}^{\mathrm{i}\omega t} \tag{5}$$

The nth order derivative of (5) when n is an integer, is

$$D^{n}(x(t)) = (\mathrm{i}\omega)^{n}X\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{6}$$

Equation (4) allows for an immediate generalization to a non-integer derivative of order v:

$$D^{v}(x(t)) = (\mathrm{i}\omega)^{v}X\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{7}$$

As $i = e^{i\pi/2}$ and, consequently, $i^v = e^{iv\pi/2}$,

$$D^v(x(t)) = \omega^v X\, e^{i(\omega t + v\pi/2)}. \tag{8}$$

It should be noted that although definition (2) is only valid for $0 < v < 1$, v may have a bigger value, as it is true that $D^{v_1}(D^{v_2}(x)) = D^{v_1 + v_2}(x)$, for v_1 integer. For example, to calculate the derivative of order 2·3, it is enough to know how to calculate the derivative of order 0·3, as $D^2(D^{0\cdot 3}(x)) = D^{2 + 0\cdot 3}(x)$. So, definition (2) does not restrict the application of (3) or (4) to just $0 < v < 1$.

From (8), it is clear that for a harmonic motion, the fractional time derivative simply means a new harmonic motion of amplitude $\omega^v X$ leading the original vector by a phase angle equal $v(\pi/2)$. It is easy to see that for the particular cases $v = 1$ and $v = 2$ one obtains the velocity and acceleration, respectively. Figure 1 illustrates the concept for $0 < v < 1$.

From (5) and (7), one can also write

$$D^v[x(t)] = (i\omega)^v x(t) \tag{9}$$

and so, one can say that a harmonic motion is characterized by

$$D^v[x(t)] - (i\omega)^v x(t) = 0. \tag{10}$$

For the particular case where $v = 2$, (10) becomes

$$\ddot{x} + \omega^2 x = 0 \tag{11}$$

corresponding to the free vibration equation for a conservative single degree-of-freedom system:

$$m\ddot{x} + kx = 0. \tag{12}$$

As the inertia and stiffness terms correspond to well-known properties, the fractional derivation seems of no interest for the case of equation (12). However

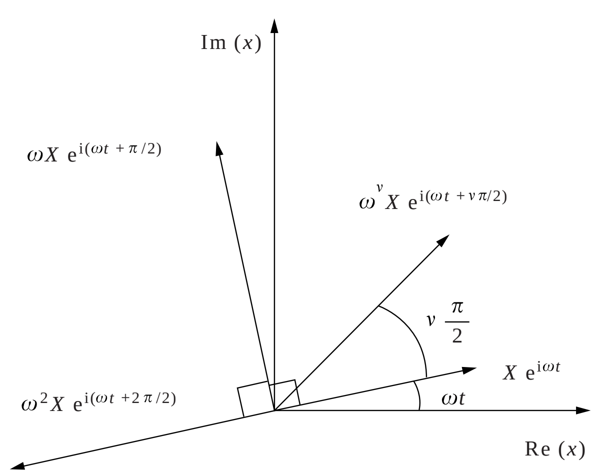

Figure 1. Physical interpretation of the fractional derivative of a harmonic motion.

considering viscous and hysteretic damping, as well as an applied force $f(t) = \mathrm{F}\,\mathrm{e}^{\mathrm{i}\omega t}$:

$$m\ddot{x} + c\dot{x} + kx + \mathrm{i}dx = F\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{13}$$

$m\ddot{x}$ and kx are well-known forces; one cannot say the same about damping forces. The viscous and hysteretic damping models are, many times, only coarse approximations of reality. In most cases, materials and structures exhibit a more complex behavior in terms of energy dissipation. It is here that fractional derivatives have a role to play, as expressed in [3–5]. Thus, instead of (13), one could write

$$m\ddot{x} + kx + \sum_{j=1}^{l} g_j D^{v_j} x = F\,\mathrm{e}^{\mathrm{i}\omega t} \tag{14}$$

where g_j are complex coefficients and l is the number of damping forces. Equation (14) can even be written in a more compact form, by including the first two terms in the summation. The dynamic equilibrium equation of a single degree-of-freedom system subjected to a harmonic force may therefore be written as:

$$\sum_{j=1}^{l'} g_j D^{v_j} x = F\,\mathrm{e}^{\mathrm{i}\omega t} \tag{15}$$

where $l' = l + 2$ and $0 \leqslant v_j \leqslant 2$. The steady-state solution of this differential equation is:

$$x(t) = \bar{X}\,\mathrm{e}^{\mathrm{i}\omega t} \tag{16}$$

where $\bar{X}$ is the complex amplitude of the motion.

Substituting (16) in (15) and taking into account (8), it follows that (making $g_j = a_j + \mathrm{i}b_j$):

$$\sum_{j=1}^{l'} (a_j + \mathrm{i}b_j)\omega^{v_j}\bar{X}\,\mathrm{e}^{\mathrm{i}(\omega t + v_j\pi/2)} = F\,\mathrm{e}^{\mathrm{i}\omega t} \tag{17}$$

$$\left(\sum_{j=1}^{l'} (a_j + \mathrm{i}b_j)\omega^{v_j}\,\mathrm{e}^{\mathrm{i}(v_j\pi/2)}\right)\bar{X} = F. \tag{18}$$

The receptance FRF will therefore be given by:

$$H = \frac{\bar{X}}{F} = \left(\sum_{j=1}^{l'} (a_j + \mathrm{i}b_j)\omega^{v_j}\,\mathrm{e}^{\mathrm{i}(v_j\pi/2)}\right)^{-1}. \tag{19}$$

For the particular case of a single degree-of-freedom system with both viscous and hysteretic damping properties, (19) would be transformed into:

$$H = (k - \omega^2 m + \mathrm{i}(d + \omega c))^{-1} \tag{20}$$

corresponding to $l' = 3$, $v_1 = 0$, $v_2 = 1$, $v_3 = 2$, $a_1 = k$, $b_1 = h$, $a_2 = c$, $b_2 = 0$, $a_3 = m$ and $b_3 = 0$.

3. AN MDOF MODEL FOR MODAL ANALYSIS

In an identification problem, it may be assumed, *a priori*, as many terms as one wishes, to estimate the values of the various parameters and thus know the nature of damping in a structure. Using fractional derivatives, the final model for the damping may be much closer to reality.

One of the most interesting applications in modal analysis is the possibility of identifying the dynamic properties of a system in the more general case above described. For the single degree-of-freedom case, the problem does not seem extremely complicated. Recalling equations (14) and (19), it is clear that to identify the unknown characteristics one has just to take into account, at least, as many frequency data points as unknowns, for a chosen value of l. For example, for $l = 3$.

$$m\ddot{x} + kx + g_1 D^{v_1}x + g_2 D^{v_2}x + g_d D^{v_3}x = F\,\mathrm{e}^{\mathrm{i}\omega t} \tag{21}$$

which means, as coefficients g_j are complex, 11 unknowns. Generalizing, there will be $3l + 2$ unknowns.

If things are not too complicated for a single DOF system, they can become really difficult for a multiple DOF system. To be aware of the difficulty, it is enough to think that while with viscously damped N DOF systems the matrix equilibrium equation in the state-space has $2N \times 2N$ matrices and complex conjugate eigenvalues and eigenvectors, now each differential equation is of non-integer order and it is not known *a priori* which order one is going to have. A simplifying possibility is to restrict the values of the non-integer order v to a rational form, i.e., $v = p/q$. Then, equation (21) becomes:

$$mD^2x + kD^0x + g_1 D^{p_1/q_1}x + g_2 D^{p_2/q_2}x + g_3 D^{p_3/q_3}x = f. \tag{22}$$

Let q^* be the least common multiple of q_1, q_2, q_3. Thus,

$$mD^{2(q^*/q^*)}x + kD^{0/q^*}x + g_1 D^{[p_1(q^*/q_1)/q^*]}x + g_2 D^{[p_2(q^*/q_2)/q^*]}x + g_3 D^{[p_3(q^*/q_3)/q^*]}x = f. \tag{23}$$

In the Laplace domain, if $f = F\,\mathrm{e}^{st}$, $x = X\,\mathrm{e}^{st}$ and so,

$$\left(ms^2 + ks^0 + g_1 s^{[p_1(q^*/q_1)/q^*]} + g_2 s^{[p_2(q^*/q_2)/q^*]} + g_3 s^{[p_3(q^*/q_3)/q^*]}\right)X = F. \tag{24}$$

However, once q^* has been chosen, it is more convenient to generalize (23) by writing:

$$X \sum_{k=0}^{2q^*} a_k s^{k/q^*} = F \tag{25}$$

which gives many more terms for the exponent v. For instance, when $q^* = 5$, $k/q^* = 0, 1/5, 2/5, 3/5, \ldots$ up to $10/5(=2)$. All the *non-playing* exponents will have a zero coefficient a_k. In the time domain, it would be:

$$\sum_{k=0}^{2q^*} a_k y_k = f \quad \text{with} \quad y_k = D^{k/q^*} x. \tag{26}$$

Normally, for a viscously damped second order system with N DOFs,

$$[\mathbf{M}]\{\ddot{\mathbf{x}}\} + [\mathbf{C}]\{\dot{\mathbf{x}}\} + [\mathbf{K}]\{\mathbf{x}\} = \{\mathbf{f}\} \tag{27}$$

leading to a state-space problem of order $2N$:

$$\begin{bmatrix} [\mathbf{0}] & [\mathbf{M}] \\ [\mathbf{M}] & [\mathbf{C}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\mathbf{x}}\} \\ \{\dot{\mathbf{x}}\} \end{Bmatrix} + \begin{bmatrix} -[\mathbf{M}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{K}] \end{bmatrix} \begin{Bmatrix} \{\dot{\mathbf{x}}\} \\ \{\mathbf{x}\} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{f}\} \end{Bmatrix} \tag{28}$$

or

$$\{\dot{\mathbf{y}}\} + [\mathbf{B}]\{\mathbf{y}\} = \{\mathbf{f}'\} \tag{29}$$

with

$$[\mathbf{B}] = \begin{bmatrix} [\mathbf{0}] & [\mathbf{M}] \\ [\mathbf{M}] & [\mathbf{C}] \end{bmatrix}^{-1} \begin{bmatrix} -[\mathbf{M}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{K}] \end{bmatrix} \quad \text{and} \quad \{\mathbf{f}'\} = \begin{bmatrix} [\mathbf{0}] & [\mathbf{M}] \\ [\mathbf{M}] & [\mathbf{C}] \end{bmatrix}^{-1} \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{f}\} \end{Bmatrix}.$$

The corresponding eigenvalue problem (of order $2N$), is:

$$[[\mathbf{B}] + \lambda[\mathbf{I}]]\{\mathbf{Y}\} = \{\mathbf{0}\}. \tag{30}$$

Now, one has instead:

$$\{\mathbf{y}\} = \begin{Bmatrix} \{\mathbf{y}_{2q^*-1}\} \\ \{\mathbf{y}_{2q^*-2}\} \\ \vdots \\ \{\mathbf{y}_1\} \\ \{\mathbf{y}_0\} \end{Bmatrix} = \begin{Bmatrix} D^{2q^*-1/q^*}\{\mathbf{x}\} \\ D^{2q^*-2/q^*}\{\mathbf{x}\} \\ \vdots \\ D^{1/q^*}\{\mathbf{x}\} \\ D^{0/q^*}\{\mathbf{x}\} \end{Bmatrix}. \tag{31}$$

One can also write

$$\begin{aligned} \{\mathbf{y}_{2q^*}\} &= D^{2q^*/q^*}\{\mathbf{x}\} = D^{1/q^*}(D^{2q^*-1/q^*}\{\mathbf{x}\}) = D^{1/q^*}\{\mathbf{y}_{2q^*-1}\} \\ &\vdots \\ \{\mathbf{y}_k\} &= D^{k/q^*}\{\mathbf{x}\} = D^{1/q^*}(D^{k-1/q^*}\{\mathbf{x}\}) = D^{1/q^*}\{\mathbf{y}_{k-1}\} \\ &\vdots \\ \{\mathbf{y}_1\} &= D^{1/q^*}\{\mathbf{x}\} = D^{1/q^*}(D^{0/q^*}\{\mathbf{x}\}) = D^{1/q^*}\{\mathbf{y}_0\}. \end{aligned} \tag{32}$$

Thus, the generalization of equation (29) is:

$$D^{1/q^*}\{\mathbf{x}\} + [\mathbf{B}]\{\mathbf{y}\} = \{\mathbf{f}'\} \tag{33}$$

where now the order is $2Nq^*$, instead of $2N$.

Taking into account relation (3), the eigenvalue problem can also be written as

$$[[\mathbf{B}] + \lambda[\mathbf{I}]]\{\mathbf{Y}\} = \{\mathbf{0}\} \tag{34}$$

but with $\lambda = s^{1/q^*}$.

For instance, let $\nu_1 = p_1/q_1 = 1/3$ and $\nu_2 = p_2/q_2 = 1/1$. Thus, $q^* = 3$, and we have, for an N DOF system:

$$\sum_{k=0}^{6} [\mathbf{a}_k]\{\mathbf{y}_k\} = \{\mathbf{f}\} \tag{35}$$

or

$$\sum_{k=0}^{6} [\mathbf{a}_k]D^{k/3}\{\mathbf{x}\} = \{\mathbf{f}\}. \tag{36}$$

Developing (36) and pre-multiplying by $[\mathbf{a}_6]^{-1}$,

$$\begin{aligned}\{\ddot{\mathbf{x}}\} + [\mathbf{a}_5']D^{5/3}\{\mathbf{x}\} + [\mathbf{a}_4']D^{4/3}\{\mathbf{x}\} + [\mathbf{a}_3']\{\dot{\mathbf{x}}\}\\ +[\mathbf{a}_2']D^{2/3}\{\mathbf{x}\} + [\mathbf{a}_1']D^{1/3}\{\mathbf{x}\} + [\mathbf{a}_0']\{\dot{\mathbf{x}}\} = \{\mathbf{f}'\}\end{aligned} \tag{37}$$

Therefore, the state vectors $\{\mathbf{y}\}$ and $D^{1/3}\{\mathbf{y}\}$ are:

$$\{\mathbf{y}\} = \begin{Bmatrix} D^{5/3}\{\mathbf{x}\} \\ D^{4/3}\{\mathbf{x}\} \\ \{\dot{\mathbf{x}}\} \\ D^{2/3}\{\mathbf{x}\} \\ D^{1/3}\{\mathbf{x}\} \\ \{\mathbf{x}\} \end{Bmatrix} \quad D^{1/3}\{\mathbf{y}\} = \begin{Bmatrix} \{\ddot{\mathbf{x}}\} \\ D^{5/3}\{\mathbf{x}\} \\ D^{4/3}\{\mathbf{x}\} \\ \{\dot{\mathbf{x}}\} \\ D^{2/3}\{\mathbf{x}\} \\ D^{1/3}\{\mathbf{x}\} \end{Bmatrix}. \tag{38}$$

The state-space equation is:

$$D^{1/3}\{\mathbf{y}\} + [\mathbf{B}]\{\mathbf{y}\} = \{\mathbf{F}\} \tag{39}$$

where

$$\{\mathbf{F}\} = \begin{Bmatrix} \{\mathbf{f}'\} \\ \{\mathbf{0}\} \\ \vdots \\ \{\mathbf{0}\} \end{Bmatrix}.$$

In the eigenvalue problem [equation (34)], $\lambda = s^{1/3}$ and $[\mathbf{B}]$ is the companion matrix, given by:

$$[\mathbf{B}] = \begin{bmatrix} [\mathbf{a}'_5] & [\mathbf{a}'_4] & [\mathbf{a}'_3] & [\mathbf{a}'_2] & [\mathbf{a}'_1] & [\mathbf{a}'_0] \\ -[\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & -[\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & -[\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & -[\mathbf{I}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & -[\mathbf{I}] & [\mathbf{0}] \end{bmatrix} \qquad (40)$$

As each matrix $[\mathbf{a}'_k]$ is $N \times N$, $[\mathbf{B}]$ is $(2q^*N) \times (2q^*N)$. The eigenvalue problem is of order $2q^*N$, in this case, $6N$. So, there will be $6N$ eigenvalues and eigenvectors, but the original system still has only N DOFs.

Therefore, the receptance FRF, in terms of rational fractional polynomials (see reference [12]) will be given by

$$H = \frac{\sum_{k=0}^{2q^*N-1} b_k (\mathrm{i}\omega)^{k/q^*}}{\sum_{k=0}^{2q^*N} c_k (\mathrm{i}\omega)^{k/q^*}} \qquad (41)$$

where b_k and c_k are, generally, complex quantities. Alternatively, it can be expressed in factorized form, as:

$$H = \frac{\prod_{k=1}^{2q^*N-1} ((\mathrm{i}\omega)^{1/q^*} - z_k)}{\prod_{k=1}^{2q^*N} ((\mathrm{i}\omega)^{1/q^*} - p_k)} \qquad (42)$$

or in terms of mode superposition, as

$$H = \sum_{r=1}^{N} \frac{\left(\sum_{k=0}^{2q^*-1} b_k (\mathrm{i}\omega)^{k/q^*} \right)_r}{\left(\sum_{k=0}^{2q^*} c_k (\mathrm{i}\omega)^{k/q^*} \right)_r}. \qquad (43)$$

It is therefore possible to obtain a generalization of the receptance expression for the case of general damping and consequently to apply modal analysis tools to identify such systems.

However, the use of expressions (41) or (43) for practical applications leads to numerical problems, as the dimension of the matrices to solve increases very easily. This problem has already been pointed out in [11]. Let $v_1 = 0{\cdot}5$ and $v_2 = 0{\cdot}7$, i.e., $v_1 = 5/10$ and $v_2 = 7/10$. This means $q^* = 10$. For a 3 DOF system, the order of the matrices would be 60, instead of 6 in the normal case. Moreover, coefficients b and c are complex and that brings extra complications. Ill-conditioning is also a problem and no easy way could be found to avoid it. So, how to deal with such a problem?

4. SIMPLIFYING APPROACHES

Instead of trying to identify the modal properties using a MDOF method, it is worthwhile beginning with a simpler approach, with a SDOF analysis. Let us start by writing expression (43) for each DOF, for an example with $q^* = 3$:

$$H = \frac{b_0 + b_1(\mathrm{i}\omega)^{1/3} + b_2(\mathrm{i}\omega)^{2/3} + b_3(\mathrm{i}\omega)^{3/3} + b_4(\mathrm{i}\omega)^{4/3} + b_5(\mathrm{i}\omega)^{5/3}}{c_0 + c_1(\mathrm{i}\omega)^{1/3} + c_2(\mathrm{i}\omega)^{2/3} + c_3(\mathrm{i}\omega)^{3/3} + c_4(\mathrm{i}\omega)^{4/3} + c_5(\mathrm{i}\omega)^{5/3} + c_6(\mathrm{i}\omega)^{6/3}}. \quad (44)$$

The objective is to evaluate coefficients b and c. If, for instance, the damping exponents associated with this DOF were around 1/3 and 5/3, it is expected that b_2, b_3 and b_4 are close to zero, as well as c_2, c_3 and c_4. It is also expected that b_0, c_0 and c_6, related to the modal stiffness and mass always exist. If a higher value for q^* is chosen, the expression is longer and more accurate results are naturally foreseeable.

However, things can be simplified further, to reduce the total number of unknowns. While the damping exponents were taken as rationals due to the theoretical development followed before, with the advantage that they were not unknowns in the MDOF approach, it seems unnecessary to keep them as rationals in a SDOF approach. Furthermore, it can be postulated that around each mode there is only one damping term. In such circumstances, the receptance FRF simplifies to:

$$H = \sum_{r=1}^{N} \frac{b_0 + b_1(\mathrm{i}\omega)^{v_r}}{c_0 + c_1(\mathrm{i}\omega)^{v_r} + c_2(\mathrm{i}\omega)^2} \quad (45)$$

where v_r is now an unknown too.

Now, do the coefficients really need to be considered as complex? b_0, c_0 and c_2 are naturally real quantities. What about b_1 and c_1? They are associated with damping and for purely dissipative systems, one can reason as follows: let c_1 be complex, equal to $u + \mathrm{i}z$. Hence, the term $c_1(\mathrm{i}\omega)^{v_r}$ becomes:

$$\begin{aligned} c_1(\mathrm{i}\omega)^{v_r} &= (u + \mathrm{i}z)(\cos v_r\pi/2 + \mathrm{i} \sin v_r\pi/2)\omega^{v_r} \\ &= (u \cos v_r\pi/2 - z \sin v_r\pi/2)\omega^{v_r} + \mathrm{i}(u \sin v_r\pi/2 + z \cos v_r\pi/2)\omega^{v_r}. \quad (46) \end{aligned}$$

For purely dissipative systems,

$$(u \cos v_r\pi/2 - z \sin v_r\pi/2)\omega^{v_r} = 0 \quad (47)$$

and thus,

$$u \cos v_r\pi/2 = z \sin v_r\pi/2 \Rightarrow z = u\,\frac{\cos v_r\pi/2}{\sin v_r\pi/2} \tag{48}$$

and $c_1(\mathrm{i}\omega)^{v_r}$ will simply be given by

$$\begin{aligned} c_1(\mathrm{i}\omega)^{v_r} &= \mathrm{i}(u \sin v_r\pi/2 + z \cos v_r\pi/2)\omega^{v_r} \\ &= i\left(u \sin v_r\pi/2 + u\,\frac{\cos v_r\pi/2}{\sin v_r\pi/2}\cos v_r\pi/2\right)\omega^{v_r} \\ &= \mathrm{i}\left(\frac{u}{\sin v_r\pi/2}\right)\omega^{v_r} = \mathrm{i}c_1'\omega^{v_r}. \end{aligned} \tag{49}$$

Therefore, expression (45) may be rewritten as:

$$H = \sum_{r=1}^{N} \frac{b_0 + \mathrm{i}b_1'\omega^{v_r}}{c_0 + \mathrm{i}c_1'\omega^{v_r} - c_2\omega^2} \tag{50}$$

where all coefficients are real quantities.

As c_0 is related to the modal stiffness and c_2 with the modal mass, the numerator and denominator of (50) may be divided by c_2, giving ($c_0/c_2 = k_r/m_r = \omega_r^2$):

$$H = \sum_{r=1}^{N} \frac{A_r + \mathrm{i}B_r\omega^{v_r}}{\omega_r^2 - \omega^2 + \mathrm{i}G_r\omega^{v_r}}\,. \tag{51}$$

This expression must include the usual hysteretic and viscous models, for $v = 0$ and $v = 1$, respectively:

$$H = \sum_{r=1}^{N} \frac{A_r + \mathrm{i}B_r}{\omega_r^2 - \omega^2 + \mathrm{i}\eta_r\omega_r^2} \tag{52}$$

$$H = \sum_{r=1}^{N} \frac{A_r + \mathrm{i}B_r\omega}{\omega_r^2 - \omega^2 + \mathrm{i}2\xi_r\omega\omega_r}\,. \tag{53}$$

Thus, we must have:

$$G_r\omega^{v_r} = \eta_r\omega_r^2 \Rightarrow G_r = \frac{\eta_r\omega_r^2}{\omega^{v_r}} \tag{54a}$$

$$G_r\omega^{v_r} = 2\xi_r\omega\omega_r \Rightarrow G_r = \frac{2\xi_r\omega\omega_r}{\omega^{v_r}}\,. \tag{54b}$$

Giving a common designation for η_r and $2\xi_r$ as γ_r, expressions (54a) and (54b) are equal when $\omega = \omega_r$, leading to

$$G_r = \frac{\gamma_r\omega_r^2}{\omega_r^{v_r}}\,. \tag{55}$$

Substituting back in (51), gives

$$H=\sum_{r=1}^{N}\frac{A_r+\mathrm{i}B_r\omega^{\nu_r}}{\omega_r^2-\omega^2+\mathrm{i}\gamma_r\omega_r^{2-\nu_r}\omega^{\nu_r}}\,. \tag{56}$$

This is a much simpler expression for the receptance FRF of a system with N DOFs with general damping: in each mode there is a damping coefficient γ_r and a ν_rth power dependence on frequency. A mode by mode identification procedure means the analysis of the following expression:

$$H=\frac{A_r+\mathrm{i}B_r\omega^{\nu_r}}{\omega_r^2-\omega^2+\mathrm{i}\gamma_r\omega_r^{2-\nu_r}\omega^{\nu_r}}+\text{Residual term.} \tag{57}$$

5. DISCUSSION

It is worthwhile to explore more deeply the nature of the first term on the right-hand side of (57), to get more insight on the physical interpretation of that expression.

5.1. Nyquist plots

Start with the representation in the Nyquist plot. Probably, this is not very important on its own, but helps understanding and framing the FRF for general damping with reference to the usual hysteretic and viscous models. It is known that for hysteretic damping, the Nyquist representation of the receptance (the derivative of zero order) $H=1/(\omega_r^2-\omega^2+\mathrm{i}\eta_r\omega_r^2)$ is a circle (assuming unit mass, without loss of generality). $A_r+\mathrm{i}B_r$, which reflects the influence of the modal mass and of other modes, increases or reduces the radius of the circle and rotates it. In the viscous damping case, the mobility (the derivative of first order) $\mathrm{i}\omega H=\mathrm{i}\omega/(\omega_r^2-\omega^2+\mathrm{i}2\xi_r\omega\omega_r)$ is also a circle. In that case, $A_r+\mathrm{i}B_r\omega$ is no longer a constant, but around resonance the frequency does not change too much and so it can still be assumed as such, having therefore the same effect of increasing or reducing the radius of the circle and rotating it. Thus, it is logical to expect that for the general damping case, the derivative of order ν_r, given by $D^{\nu_r}H=(\mathrm{i}\omega)^{\nu_r}/(\omega_r^2-\omega^2+\mathrm{i}\gamma_r\omega_r^{2-\nu_r}\omega^{\nu_r})$, is a circle too. About the term $A_r+\mathrm{i}B_r\omega^{\nu_r}$, it may be assumed that it is a constant provided that ν_r stays smaller than 1, an intermediate situation between the hysteretic and viscous cases. If ν_r is higher than 1, nothing can be said *a priori*.

Proposition 1

Let the receptance of a SDOF system with general damping and unit mass be given by

$$H=\frac{1}{\omega_r^2-\omega^2+\mathrm{i}\gamma_r\omega_r^{2-\nu_r}\omega^{\nu_r}}\,. \tag{58}$$

Then, the derivative of order ν_r will be a circle in the Argand plane (Nyquist plot).

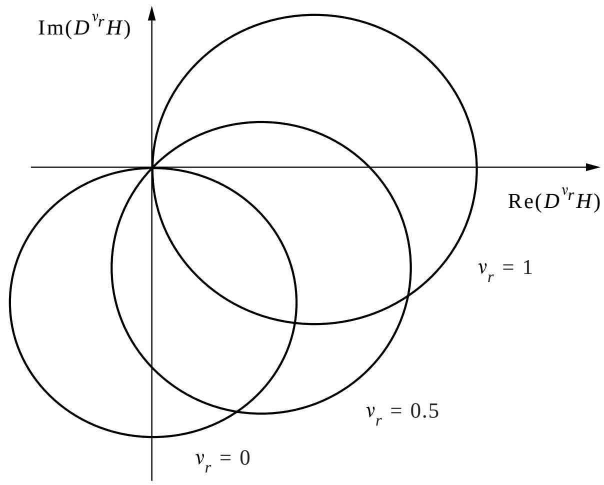

Figure 2. Nyquist plot for $v_r = 0$, $v_r = 1$ and $v_r = 0{\cdot}5$.

Proof

$$D^{v_r}H = \frac{(\mathrm{i}\omega)^{v_r}}{\omega_r^2 - \omega^2 + \mathrm{i}\gamma_r\omega_r^{2-v_r}\omega^{v_r}} \tag{59}$$

$$\mathrm{Re} = \frac{(\omega_r^2 - \omega^2)\omega^{v_r}\cos v_r\pi/2 + \gamma_r\omega_r^{2-v_r}\omega^{2v_r}\sin v_r\pi/2}{(\omega_r^2 - \omega^2)^2 + (\gamma_r\omega_r^{2-v_r}\omega^{v_r})^2} \tag{60}$$

$$\mathrm{Im} = \frac{(\omega_r^2 - \omega^2)\omega^{v_r}\sin v_r\pi/2 - \gamma_r\omega_r^{2-v_r}\omega^{2v_r}\cos v_r\pi/2}{(\omega_r^2 - \omega^2)^2 + (\gamma_r\omega_r^{2-v_r}\omega^{v_r})^2}. \tag{61}$$

Hence,

$$\mathrm{Re}^2 + \mathrm{Im}^2 = \frac{\omega^{2v_r}}{(\omega_r^2 - \omega^2)^2 + (\gamma_r\omega_r^{2-v_r}\omega^{v_r})^2} \tag{62}$$

and

$$\mathrm{Re}\cdot\sin v_r\pi/2 - \mathrm{Im}\cdot\cos v_r\pi/2 = \frac{\gamma_r\omega_r^{2-v_r}\omega^{2v_r}}{(\omega_r^2 - \omega^2)^2 + (\gamma_r\omega_r^{2-v_r}\omega^{v_r})^2}. \tag{63}$$

From (62) and (63),

$$(\mathrm{Re}^2 + \mathrm{Im}^2)\gamma_r\omega_r^{2-v_r} = \mathrm{Re}\cdot\sin v_r\pi/2 - \mathrm{Im}\cdot\cos v_r\pi/2 \tag{64}$$

from which,

$$\left(\mathrm{Re} - \frac{\sin v_r\pi/2}{2\gamma_r\omega_r^{2-v_r}}\right)^2 + \left(\mathrm{Im} + \frac{\cos v_r\pi/2}{2\gamma_r\omega_r^{2-v_r}}\right)^2 = \left(\frac{1}{2\gamma_r\omega_r^{2-v_r}}\right)^2 \tag{65}$$

which is the equation of a circle, with center coordinates $x_0 = (\sin v_r\pi/2)/(2\gamma_r \omega_r^{2-v_r})$ and $y_0 = -(\cos v_r\pi/2)/(2\gamma_r\omega_r^{2-v_r})$ and radius $r_0 = 1/(2\gamma_r\omega_r^{2-v_r})$. □

A plot of this circle for $v_r = 0$ (hysteretic damping), $v_r = 1$ (viscous damping) and $v_r = 0{\cdot}5$ is given in Figure 2.

5.2. INVERSE OF RECEPTANCE

The inverse of the receptance may also be analyzed:

$$\frac{1}{H} = \omega_r^2 - \omega^2 + \mathrm{i}\gamma_r \omega_r^{2-\nu_r}\omega^{\nu_r}. \tag{66}$$

Thus,

$$\mathrm{Re}\left(\frac{1}{H}\right) = \omega_r^2 - \omega^2 \tag{67}$$

$$\mathrm{Im}\left(\frac{1}{H}\right) = \gamma_r \omega_r^{2-\nu_r}\omega^{\nu_r}. \tag{68}$$

Taking logarithms, it follows that

$$\log \mathrm{Re}\left(\frac{1}{H}\right) = \log(\omega_r^2) - 2\log\omega \tag{69}$$

$$\log \mathrm{Im}\left(\frac{1}{H}\right) = \log(\gamma_r \omega_r^{2-\nu_r}) + \nu_r \log\omega. \tag{70}$$

These expressions are straight lines in $\log\omega$ and the slope of the imaginary part is the order of the existing damping. In Figure 3, the three cases presented in Figure 2 are illustrated.

Whereas in the Nyquist plot one cannot, in practice, represent the ν_rth order derivative of the measured data because ν_r is unknown, in the inverse method the slope of the imaginary part can indicate the type of existing damping. Unfortunately, in real cases, the influence of neighboring modes can distort those graphs and therefore give a false indication about the damping. It would be necessary to remove that influence before identifying the order of damping affecting a particular mode.

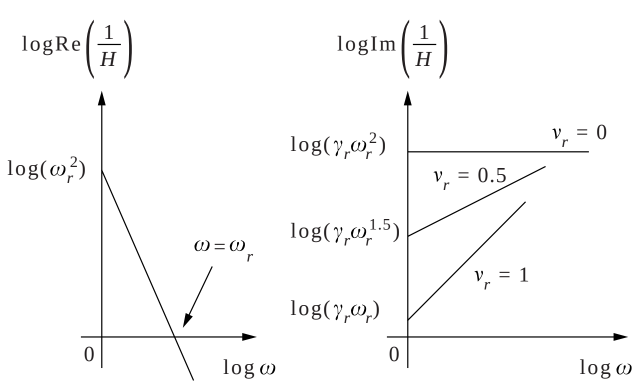

Figure 3. Plots of the inverse of the receptance for $\nu_r = 0$, $\nu_r = 1$ and $\nu_r = 0{\cdot}5$.

5.3. MODULUS OF THE FRE

Another valuable discussion concerns the interpretation of the representation of the absolute value of the FRF. Writing the absolute value of receptance for hysteretic damping as

$$|H| = \frac{1}{\sqrt{[\omega_r^2(1-\beta^2)]^2 + (\eta_r\omega_r^2)^2}} = \frac{1/\omega_r^2}{\sqrt{(1-\beta^2)^2 + \eta_r^2}} \tag{71}$$

where β is the frequency ratio ($=\omega/\omega_r$).

For viscous damping,

$$|H| = \frac{1}{\sqrt{[\omega_r^2(1-\beta^2)]^2 + (2\xi_r\beta\omega_r^2)^2}} = \frac{1/\omega_r^2}{\sqrt{(1-\beta^2)^2 + (2\xi_r\beta)^2}}. \tag{72}$$

For general damping,

$$|H| = \frac{1}{\sqrt{[\omega_r^2(1-\beta^2)]^2 + (\gamma_r\beta^{v_r}\omega_r^2)^2}} = \frac{1/\omega_r^2}{\sqrt{(1-\beta^2)^2 + (\gamma_r\beta^{v_r})^2}}. \tag{73}$$

The graphical representation of $|H|$ versus β for the three cases is illustrated in Figure 4.

As it is known, for hysteretic damping, the maxima always occur at $\beta = 1$ ($\omega = \omega_r$) and for viscous damping they occur on the left of ω_r, at $\beta = \sqrt{1 - 2\xi_r^2}$. It is natural that the maxima for the general case also happen on the left of ω_r, more or less, according to the value of v_r.

It is also natural to expect that such a deviation results from the fact that one is always representing the receptance, for the various types of damping. If one represents, instead of $|H|$, the modulus of the v_rth order derivative, the maxima should always happen at $\beta = 1$ ($\omega = \omega_r$).

Proposition 2

Let the v_rth order derivative of the receptance for a SDOF system with general damping be given by

$$D^{v_r}H = \frac{(\mathrm{i}\omega)^{v_r}}{\omega_r^2 - \omega^2 + \mathrm{i}\gamma_r\omega_r^{2-v_r}\omega^{v_r}}. \tag{74}$$

Then, the absolute value of $D^{v_r}H$ will have a maximum at $\omega = \omega_r$, for every level of damping γ_r.

Proof

$$|D^{v_r}H| = \frac{\omega^{v_r}}{\sqrt{(\omega_r^2 - \omega^2)^2 + (\gamma_r\omega_r^{2-v_r}\omega^{v_r})^2}} = \frac{\beta^{v_r}\omega_r^{2-v_r}}{\sqrt{(1-\beta^2)^2 + (\gamma_r\beta^{v_r})^2}} \tag{75}$$

$$\frac{\partial|D^{v_r}H|}{\partial\beta} = 0 \Rightarrow [(2-v_r)\beta^2 + v_r](1-\beta^2) = 0 \Rightarrow \beta = 1 \quad \text{for} \quad 0 \leqslant v_{vr} < 2. \tag{76}$$

□

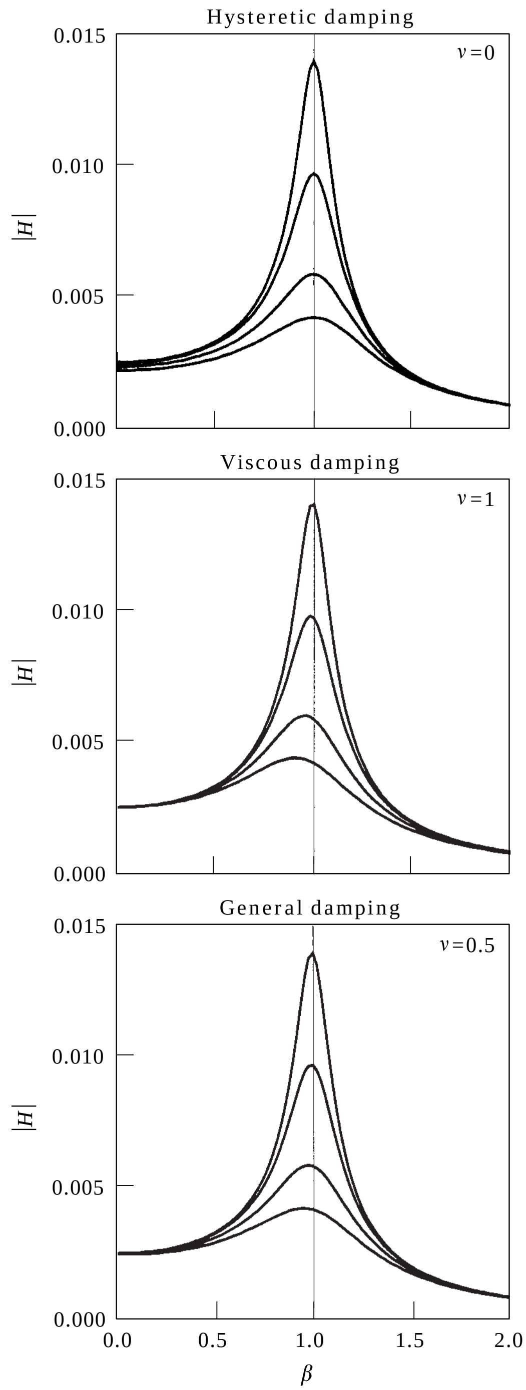

Figure 4. $|H|$ versus β for $\nu_r = 0$, $\nu_r = 1$ and $\nu_r = 0{\cdot}5$ and different values of the damping ratio.

It should be remembered that $\nu_r = 2$ is the order of the derivative affecting the mass, so $0 \leqslant \nu_r < 2$ is the range one may be dealing with. Hence, it can be said that, for general damping, the maxima always occur at $\beta = 1$ when the absolute value of the ν_rth order derivative of the receptance is represented. Figure 5 illustrates the result just given, for $\nu_r = 0{\cdot}5$ and $\nu_r = 1$.

For $\beta = 1$, the maximum values are

$$|D^{\nu_r}H|_{\max} = \frac{\omega_r^{2-\nu_r}}{\gamma_r}. \tag{77}$$

5.4. MODULUS OF THE FRF AT $\omega = 0$

Recalling equation (73),

$$|H| = \frac{1/\omega_r^2}{\sqrt{(1-\beta^2)^2 + (\gamma_r \beta^{\nu_r})^2}}. \tag{73}$$

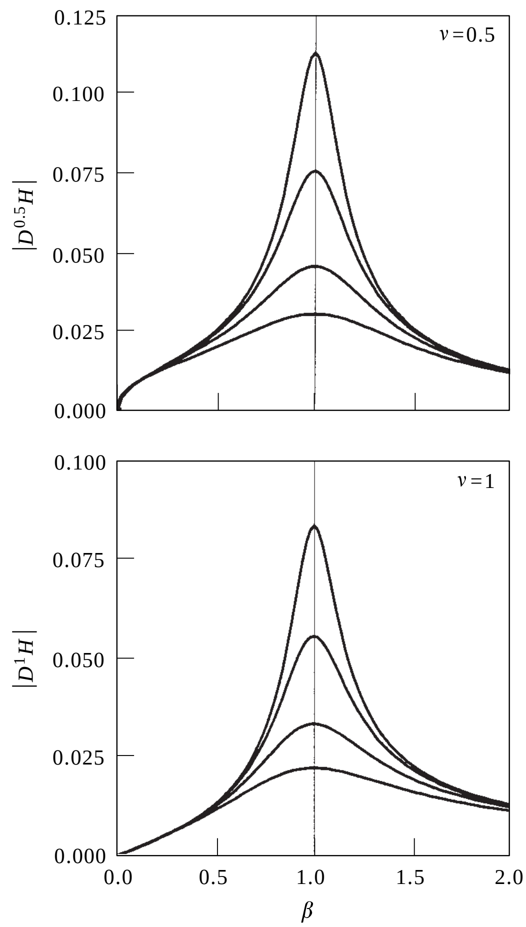

Figure 5. $|D^{\nu_r}H|$ versus β for $\nu_r = 0{\cdot}5$ and $\nu_r = 1$.

For any $v_r \neq 0$, the response at $\omega = 0$, i.e., $\beta = 0$, is the static response (it is recalled that $m = 1$ for simplicity):

$$|H| = 1/\omega_r^2. \tag{78}$$

For the hysteretic case $v_r = 0$ and if the usual expression [equation (71)] is taken directly, it follows that

$$|H||_{\beta=0} = \left.\frac{1/\omega_r^2}{\sqrt{(1-\beta^2)^2 + \eta_r^2}}\right|_{\beta=0} = \frac{1/\omega_r^2}{\sqrt{1+\eta_r^2}} \tag{79}$$

which is the commonly known result, although quite a strange one, as it states that the static response depends on the value of the damping itself! As a consequence [see Figure 4(a)], the curves do not start at the same point. This is commonly disregarded by simply stating that the hysteretic damping model is only valid if there is harmonic motion. However, the present approach allows for an explanation to this apparent physical incoherence. In fact, from equation (73), it can be observed that $v_r = 0$ and $\beta = 0$ leads to the indetermination 0^0. This means that the common result as in equation (79) corresponds to arbitrarily have set $0^0 = 1$, while it should be made zero to comply with the physical result that the static response should always be the same [as in equation (78)], no matter which value of damping there is.

It is therefore concluded that the hysteretic damping is a limit case, since for $\beta = 0$ the quantity β^{v_r} is always zero except exactly for $v_r = 0$, where an indetermination exists, but where one can now state that it should also be zero.

It should be noted that from a pure mathematical point of view things are a bit more complex. One is looking for a double limit, when both β and v approach zero, $\lim_{\beta\to 0, v_r\to 0} \beta^{v_r}$. As β and v are independent from each other, the limit is directional, and according to the direction taken the result is different. This means that in fact the limit does not exist. The classical approach corresponds to evaluate $\lim_{v_r\to 0} \beta^{v_r}$, which gives 1, and from the above discussion it is concluded that for a meaningful physical explanation the limit should be taken as $\lim_{\beta\to 0} \beta^{v_r}$, giving zero.

6. CONCLUSIONS

The behavior of some types of materials with respect to energy dissipation is, sometimes, not entirely satisfied by the use of the common hysteretic or viscous models. In this paper the authors propose a generalization of the FRF receptance of a system, when damping terms dependent on non-integer powers of frequency are present. The theory for the MDOF case is developed in detail, but the final result is difficult to implement in practice. The single degree-of-freedom approach seems to be the one with more possibilities for practical applications.

The main developments in this work concern the proof that the general FRF expression obtained makes sense, as the Nyquist diagram, the inverse FRF representation and the plot of the modulus of the FRF constitute natural extensions of the typical hysteretic and viscous models, as these are just particular cases. The general model also allows one to explain the particular behavior of those current damping models in a clearer fashion. It is believed that this article may contribute to a more accurate identification of systems where damping forces are functions of non-integer powers of the exciting frequency, although a specific efficient identification procedure still has to be further investigated.

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APPENDIX: NOTATION

$[\mathbf{a}_k]$, $[\mathbf{a}'_k]$	system matrices
b_k, c_k	coefficients
$[\mathbf{B}]$	system matrix, companion matrix
c, d	viscous and hysteretic damping coefficient
g_j	complex coefficients
D^v	derivative of order v
$\{\mathbf{f}\}$, $\{\mathbf{f}'\}$	force vectors
$\{\mathbf{F}\}$	state space force vector
i	imaginary unit
$[\mathbf{M}]$, $[\mathbf{K}]$, $[\mathbf{C}]$	mass, stiffness and viscous damping matrices
p, q	numerator and denominator of v, in its rational form

q^*	least common multiple of the several q
Re, Im	real and imaginary parts
s	Laplace variable
x	displacement
$\bar{X}$	complex amplitude
y_k	derivative of x, of order k/q^*
$\{\mathbf{y}\}$	state vector
z_k, p_k	zeros and poles
H	receptance FRF
β	frequency ratio
v	order of fractional derivative
γ	general damping ratio
Γ	Gamma function
λ	eigenvalue
ω	circular frequency
$\mathscr{L}$, $\mathscr{F}$	Laplace and Fourier transforms