Journal of Sound and Vibration (1998) **218**(5), 791–798
Article No. sv981860

PROPAGATION OF FINITE AMPLITUDE ACOUSTIC WAVES IN A TWO-DIMENSIONAL RECTANGULAR DUCT

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(*Received* 19 *November* 1996, *and in final form* 22 *June* 1998)

Finite amplitude acoustic waves (FAAWs) that propagate in a two-dimensional rectangular duct of semi-infinite length as a result of periodic excitation are determined by using second-order perturbation, based on the partial wave analysis method. With second-harmonic boundary and initial conditions of excitation, second-harmonic analytical expressions, which are applicable to quantitative analysis, have been derived. In this manner, a physical mechanism of second-harmonic generation and propagation in the process of propagation of FAAWs is clearly displayed. Based on the formula, some numerical calculations are performed. The numerical results clearly exhibit the distortion and symmetry of second-harmonic field pattern for a given source of excitation.

1. INTRODUCTION

Finite amplitude acoustic waves (FAAWs) that propagate in a duct have attracted considerable attention due to their practical applications. For the problem of FAAW propagation in a duct, in addition to plane wave solutions, some multidimensional responses have already been examined theoretically by a number of authors using a variety of approaches [1–6]. If the thickness of duct is much less than the transverse width, for simplicity, the duct can be assumed to be two-dimensional. Thus, the examination of FAAW propagation in a two-dimensional rectangular duct can be of practical significance. The present article is based on the fact that previous analyses lack the second-harmonic analytical expressions, the physical models of second-harmonic generation and propagation, and the second-harmonic field pattern that is required in practical applications.

The main purpose of the present article is to study second-harmonic generation and propagation in the process of propagation of FAAWs in a two-dimensional rectangular duct, in which FAAWs are induced by a source of periodic excitation, and to offer a straightforward physical model useful for the explanation of the process of second-harmonic generation and propagation. To simplify the process of analysis, we assume that second-harmonic generation is due to the bulk nonlinearity of fluid in the duct, and that the fluid is irrotational, inviscid, and

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compressible, and that the waves propagate in a hard-walled uniform duct with no mean flow. Here, we will be concerned only with second-order perturbation, and will assume that the second-harmonic amplitude is much smaller than the fundamental amplitude which we assume constant. Thus, the solution does not take into account energy exchange between the fundamental and the higher harmonics.

2. THEORETICAL FUNDAMENTALS

One of the three physical quantities, mechanical displacement vector $\mathbf{u}$, particle velocity $\mathbf{v}$ or scale potential ϕ, may be employed to describe the process of acoustic propagation in a fluid [5, 6]. There are relationships among $\mathbf{u}$, $\mathbf{v}$ and ϕ, i.e., $\mathbf{v} = \mathbf{u}_{,t}$, $\mathbf{v} = \nabla\phi$. When we take second-order approximation, ϕ can be expanded as $\phi = \phi^{(1)} + \phi^{(2)}$, and $\phi^{(1)}$, $\phi^{(2)}$ are scale potentials corresponding to the fundamental and second harmonic, respectively. Two linear wave equations can be obtained from the non-linear wave equation [5, 6],

$$c_L^2 \nabla^2 \phi^{(1)} - \phi^{(1)}_{,tt} = 0 \tag{1}$$

and

$$c_L^2 \nabla^2 \phi^{(2)} - \phi^{(2)}_{,tt} = F(\phi^{(1)}), \tag{2}$$

where $F(\phi^{(1)})$ is the driving force caused by the fundamental $\phi^{(1)}$ due to the bulk non-linearity of fluid, and c_L is the longitudinal velocity of fluid.

A Cartesian coordinates system whose oz axis coincides with the center of a two-dimensional rectangular duct, and its oy axis is normal to the walls of the duct (see Figure 1) is established. According to the partial wave analysis method, each

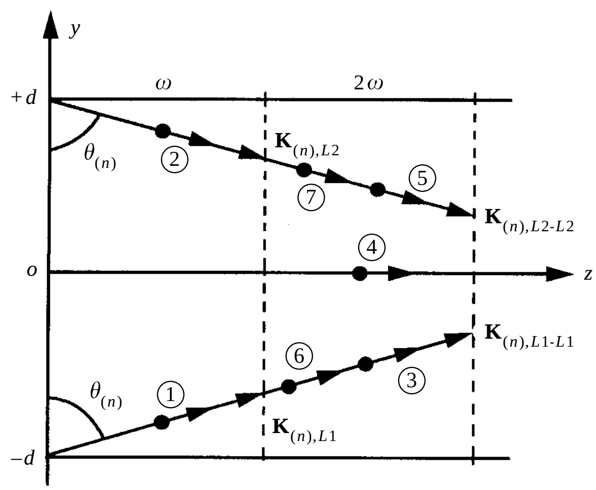

Figure 1. Fundamental and second-harmonic acoustic fields of the nth fundamental mode in the duct. ① and ②are longitudinal wave displacement vectors corresponding to $\phi^{(1)}_{(n),1}$ and $\phi^{(1)}_{(n),2}$; ③, ④ and ⑤ driven second-harmonic displacement vectors corresponding to $\phi^{(2D)}_{(n),L1-L1}$, $\phi^{(2D)}_{(n),L1-L2}$ and $\phi^{(2D)}_{(n),L2-L2}$; ⑥ and ⑦ freely propagating second-harmonic displacement vectors corresponding to $\phi^{(2F)}_{(n),1}$ and $\phi^{(2F)}_{(n),2}$.

acoustic propagation mode in the duct consists of two partial longitudinal waves, which are reflected at the upper and lower walls of the duct [6, 7]. Based on the phase matching of two partial longitudinal waves along the *oz* axis, two scale potential solutions of the *n*th fundamental mode, which satisfy equation (1) and correspond to two partial longitudinal waves, can be written as [1, 7]:

$$\begin{cases} \phi^{(1)}_{(n),1} = \phi_{(n),1} \exp j(\mathbf{K}_{(n),L1} \cdot \mathbf{r}_1 - \omega t) \\ \phi^{(1)}_{(n),2} = \phi_{(n),2} \exp j(\mathbf{K}_{(n),L2} \cdot \mathbf{r}_2 - \omega t) \end{cases}, \tag{3}$$

where the subscript (n) is used to identify a physical quantity associated with the nth fundamental mode, $\mathbf{K}_{(n),L1}$ and $\mathbf{K}_{(n),L2}$ are wave vectors of two partial longitudinal waves ① and ② (see Figure 1), and ω is angular frequency associated with excitation frequency f.

There is a boundary condition that requires that the normal components of particle velocities at the two rigid walls, $y = \pm d$, should be zero. It follows that

$$[M(\omega)]\begin{pmatrix} \phi_{(n),1} \\ \phi_{(n),2} \end{pmatrix} = 0. \tag{4}$$

For a non-trivial solution the coefficient determinant of equation (4) must be zero, i.e., $|M(\omega)| = 0$, which leads to a dispersion equation [6],

$$c_{(n)} = \frac{4c_L \cdot fd}{\sqrt{(4fd)^2 - (nc_L)^2}}, \tag{5}$$

where $c_{(n)}$ is the phase velocity of the nth fundamental mode, fd is the product between the excitation frequency and the half thickness of the duct. If the nth fundamental mode with propagation characteristic is to be generated, the condition of excitation frequency, $fd > (nc_L)/4$, must be satisfied. Combining equations (4) and (5) yields $\phi_{(n),1} = (-1)^n \phi_{(n),2} = \phi_{(n)}$.

The driving force, $F(\phi^{(1)}_{(n)})$, can be obtained by substituting the sum of $\phi^{(1)}_{(n),1}$ and $\phi^{(1)}_{(n),2}$ into the right-hand side of equation (2) [6, 8–10]. It follows that

$$\begin{cases} F(\phi^{(1)}_{(n)}) = F_{(n),L1-L1} \exp[j(\mathbf{K}_{(n),L1-L1} \cdot \mathbf{r}_1 - 2\omega t)] \\ \qquad + F_{(n),L2-L2} \exp[j(\mathbf{K}_{(n),L2-L2} \cdot \mathbf{r}_2 - 2\omega t)] \\ \qquad + F_{(n),L1-L2} \exp[j(\mathbf{K}_{(n),L1-L2} \cdot z\hat{z} - 2\omega t)] \end{cases} \tag{6}$$

with [5, 6]

$$\begin{cases} F_{(n),Lp-Lp} = j(\gamma + 1)\omega K_L^2 \phi^2_{(n),p} \\ \mathbf{K}_{(n),Lp-Lp} = \mathbf{K}_{(n),Lp} + \mathbf{K}_{(n),Lp}, \quad p = 1, 2 \\ F_{(n),L1-L2} = 2j\omega K_L^2 \left[(\gamma - 1) + \frac{\mathbf{K}_{(n),L1} \cdot \mathbf{K}_{(n),L2}}{K_L^2} \right] \phi_{(n),1} \phi_{(n),2} \\ \mathbf{K}_{(n),L1-L2} = \mathbf{K}_{(n),L1} + \mathbf{K}_{(n),L2}, \end{cases}. \tag{7}$$

In equations (6) and (7), $\hat{z}$ is a unit vector along the oz axis, $K_L = \omega/c_L$ the magnitude of wave vector $\mathbf{K}_{(n),Lp}$, and γ the ratio of specific heats for gas or $(1 + B/A)$ for liquid. The driven second harmonics of the nth fundamental mode (see Figure 1) are obtained from equations (2), (6) and (7) as [6, 10]

$$\phi^{(2D)}_{(n),Lp-Lp} = \frac{F_{(n),Lp-Lp}}{c_L^2[(2K_L)^2 - |\mathbf{K}_{(n),Lp-LP}|^2]} \exp[j(\mathbf{K}_{(n),Lp-Lp} \cdot \mathbf{r}_p - 2\omega t)], \quad p = 1, 2 \quad (8)$$

and

$$\phi^{(2D)}_{(n),L1-L2} = \frac{F_{(n),L1-L2}}{c_L^2[(2K_L)^2 - |\mathbf{K}_{(n),L1-L2}|^2]} \exp[j(\mathbf{K}_{(n),L1-L2} \cdot z\hat{z} - 2\omega t)]. \quad (9)$$

In equations (8) and (9) the superscript $(2D)$ is used to identify the scale potentials associated with the driving force $F(\phi^{(1)}_{(n)})$, and the subscript $Lp - Lq$ $(p, q = 1, 2)$ means that a physical quantity relates to the non-linear interaction between partial longitudinal waves p and q.

Because of the absence of dispersion of the fluid in the duct, there is a relationship, i.e., $|\mathbf{K}_{(n),Lp-Lp}| \to 2K_L$ $(p = 1, 2)$. From the expression of equation (8), we obtain a conclusion, $\phi^{(2D)}_{(n),Lp-Lp} \to \infty$ $(p = 1, 2)$. There also exists the relationship $|\mathbf{K}_{(n),L1-L2}| \to 2K_L$ as $n = 0$. Thus, $\phi^{(2D)}_{(0),L1-L2} \to \infty$. But $n = 0$ corresponds to plane wave solution which has been the subject of intensive study. Therefore, in the present article only the case $n \geqslant 1$ will be discussed. The amplitude of the driven second harmonic $\phi^{(2D)}_{(n),L1-L2}$ can be neglected when compared with that of $\phi^{(2D)}_{(n),L1-L1}$ and $\phi^{(2D)}_{(n),L2-L2}$.

There is a boundary condition that requires that the normal components of second-harmonic particle velocities equal zero at the two rigid walls, $y = \pm d$. However, the boundary condition, in general, cannot be satisfied when $\phi^{(2D)}_{(n),L1-L1}$ and $\phi^{(2D)}_{(n),L2-L2}$ are only considered [8–10]. To satisfy this boundary condition, the second harmonics accompanying the nth fundamental mode, i.e., the general solution of equation (2) when the driving force $F(\phi^{(1)}_{(n)})$ is zero, must be introduced in the duct. We call the second harmonics freely propagating since there is no driving force. Because of the absence of dispersion of the fluid and phase matching, along the oz axis, between the driven and freely propagating second harmonics, we can describe freely propagating second harmonics by (see Figure 1) [9, 10]

$$\phi^{(2F)}_{(n)} = \phi^{(2F)}_{(n),1} \exp[j(2\mathbf{K}_{(n),L1} \cdot \mathbf{r}_1 - 2\omega t)] + \phi^{(2F)}_{(n),2} \exp[j(2\mathbf{K}_{(n),L2} \cdot \mathbf{r}_2 - 2\omega t)], \quad (10)$$

where the superscript $(2F)$ is used to denote the freely propagating second harmonic without the driving force. In fact, $\phi^{(2F)}_{(n)}$ occurs synchronously along with $\phi^{(2D)}_{(n),Lp-Lp}$ $(p = 1, 2)$, and $\phi^{(2F)}_{(n)} \cdot \phi^{(2D)}_{(n),Lp-Lp}$ are dependent of the non-linear interaction of partial longitudinal waves of the nth fundamental mode.

The ultimate second harmonics (including the driven and freely propagating second harmonics) in the process of propagation of FAAWs, described by scale potential $\phi^{(2\omega)}_{(n)}$, are determined as a sum of $\phi^{(2D)}_{(n)}$ (including $\phi^{(2D)}_{(n),L1-L1}$ and $\phi^{(2D)}_{(n),L2-L2}$)

and $\phi_{(n)}^{(2F)}$ (including $\phi_{(n),1}^{(2F)}$ and $\phi_{(n),2}^{(2F)}$), i.e., $\phi_{(n)}^{(2\omega)} = \phi_{(n)}^{(2D)} + \phi_{(n)}^{(2F)}$. We define the variables $\phi_{(n),p}^{(2\omega)}$ as follows (neglecting the factor $\exp[j(2\mathbf{K}_{(n),Lp} \cdot \mathbf{r}_p - 2\omega t)]$):

$$\phi_{(n),p}^{(2\omega)} = \phi_{(n),Lp-Lp}^{(2D)} + \phi_{(n),p}^{(2F)} \exp[j(2\mathbf{K}_{(n),Lp} \cdot \mathbf{r}_p - 2\omega t)]$$

$$= \left\{ \frac{F_{(n),Lp-Lp}}{C_L^2[(2K_L)^2 - |\mathbf{K}_{(n),Lp-Lp}|^2]} \exp[j(\mathbf{K}_{(n),Lp-Lp} - 2\mathbf{K}_{(n),Lp}) \cdot \mathbf{r}_p] + \phi_{(n),p}^{(2F)} \right\}$$

$$\times \exp[j(2\mathbf{K}_{(n),Lp} \cdot \mathbf{r}_p - 2\omega t)], \quad p = 1, 2. \tag{11}$$

On the basis of the fact that the fluid in the duct lacks dispersion, the value in { } of equation (11) can be considered as a constant. For simplicity we still use $\phi_{(n),p}^{(2\omega)}$ to denote the value in { } of equation (11). From second-harmonic boundary condition [6], we have

$$\left. \frac{\partial[\phi_{(n),1}^{(2\omega)} + \phi_{(n),2}^{(2\omega)}]}{\partial y} \right|_{y=\pm d} = 0. \tag{12}$$

It follows that

$$[M(2\omega)]\begin{pmatrix} \phi_{(n),1}^{(2\omega)} \\ \phi_{(n),2}^{(2\omega)} \end{pmatrix} = 0. \tag{13}$$

Usually, the propagation mode in a duct is dispersive, i.e., there is phase mismatch between the fundamental and second-order modes, denoted by ω- and 2ω-modes. For convenience we still let $[M(\omega)]$ and $[M(2\omega)]$ be the coefficient matrixes of ω- and 2ω-modes, respectively, which are determined by the corresponding boundary conditions. The relationship, $|M(\omega) = 0|$ or $|M(2\omega)| = 0$ determines the phase velocity of ω- or 2ω-mode. Although $|M(\omega)| = 0$, in general, $|M(2\omega)| \neq 0$. Conversely, $|M(\omega)| = 0$ cannot be deduced from $|M(2\omega)| = 0$.

However, there is an exception for the nth fundamental mode in a two-dimensional rectangular duct. Although $|M(\omega)| = 0$ cannot be obtained from $|M(2\omega)| = 0$, it is easy to show that $|M(\omega)| = 0$ can lead to $|M(2\omega)| = 0$. Thus equation (13) has non-trivial solution, i.e., there does exist the solution of the ultimate second harmonics accompanying the nth fundamental mode.

With the relationship among mechanical displacement vector, scale potential and particle velocity, we can obtain the corresponding ultimate second-harmonic mechanical displacement vector $\mathbf{u}_{(n),p}^{(2\omega)}$ ($p = 1, 2$),

$$\mathbf{u}_{(n),p}^{(2\omega)} = -\frac{1}{2j\omega} \nabla \phi_{(n),p}^{(2\omega)}. \tag{14}$$

The second-harmonic analytical expressions in the duct can be obtained by using the following method [9, 10]. Firstly, let us substitute equation (11) into equation (14),

$$\mathbf{u}_{(n),p}^{(2\omega)} = -\frac{1}{2\omega}\left\{\begin{array}{l}\left[\dfrac{-F_{(n),Lp-Lp}}{(2K_L + |\mathbf{K}_{(n),Lp-Lp}|)c_L^2} + \dfrac{2K_L F_{(n),Lp-Lp}}{c_L^2[(2K_L)^2 - |\mathbf{K}_{(n),Lp-Lp}|^2]}\right] \\ \quad \times \exp[j(\mathbf{K}_{(n),Lp-Lp} - 2\mathbf{K}_{(n),Lp})\cdot\mathbf{r}_p] \\ \quad + 2K_{(n),Lp}\phi_{(n),p}^{(2F)}\end{array}\right\}$$

$$\times\frac{\mathbf{K}_{(n),Lp}}{K_L}\exp[j(2\mathbf{K}_{(n),Lp}\cdot\mathbf{r}_p - 2\omega t)]. \tag{15}$$

Further, equation (15) can be rewritten as

$$\mathbf{u}_{(n),p}^{(2\omega)} = \frac{1}{2\omega}\left\{\begin{array}{l}\dfrac{F_{(n),Lp-Lp}}{4K_L c_L^2}\left[1 + \dfrac{2K_{(n),Lp}}{(|\mathbf{K}_{(n),Lp-Lp}| - 2K_L)}\right] \\ \quad \times[1 + j(\mathbf{K}_{(n),LP-Lp} - 2\mathbf{K}_{(n),Lp})\cdot\mathbf{r}_p + \cdots] - 2K_{(n),Lp}\phi_{(n),p}^{(2F)}\end{array}\right\}$$

$$\times\frac{\mathbf{K}_{(n),Lp}}{K_L}\exp[j(2\mathbf{K}_{(n),Lp}\cdot\mathbf{r}_p - 2\omega t)]. \tag{16}$$

Consider the initial condition of excitation [9, 10], i.e., the second-harmonic amplitude equals zero at the initial position $z = 0$. From equation (16), it is easy to see that this condition is satisfied if we set

$$\phi_{(n),p}^{(2F)} = \frac{F_{(n),Lp-Lp}}{8K_L^2 c_L^2}\left[1 + \frac{2K_L}{(|\mathbf{K}_{(n),Lp-Lp}| - 2K_L)}\right]. \tag{17}$$

Hence, combining equations (16) and (17) yields

$$\mathbf{u}_{(n),p}^{(2\omega)} = -\frac{(\gamma+1)K_L^2}{4c_L^2}\phi_{(n),p}^2[z\sin\theta_{(n)} + (-1)^{p-1}y\cos\theta_{(n)}]$$

$$\times\frac{\mathbf{K}_{(n),Lp}}{K_L}\exp[j(2\mathbf{K}_{(n),Lp}\cdot\mathbf{r}_p - 2\omega t)]. \tag{18}$$

Equation (18) shows that the ultimate second harmonic $\mathbf{u}_{(n),p}^{(2\omega)}$ ($p = 1, 2$) due to both the bulk non-linearity of the fluid and the restriction of boundary of the duct grows linearly with propagation distance. Because of $\phi_{(n),1} = \pm\phi_{(n),2}$, there exists the relationship $|\mathbf{u}_{(n),1}^{(2\omega)}| = |\mathbf{u}_{(n),2}^{(2\omega)}|$ at the duct walls $y = \pm d$. From equation (18), the ultimate second harmonics (including $\mathbf{u}_{(n),1}^{(2\omega)}$ and $\mathbf{u}_{(n),2}^{(2\omega)}$) associated with the nth fundamental mode in the duct possess the characteristic of symmetry.

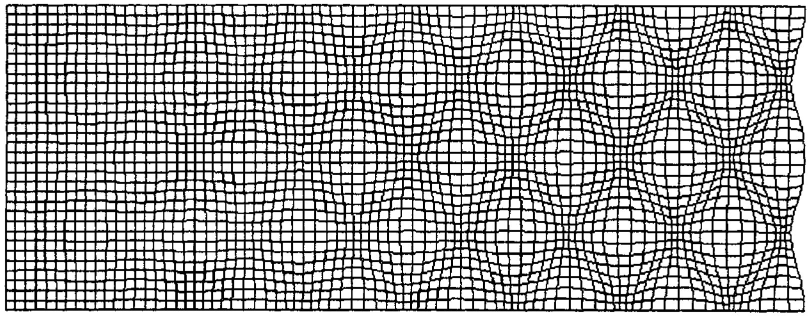

Figure 2. Second-harmonic field pattern for a given source of excitation.

3. NUMERICAL ANALYSIS

We use quantitative computation to illustrate the previous results. For simplicity, we assume that the fluid in the duct is an ideal gas, and have the constant values as: $c_L = 334\ \mathrm{m\ s^{-1}}$, $\gamma = 1{\cdot}4$. We assume that $fd = 0{\cdot}3\ \mathrm{MHz \cdot mm}$ and the vibration distribution function of the source is given by $\phi(y) = \phi_0 \cos(\pi y/d)$, $-d \leqslant y \leqslant d$, $z = 0$, so that we have $\phi_{(2)} = \phi_0$. From equation (5) it is easy to find $c_{(2)} = 419{\cdot}9\ \mathrm{m\ s^{-1}}$. We obtain the second-harmonic field pattern (relative to $\phi_0^2 c_L^{-2} d^{-1}$) shown in Figure 2. The characteristics of distortion and symmetry in the field pattern are clearly exhibited.

4. CONCLUSION

As described above, we have studied the propagation of FAAWs in a two-dimensional rectangular duct with two rigid walls by applying second-order perturbation and partial wave analysis method. The results show that the second harmonics in the process of propagation of FAAWs arise from the bulk non-linearity of fluid of the duct and the non-linear self-interaction of partial longitudinal wave of fundamental mode, and that the cross-interaction between two partial longitudinal waves does not retain second-order non-linearity (i.e., this cross-interaction cannot induce second-harmonic growth effect). Moreover, a second-harmonic field pattern associated with an arbitrary fundamental mode displays characteristics of symmetry. The process of analysis of FAAW propagation involved in this article exhibits a clearly physical mechanism, based on the partial wave method and second-order perturbation. Furthermore, the present results permit quantitative discussion of the physical process involved in FAAW propagation, and lay a foundation for further study of propagation of FAAWs in a duct.

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APPENDIX: COEFFICIENT MATRICES IN EQUATIONS (4) AND (13)

$[M(\omega)]$ and $[M(2\omega)]$ in equations (4) and (13) are given by

$$[M(\omega)] = \begin{pmatrix} \exp[j\sqrt{C_L^{-2} - C^{-2}} \cdot \omega d] & -\exp[-j\sqrt{C_L^{-2} - C^{-2}} \cdot \omega d] \\ \exp[-j\sqrt{C_L^{-2} - C^{-2}} \cdot \omega d] & -\exp[j\sqrt{C_L^{-2} - C^{-2}} \cdot \omega d] \end{pmatrix}$$

and

$$[M(2\omega)] = \begin{pmatrix} \exp[j\sqrt{C_L^{-2} - C^{-2}} \cdot 2\omega d] & -\exp[-j\sqrt{C_L^{-2} - C^{-2}} \cdot 2\omega d] \\ \exp[-j\sqrt{C_L^{-2} - C^{-2}} \cdot 2\omega d] & -\exp[j\sqrt{C_L^{-2} - C^{-2}} \cdot 2\omega d] \end{pmatrix}.$$