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# AN INTEGRAL REPRESENTATION FORMULATION OF SOME ONE-DIMENSIONAL PROBLEMS

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An Integral Representation Formulation (IRF) has been developed to solve some one-dimensional problems: acoustic problems in pipes, longitudinal and transverse vibrations of pipes, including mean flow effects. The fluid–structure interaction is analyzed in the case of a straight pipe. The fundamental solutions and the set of equations governing problems are determined in each case. To demonstrate the efficiency and the simplicity of the use of the IRF method for one-dimensional problems, examples are presented.

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## 1. INTRODUCTION

Various methods have been used for dynamic analysis. Such methods are: the Modal Method (MM) using the Finite Element Method (FEM), the Transfer Matrix Method (TMM), the Integral Representation Formulation (IRF).

When the frequency domain analysis contains a large number of modes, the use of the FEM is not appropriate and the numerical determination of modes contains many errors.

The Transfer Matrix Method (TMM) is often used for one-dimensional acoustic problems. For one-dimensional problems of beam structures, the TMM is not frequently used; since the analytical solutions are complicated.

The IRF is very often used for three-dimensional problems dealing with an infinite space or half-space: acoustic problems, seismic analysis of underground structures. In this case, the method requires only the discretization of the surface of the domain under consideration.

The IRF method is easily used for one-dimensional problems due to the existence of the fundamental solutions and the fact that the boundary of a domain is a point and so no discretization is required. In fact the IRF only requires the discretization of the one-dimensional element under consideration, regardless of the frequency range of analysis.

In this present work, an integral representation formulation for some one-dimensional problems is proposed: • an acoustic problem: a comparison will be made with the transfer matrix method; • a beam structure problem: a comparison will be made with the modal method; • a fluid–structure interaction

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problem: considered here is the fluid–structure interaction taking into account the mean flow effects. Two cases are analysed: the longitudinal motion of a pipe with a restriction of the flow area and the transverse motion of a pipe with a fluid characterised by the mean flow velocity and the mean pressure.

For each problem, the system is modelled by using a frequency-domain integral representation formulation based on the fundamental solutions for each one-dimensional problem. In such a way, the set of equations governing each problem is determined. In order to show the simplicity and the efficiency of the IRF method for one-dimensional problems, examples are carried out and compared with other available methods.

## 2. GOVERNING EQUATIONS

### 2.1. STRUCTURAL EQUATION, INCLUDING MEAN FLOW EFFECTS

In the case of zero mean flow, continuity across the interface between fluid and structure requires that the positive normal velocity of the fluid equals the displacement rate of the solid in that same direction. Similarly, equilibrium necessitates that the fluid pressure is balanced by the normal surface traction on the contact surface. For flexible pipeline systems the junction coupling (restriction of the fluid area or elbow . . .) is mostly the predominant type of coupling [1–3]. Only the restriction of the fluid area case is considered here.

For harmonic analysis, the governing lateral and longitudinal vibration equations of a straight elastic pipe can be written, for localised forces as follows [1, 2, 4]:

for harmonic analysis and longitudinal motion of the pipe:

$$EA_t \frac{\partial^2 U_x}{\partial x^2} + \rho_t A_t \omega^2 U_x - F_x(\omega)\delta(x - x_1) - p(\Delta A_f)\delta(x - x_3) = 0; \tag{1a}$$

for harmonic analysis and lateral motion of the pipe:

$$EI_y \frac{\partial^4 U_z}{\partial x^4} + A_f(\Delta P + \rho_f \bar{V}) \frac{\partial^2 U_z}{\partial x^2} + 2\rho_f A_f \bar{V} \mathrm{i}\omega \frac{\partial U_z}{\partial x} - \omega^2(\rho_t A_t + \rho_f A_f) U_z + F_z(\omega)\delta(x - x_2) = 0, \tag{1b}$$

or

$$\frac{\partial^4 U_z}{\partial x^4} + \beta^2 \frac{\partial^2 U_z}{\partial x^2} + 2\mathrm{i}\omega\gamma^2 \frac{\partial U_z}{\partial x} - \lambda_0^4 U_z + \frac{F_z}{EI_y}\delta(x - x_2) = 0,$$

with

$$\beta^2 = \frac{A_f(\Delta P + \rho_f \bar{V})}{EI_y}; \qquad \gamma^2 = \frac{\rho_f A_f \bar{V}}{EI_y}; \qquad \lambda_0^4 = \omega^2 \frac{(\rho_t A_t + \rho_f A_f)}{EI_y},$$

where $U_x(\omega)$ and $U_z(\omega)$ are respectively the Fourier transforms of longitudinal and lateral displacement of the pipe, $x$ is the longitudinal co-ordinate, $(y, z)$ are the lateral co-ordinates, $\omega$ is the frequency, $A_f$ is the cross-sectional discharge area, $A_t$ is the cross-sectional pipe wall area, $\Delta A_f$ is the local restriction of the cross-sectional discharge area, $E$ is the Young's modulus, $I_y$ is the $y$-moment of inertia, $\Delta P$ is the pressure drop in the pipe, $\bar{V}$ is the fluid velocity, $\rho_f$ is the fluid density, $\rho_t$ is the pipe wall density, $x_1$ and $x_2$ are respectively the position of the localised forces $F_x$ and $F_z$ and finally $x_3$ is the position of the restriction of the cross-sectional discharge area.

### 2.2. FLUID EQUATIONS, INCLUDING MEAN FLOW EFFECTS

For one-dimensional flow, with velocity $\bar{V}$ in the $x$ direction, and by introducing small perturbations into the governing equations (momentum and mass equations for inviscid and isotropic flow) and by combining the total time derivative of the momentum equation and the divergence of the mass equation, one obtains the following Fourier transformed equation [2, 5]:

$$\left(\frac{\partial^2}{\partial x\,\partial x} + k_p^2 + 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\frac{\partial}{\partial x} - \frac{\bar{V}^2}{c^2}\frac{\partial^2}{\partial x\,\partial x}\right)p - \frac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x\delta(x - x_3) = 0, \quad (1c)$$

where $k_p = \omega/c$ is the acoustic wave number. In equation (1c), it is assumed that the fluid has attained a steady flow characterised by the mean velocity $\bar{V}$ and speed of sound squared $c^2$ along with zero body forces. The field quantities $\bar{V}$ and $c^2$ are assumed to be constant within a given region of the fluid domain. If one permits the mean flow velocity to vary within a region of space, then an equation formally similar to (1c) is obtained. However, the expression becomes more cumbersome [4]. Note that for zero mean flow velocity and in the absence of body forces, this becomes the classical Helmholtz equation.

## 3. INTEGRAL REPRESENTATION

The use of the integral formulation for periodic processes in acoustics have a long history beginning with the work of Helmholtz [6]. For three-dimensional domain problems, the fundamental solution can be derived by utilising a triple exponential Fourier transform [7]. For an elastic solid, an integral formulation can be derived in a manner similar to that utilized for an acoustical fluid (see for example reference [8]).

For one-dimensional fluid–structure interaction problems, the following development is proposed.

### 3.1. FLUID INTEGRAL REPRESENTATION, INCLUDING MEAN FLOW EFFECTS

For equation (1c), an integral representation can be developed in a straightforward manner. First multiply (1c) by an arbitrary function $\tilde{G}_p$ and integrate over the length of the pipe. For zero body forces, one obtains:

$$\int_0^L A_f\left(\frac{\partial^2}{\partial x\,\partial x} + k_p^2 + 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\frac{\partial}{\partial x} - M^2\frac{\partial^2}{\partial x\,\partial x}\right)p\tilde{G}_p\,\mathrm{d}x$$

$$-\int_0^L \frac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x\delta(x - x_3)\tilde{G}_p\,\mathrm{d}x = 0, \quad (2)$$

where $M$ is the Mach number; $M = \bar{V}/c$.

Then after using integration by parts on the various terms in equation (2) to transfer derivatives from $p$ to $\tilde{G}_p$, the following integral equation is produced:

$$\int_0^L \left((1-M^2)\frac{\partial^2\tilde{G}_p}{\partial x^2} + k_p^2\tilde{G}_p - 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\frac{\partial\tilde{G}_p}{\partial x}\right)p\,\mathrm{d}x$$

$$+\left[(1-M^2)\left(\frac{\partial p}{\partial x}\tilde{G}_p - \frac{\partial\tilde{G}_p}{\partial x}p\right) + 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\tilde{G}_p p\right]_0^L$$

$$-\frac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x(x_3)\tilde{G}_p(x_3) = 0. \tag{3}$$

By selecting $\tilde{G}_p$ to be related to the fundamental solution of the adjoint convective Helmholtz operator:

$$(1-M^2)\frac{\partial^2\tilde{G}_p}{\partial x^2} + k_p^2\tilde{G}_p - 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\frac{\partial\tilde{G}_p}{\partial x} = -\delta(x-x_0), \tag{4}$$

equation (3) simplifies to the form:

$$\left[(1-M^2)\left(\frac{\partial p}{\partial x}\tilde{G}_p - \frac{\partial\tilde{G}_p}{\partial x}p\right) + 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\tilde{G}_p p\right]_0^L - \frac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x(x_3)\tilde{G}_p(x_3) = p(x_0), \tag{5}$$

where $x_0$ represents a point on [0, $L$] and $p(x_0)$ is the fluctuated pressure at $x = x_0$. Equation (5) is applied for $x_0 = 0$ (then $\tilde{G}_p \equiv \tilde{G}_p^0$) and $x_0 = L$ (then $\tilde{G}_p \equiv \tilde{G}_p^L$), one obtains:

$$\begin{bmatrix} -F_p(\tilde{G}_p^0(0));\ \{-(1-M^2)(\tilde{G}_p^0(0))\};\ F_p(\tilde{G}_p^0(L));\ \{(1-M^2)(\tilde{G}_p^0(L))\} \\ -F_p(\tilde{G}_p^0(0));\ \{-(1-M^2)(\tilde{G}_p^0(0))\};\ F_p(\tilde{G}_p^0(L));\ \{(1-M^2)(\tilde{G}_p^0(L))\} \end{bmatrix}$$

$$\times \begin{bmatrix} p(0) \\ \dfrac{\partial p(0)}{\partial x} \\ p(L) \\ \dfrac{\partial p(L)}{\partial x} \end{bmatrix} - \begin{bmatrix} \dfrac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x(x_3)\tilde{G}_p^0(x_3) \\ \dfrac{\omega^2\rho_f}{A_f}(\Delta A_f)U_x(x_3)\tilde{G}_p^L(x_3) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(L) \end{bmatrix}, \tag{6}$$

where

$$F_p(\tilde{G}_p) = \left\{-(1-M^2)\left(\frac{\partial\tilde{G}_p}{\partial x}\right) + 2\mathrm{i}\omega\frac{\bar{V}}{c^2}\tilde{G}_p\right\}. \tag{7}$$

Using the development of Appendix A, the fundamental function $\tilde{G}_p$ is written as:

$$\tilde{G}_p^{x_0}(x) = \frac{e^{\lambda_1(x-x_0)}}{2k_0} \qquad \text{for } (x-x_0) \geqslant 0,$$

$$\tilde{G}_p^{x_0}(x) = \frac{e^{\lambda_2(x-x_0)}}{2k_p} \qquad \text{for } (x-x_0) \leqslant 0, \tag{8}$$

where

$$\lambda_1 = -\mathrm{i}(k_p/1+M); \qquad \lambda_2 = \mathrm{i}(k_p/1-M).$$

### 3.2. STRUCTURAL INTEGRAL REPRESENTATION, INCLUDING MEAN FLOW EFFECTS

For elastic pipes, an integral formulation can be derived in a manner similar to that utilized for fluid domain. First multiply equation (1a) by an arbitrary function $\tilde{G}_x$, multiply (1b) by an arbitrary function $\tilde{G}_z$ and integrate over the length of the pipe, one obtains:

$$\int_0^L \left[ EA_t \frac{\partial^2 U_x}{\partial x^2} + \rho_t A_t \omega^2 U_x - F_x(\omega)\delta(x-x_1) - p(\Delta A_f)\delta(x-x_3) \right] \tilde{G}_x \, \mathrm{d}x = 0, \tag{9a}$$

$$\int_0^L \left[ \frac{\partial^4 U_z}{\partial x^4} + \beta^2 \frac{\partial^2 U_z}{\partial x^2} + 2\mathrm{i}\omega\gamma^2 \frac{\partial U_z}{\partial x} - \lambda_0^4 U_z + \frac{F_z}{EI_y}\delta(x-x_2) \right] \tilde{G}_z \, \mathrm{d}x = 0. \tag{9b}$$

Then after using integration by parts on the various terms in equations (9) to transfer derivatives from displacements to fundamental functions, the following integral equations are produced:

$$\int_0^L \left[ EA_t \frac{\partial^2 \tilde{G}_x}{\partial x^2} + \rho_t A_t \omega^2 \tilde{G}_x \right] U_x \, \mathrm{d}x + \left[ EA_t \left( \frac{\partial U_x}{\partial x} \tilde{G}_x - \frac{\partial \tilde{G}_x}{\partial x} U_x \right) \right]_0^L$$
$$- F_x(\omega)\tilde{G}_x(x_1) - p(x_3)(\Delta A_f)\tilde{G}_x(x_3) = 0, \tag{10a}$$

$$\int_0^L \left[ \frac{\partial^4 \tilde{G}_z}{\partial x^4} + \beta^2 \frac{\partial^2 \tilde{G}_z}{\partial x^2} - 2\mathrm{i}\omega\gamma^2 \frac{\partial \tilde{G}_z}{\partial x} - \lambda_0^4 \tilde{G}_z \right] U_z \, \mathrm{d}x$$
$$+ \left[ \left( \frac{\partial^3 U_z}{\partial x^3} \tilde{G}_z - \frac{\partial^3 \tilde{G}_z}{\partial x^3} U_z - \frac{\partial^2 U_z}{\partial x^2} \frac{\partial \tilde{G}_z}{\partial x} + \frac{\partial^2 \tilde{G}_z}{\partial x^2} \frac{\partial U_z}{\partial x} \right) \right]_0^L$$
$$+ \left[ \beta^2 \left( \frac{\partial U_z}{\partial x} \tilde{G}_z - \frac{\partial \tilde{G}_z}{\partial x} U_z \right) \right]_0^L + \frac{F_z}{EI_y} \tilde{G}_z(x_2) = 0. \tag{10b}$$

By selecting $\tilde{G}_x$ and $\tilde{G}_z$ to be related to the fundamental solutions of the corresponding adjoint operator (see Appendix A), equations (10) simplify to the form:

$$\left[ EA_t \left( \frac{\partial U_x}{\partial x} \tilde{G}_x - \frac{\partial \tilde{G}_x}{\partial x} U_x \right) \right]_0^L - F_x(\omega)\tilde{G}_x(x_1) - p(x_3)(\Delta A_f)\tilde{G}_x(x_3) = U_x(x_0), \tag{11a}$$

$$\left[\left(\frac{\partial^3 U_z}{\partial x^3}\tilde{G}_z - \frac{\partial^3 \tilde{G}_z}{\partial x^3}U_z - \frac{\partial^2 U_z}{\partial x^2}\frac{\partial \tilde{G}_z}{\partial x} + \frac{\partial^2 \tilde{G}_z}{\partial x^2}\frac{\partial U_z}{\partial x}\right)\right]_0^L$$

$$+\left[\beta^2\left(\frac{\partial U_z}{\partial x}\tilde{G}_z - \frac{\partial \tilde{G}_z}{\partial x}U_z\right)\right]_0^L + \frac{F_z}{EI_y}\tilde{G}_z(x_2) = -\frac{U_z(x_0)}{EI_y}, \quad (11b)$$

where $x_0$ represents a point on [0, $L$]. By taking the $x_0$ derivative of equation (11b) ($\partial$(eq. (11b)/$\partial x_0$), one obtains:

$$\left[\left(-\frac{\partial^3 U_z}{\partial x^3}\frac{\partial \tilde{G}_z}{\partial x} + \frac{\partial^4 \tilde{G}_z}{\partial x^4}U_z + \frac{\partial^2 U_z}{\partial x^2}\frac{\partial^2 \tilde{G}_z}{\partial x^2} - \frac{\partial^3 \tilde{G}_z}{\partial x^3}\frac{\partial U_z}{\partial x}\right)\right]_0^L$$

$$+\left[\beta^2\left(-\frac{\partial U_z}{\partial x}\frac{\partial \tilde{G}_z}{\partial x} + \frac{\partial^2 \tilde{G}_z}{\partial x^2}U_z\right)\right]_0^L - \frac{F_z}{EI_y}\frac{\partial \tilde{G}_z}{\partial x}(x_2) = -\frac{1}{EI_y}\frac{\partial U_z}{\partial x}(x_0). \quad (12)$$

Note:

$$\frac{\partial \tilde{G}_z}{\partial x_0} = -\frac{\partial \tilde{G}_z}{\partial x}; \qquad \frac{\partial [U_z(x_0)]}{\partial x_0} = \frac{\partial [U_z(x)]}{\partial x}(x_0).$$

Equations (11) and (12) are applied for $x_0 = 0$ (then $\tilde{G}_x \equiv \tilde{G}_x^0$ and $\tilde{G}_z \equiv \tilde{G}_z^0$) and $x_0 = L$ (then $\tilde{G}_x \equiv \tilde{G}_x^L$ and $\tilde{G}_x \equiv \tilde{G}_z^L$), giving

$$\left[\begin{matrix} EA_t\dfrac{\partial \tilde{G}_x^0}{\partial x}(0) - 1; \ -EA_t(\tilde{G}_x^0(0); \ -EA_t\dfrac{\partial \tilde{G}_x^0}{\partial x}(L); \ EA_t\tilde{G}_x^0(L) \\ EA_t\dfrac{\partial \tilde{G}_x^L}{\partial x}(0); \ -EA_t\tilde{G}_x^L(0); \ -EA_t\dfrac{\partial \tilde{G}_x^L}{\partial x}(L) - 1; \ EA_t\tilde{G}_x^L(L) \end{matrix}\right]_0^L$$

$$\times \begin{bmatrix} U_x(0) \\ \dfrac{\partial U_x}{\partial x}(0) \\ U_x(L) \\ \dfrac{\partial U_x}{\partial x}(L) \end{bmatrix} - \begin{bmatrix} p(x_3)\Delta A_f\tilde{G}_x^0(x_3) \\ p(x_3)\Delta A_f\tilde{G}_x^L(x_3) \end{bmatrix} = \begin{bmatrix} F_x(\omega)\tilde{G}_x^0(x_1) \\ F_x(\omega)\tilde{G}_x^L(x_1) \end{bmatrix}, \quad (13a)$$

$$\{[\mathbf{M}_1] + \beta^2[\mathbf{M}_2\} \cdot [\mathbf{q}] = [\mathbf{F}]. \quad (13b)$$

Matrices $[\mathbf{M}_1]$, $[\mathbf{M}_2]$ and vectors $[\mathbf{q}]$, $[\mathbf{F}]$ are defined in Appendix B.

Using the development of Appendix A, the fundamental solution $\tilde{G}_x$ for longitudinal motion is written as

$$\tilde{G}_x = \frac{e^{\lambda_x |x - x_0|}}{2EA_1\lambda_x}; \qquad \lambda_x = i\omega\sqrt{\frac{\rho}{E}} \quad (14)$$

and the fundamental solution $\tilde{G}_z$ for transverse motion is written as:

$$\tilde{G}_z = \frac{1}{2EI_y}\left[\frac{e^{\lambda_{z1}(x-x_0)}}{\lambda_{z1}(\lambda_1^2-\lambda_{z3}^2)} - \frac{e^{\lambda_{z3}(x-x_0)}}{\lambda_{z3}(\lambda_{z1}^2-\lambda_{z3}^2)}\right] \qquad \text{for } x - x_0 \leqslant 0,$$

$$\tilde{G}_z = \frac{1}{2EI_y}\left[\frac{e^{-\lambda_{z1}(x-x_0)}}{\lambda_{z1}(\lambda_{z1}^2-\lambda_{z3}^2)} - \frac{e^{-\lambda_{z3}(x-x_0)}}{\lambda_{z3}(\lambda_{z1}^2-\lambda_{z3}^2)}\right] \qquad \text{for } x - x_0 \geqslant 0, \tag{15}$$

where

$$\lambda_{z1} = \mathrm{i}\,\frac{\sqrt{2}}{2}\sqrt{2\sqrt{\beta^4+\lambda_0^4}+\beta^2}; \qquad \lambda_{z2} = -\lambda_{z1},$$

$$\lambda_{z3} = \frac{\sqrt{2}}{2}\sqrt{2\sqrt{\beta^4+\lambda_0^4}-\beta^2}; \qquad \lambda_{z4} = -\lambda_{z3}.$$

## 4. NUMERICAL IMPLEMENTATION

The numerical implementation is classical, it uses a multi-pipe part approach. It consists of writing the difference equations that arise from the assembly of the elements of the pipe and the natural imposition of the boundary conditions. The IRF discretization results in matrices that are non-symmetric and complex valued. To solve efficiently the coupled system of equations arising from the IRF, the Gauss elimination with partial pivoting can be used without any problem. The numerical implementation of the IRF is similar to that used by the Transfer Matrix Method.

## 5. NUMERICAL APPLICATIONS

The following numerical applications will be used only to demonstrate the high accuracy, efficiency and the easy use of the IRF for the one-dimensional problems, especially the fluid–structure problem in pipes.

Three numerical examples are presented in this section to illustrate the use of the IRF.

### 5.1. ONE-DIMENSIONAL FLOW IN PIPES

Consider as a first example, a uniform pipe of length $L$ with pressure nodes at each end, as shown in Figure 1. The pipe is submitted to a Dirac pressure applied at the pipe's middle.

Using the IRF method, the modulus of the response function $\partial p/\partial x(0)$ versus the wave number $k_p$ is plotted in Figure 2(a), for a Dirac pressure localised at $x = L/2$ and for various values of Mach number $M$. The IRF and the Transfer Matrix Method (TMM) response are plotted in Figure 2(b). The frequency range considered contains the first three natural frequencies.

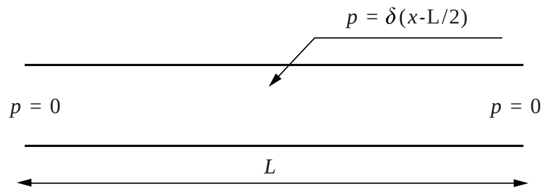


Figure 1. Pipe submitted to Dirac pressure.

The IRF Solution is written as follows:

$$\frac{\partial p}{\partial x}(0) = \frac{\tilde{G}_p^L(L)\tilde{G}_p^0(L/2) - \tilde{G}_p^0(L)\tilde{G}_p^L(L/2)}{(1-M^2)[-\tilde{G}_p^0(0)\tilde{G}_p^L(L) + \tilde{G}_p^0(L)\tilde{G}_p^L(0)]}.$$

Figure 2(a) shows the effect of the Mach number on the natural frequencies and on the response of the pipe. When the Mach number increases the natural frequencies decrease. The comparison between the IRF and the TMM in Figure 2(b) shows a very good agreement between the two methods.

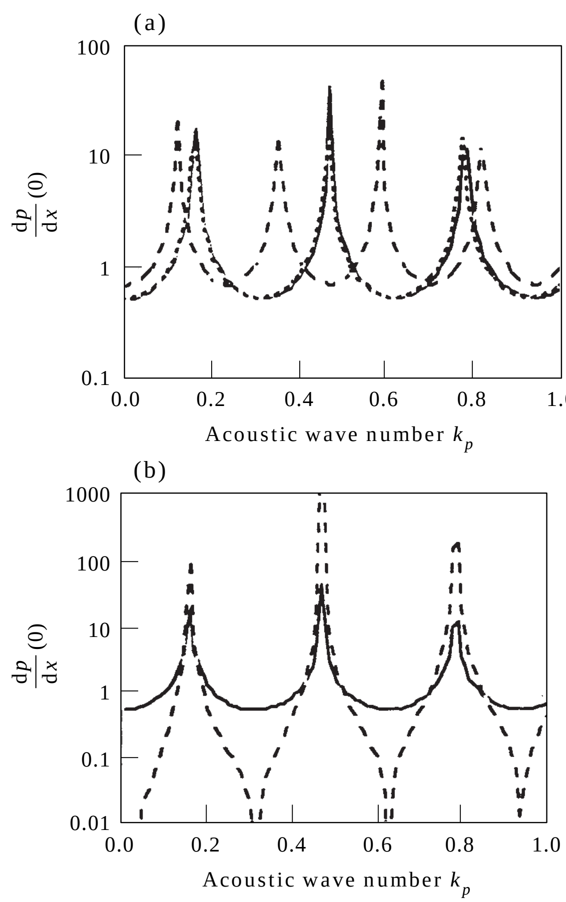


Figure 2. (a) $\partial p/\partial x(0)$, for localised Dirac pressure in $x = L/2$, for various values of Mach number: ——, $M = 0{\cdot}0$; ---, $M = 0{\cdot}1$; –––, $M = 0{\cdot}5$. (b) Comparison of the IRF and the Transfer Matrix Method (TTM): ——, IRF; –––, TMM.

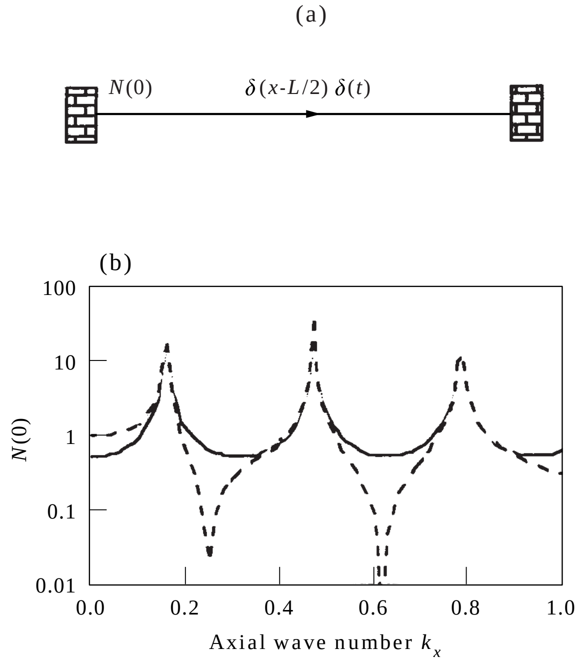


Figure 3. (a) Longitudinal vibration of the pipe. (b) Comparison of the IRF and the Model Methods (MM): ——, IRF; – – –, Modal.

### 5.2. LONGITUDINAL VIBRATION OF A PIPE

Consider a pipe of length $L$ and uniform cross-section $A_t$, as shown in Figure 3(a). The pipe is stretched by a Dirac longitudinal force applied at $x = L/2$. Both ends of the pipe are fixed. Figure 3(b) gives the normal stress resultant $N(0)$, at $x = 0$, as a function of longitudinal wave number $k_x = \omega\sqrt{\rho_t/E}$.

The modal solution for $N(0)$ is:

$$N(0) = \sum_{p=0}^{\infty} \frac{2(2p+1)\pi}{L^2\left[\left(\frac{(2p+1)\pi}{L}\right)^2 - k_x^2\right]}.$$

The IRF Solution is:

$$N(0) = \frac{\tilde{G}_x^0\left(\frac{L}{2}\right)}{\tilde{G}_x^0(L) + \tilde{G}_x^0(0)}.$$

The comparison of the IRF and the modal method (MM) in Figure 3(b) shows a very good agreement between the two methods.

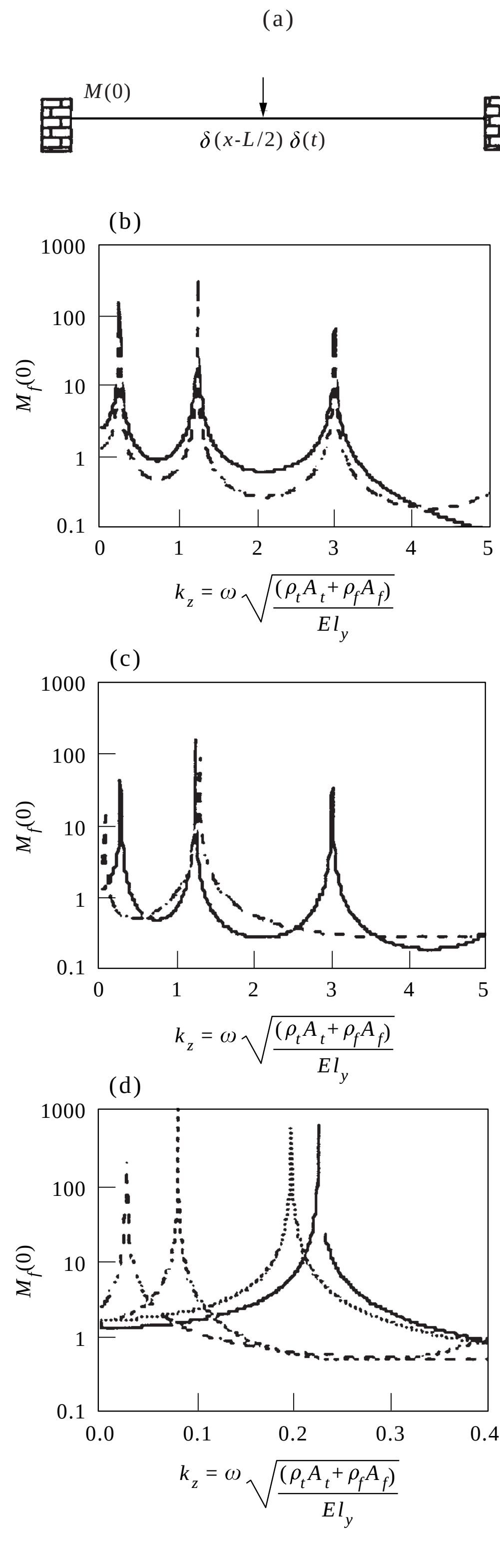


Figure 4. (a) Transverse vibration of a pipe. (b) Comparison of the IRF and the model methods; $\beta = 0$: ——, Modal; –––, IRF. (c) Effects of permanent velocity on the pipe response: $\beta = 0{\cdot}00$; ——, IRF; $\beta = 0{\cdot}20$; –––, IRF. (d) Effects of permanent velocity on the pipe response: ——, $\beta = 0{\cdot}00$; · · ·, $\beta = 0{\cdot}10$; -–-, $\beta = 0{\cdot}15$; –––, $\beta = 0{\cdot}20$.

### 5.3. TRANSVERSE VIBRATION OF A PIPE INCLUDING MEAN FLOW EFFECTS

Let a pipe of length $L$ and uniform cross-section $A_t$ be as shown in Figure 4(a). The pipe is stretched by a Dirac transverse force applied at $x = L/2$. The flow in the pipe is characterised by a permanent flow parameter $\beta$. Both ends of the pipe are fixed.

Using the symmetry of the structure:

$$EI_y \frac{\mathrm{d}^2 U_z(0)}{\mathrm{d}x^2} = EI_y \frac{\mathrm{d}^2 U_z(L)}{\mathrm{d}x^2}; \qquad EI_y \frac{\mathrm{d}^3 U_z(0)}{\mathrm{d}x^3} = -EI_y \frac{\mathrm{d}^3 U_z(L)}{\mathrm{d}x^3}$$

and the boundary conditions:

$$U_z(0) = U_z(L) = 0; \qquad \frac{\mathrm{d}U_z(0)}{\mathrm{d}x} = \frac{\mathrm{d}U_z(L)}{\mathrm{d}x}$$

the moment $M_f(0)$ can be determined.

Figure 4(b) gives the flexural moment $M_f(0)$, at $x = 0$, as a function of the "transverse wave number" $k_z = \omega\sqrt{(\rho_t A_t + \rho_f A_f)/EI_y}$.

The comparison of the IRF and the MM in Figure 4(b) for $\beta = 0$ shows a very good agreement between the two methods. Figures 4(c) and (d) show the effect of the mean flow parameter $\beta$ on the natural frequencies and on the response of the pipe. When the parameter $\beta$ increases the natural frequencies decrease.

## 6. CONCLUSIONS

The set of equations governing the vibration of the fluid–pipe system was formulated. An integral formulation was derived and the fundamental solutions were determined for: an acoustic medium in a pipe, including mean flow effects; a longitudinal vibration of a bar; a transverse vibration of a pipe, including mean flow effects; a longitudinal fluid–structure interaction in pipes with a severe restriction.

The numerical implementation is classical. It uses a multi-pipe approach. However, with a proper numerical calculation, very precise solutions can be obtained, as demonstrated in the numerical examples of section 5.

The use of the IRF to solve fluid–pipe problems is characterized by high accuracy and efficiency and can easily take into account interaction phenomena.

This can be used as the appropriate tool for studying the dynamic behavior of mechanical systems for one-dimensional problems.

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## APPENDIX A: FUNDAMENTAL SOLUTIONS

### A.1. ACOUSTIC PROBLEM

For the acoustic problem, the fundamental function $\tilde{G}_p$ is the solution of the following equation:

$$(1 - M^2)\frac{\partial^2 \tilde{G}_p}{\partial x^2} + k_p^2 \tilde{G}_p - 2\mathrm{i}\omega \frac{\bar{V}}{c^2}\frac{\partial \tilde{G}_p}{\partial x} = -\delta(x - x_0).$$

The function $\tilde{G}_p$ can be written as:

$$\tilde{G}_p + A\,\mathrm{e}^{\lambda_{p1}(x - x_0)} + B\,\mathrm{e}^{\lambda_{p2}(x - x_0)}.$$

$\lambda_1$ and $\lambda_2$ are solutions of:

$$(1 - M^2)\lambda^2 + k_p^2 - 2\mathrm{i}\omega \frac{\bar{V}}{c^2}\lambda = 0.$$

One finds:

$$\lambda_{p1} = -\mathrm{i}\frac{k_p}{1 + M}; \qquad \lambda_{p2} = \mathrm{i}\frac{k_p}{1 - M}.$$

Using the continuity of $\tilde{G}_p$ and the discontinuity of $(1 - M^2)\,\mathrm{d}\tilde{G}_p/\mathrm{d}x$ at $x = x_0$, the fundamental function $\tilde{G}_p$ is written as:

$$\tilde{G}_p^{x_0}(x) = \frac{\mathrm{e}^{\lambda_{p1}(x - x_0)}}{2k_p} \qquad \text{for } (x - x_0) \geqslant 0$$

$$\tilde{G}_p^{x_0}(x) = \frac{\mathrm{e}^{\lambda_{p2}(x - x_0)}}{2k_p} \qquad \text{for } (x - x_0) \leqslant 0$$

### A.2. LONGITUDINAL MOTION OF THE PIPE

For equation (1a) and for zero localized forces, the characteristic polynomial is written:

$$EA_t\lambda^2 + \rho A_t\omega^2 = 0.$$

One finds:

$$\pm\lambda_x = \pm \mathrm{i}\omega\sqrt{\frac{\rho}{E}}.$$

The fundamental function $\tilde{G}_x$ is the solution of the following equation:

$$EA_t \frac{\partial^2 \tilde{G}_x}{\partial x^2} + \rho_f A_t \omega^2 \tilde{G}_x = -\delta(x - x_0).$$

The fundamental solution $\tilde{G}_x$ is written:

$$\begin{cases} \tilde{G}_x = A_1\,\mathrm{e}^{\lambda_x(x - x_0)} & \text{for } x - x_0 \leqslant 0 \\ \tilde{G}_x = A_2\,\mathrm{e}^{-\lambda_x(x - x_0)} & \text{for } x - x_0 \geqslant 0 \end{cases}.$$

Using the continuity of $\tilde{G}_x$ and the discontinuity of $EA_t\,(\partial\tilde{G}_x/\partial x)$ at $x = x_0$, the function $\tilde{G}_x$ is written as:

$$\tilde{G}_x = \frac{\mathrm{e}^{\lambda_x|x - x_0|}}{2EA_t\lambda_x}.$$

A.3. TRANSVERSE MOTION OF THE PIPE

For equation (1b), for zero force $f_z$ and by neglecting the damping term, the characteristic polynomial is written:

$$\lambda^4 + \beta^2\lambda^2 - \lambda_0^4 = 0$$

where:

$$\beta^2 = \frac{A_f(\Delta P + \rho_f \bar{V}^2)}{EI_y}; \qquad \lambda_0^4 = \frac{\omega^2(\rho_t A_t + \rho_f A_f)}{EI_y}.$$

One finds:

$$\lambda_{z1} = \mathrm{i}\,\frac{\sqrt{2}}{2}\sqrt{2\sqrt{\beta^4 + \lambda_0^4} + \beta^2}; \qquad \lambda_{z2} = -\lambda_{z1},$$

$$\lambda_{z3} = \frac{\sqrt{2}}{2}\sqrt{2\sqrt{\beta^4 + \lambda_0^4} - \beta^2}; \qquad \lambda_{z4} = -\lambda_{z3}.$$

The fundamental function $\tilde{G}_z$ is the solution of the following equation:

$$\frac{\partial^4 \tilde{G}_z}{\partial x^4} + \beta^2 \frac{\partial^2 \tilde{G}_z}{\partial x^2} + 2\mathrm{i}\omega\gamma^2 \frac{\partial \tilde{G}_z}{\partial x} - \lambda_0^4 \tilde{G}_z = \frac{1}{EI_y}\,\delta(x - x_0).$$

The transverse fundamental solution $\tilde{G}_z$ is:

$$\begin{cases} \tilde{G}_z = A_1\,\mathrm{e}^{\lambda_{z1}(x - x_0)} + A_3\,\mathrm{e}^{\lambda_{z3}(x - x_0)} & \text{for } x - x_0 \leqslant 0 \\ \tilde{G}_z = A_2\,\mathrm{e}^{\lambda_{z2}(x - x_0)} + A_4\,\mathrm{e}^{\lambda_{z4}(x - x_0)} & \text{for } x - x_0 \geqslant 0 \end{cases}$$

Using the continuity of $(\tilde{G}_z, (\partial\tilde{G}_z/\partial x)(\partial^2\tilde{G}_z/\partial x^2))$ and the discontinuity of $\partial^3\tilde{G}_z/\partial x^3$ at $x = x_0$, the function $\tilde{G}_z$ is written as:

$$\tilde{G}_z = \frac{1}{2EI_y}\left[\frac{e^{\lambda_{z1}(x-x_0)}}{\lambda_{z1}(\lambda_1^2-\lambda_{z3}^2)} - \frac{e^{\lambda_{z3}(x-x_0)}}{\lambda_{z3}(\lambda_{z1}^2-\lambda_{z3}^2)}\right] \qquad \text{for } x - x_0 \leqslant 0,$$

$$\tilde{G}_z = \frac{1}{2EI_y}\left[\frac{e^{-\lambda_{z1}(x-x_0)}}{\lambda_{z1}(\lambda_{z1}^2-\lambda_{z3}^2)} - \frac{e^{-\lambda_{z3}(x-x_0)}}{\lambda_{z3}(\lambda_{z1}^2-\lambda_{z3}^2)}\right] \qquad \text{for } x - x_0 \geqslant 0.$$

## APPENDIX B

$$[\mathbf{M}_1] = \begin{bmatrix} \left(\dfrac{\partial^3\tilde{G}_z^0}{\partial x^3}(0) - \dfrac{1}{EI_y};\ -\dfrac{\partial^2\tilde{G}_z^0}{\partial x^2}(0);\ \dfrac{\partial\tilde{G}_z^2}{\partial x}(0);\ -\tilde{G}_z^0(0);\right. \\ \left. -\dfrac{\partial^3\tilde{G}_z^0}{\partial x^3}(L);\ \dfrac{\partial^2\tilde{G}_z^0}{\partial x^2}(L);\ -\dfrac{\partial\tilde{G}_z^0}{\partial x}(L);\ \tilde{G}_z^0(L)\right) \\ \left(\dfrac{\partial^3\tilde{G}_z^L}{\partial x^3}(0);\ -\dfrac{\partial^2\tilde{G}_z^L}{\partial x^2}(0);\ \dfrac{\partial\tilde{G}_z^L}{\partial x}(0);\ -\tilde{G}_z^L(0);\ -\dfrac{\partial^3\tilde{G}_z^L}{\partial x^3}(L) - \dfrac{1}{EI_y};\right. \\ \left. \dfrac{\partial^2\tilde{G}_z^L}{\partial x^2}(L);\ -\dfrac{\partial\tilde{G}_z^L}{\partial x}(L);\ \tilde{G}_z^L(L)\right) \\ \left(-\dfrac{\partial^4\tilde{G}_z^0}{\partial x^4}(0);\ \dfrac{\partial^3\tilde{G}_z^0}{\partial x^3}(0) - \dfrac{1}{EI_y};\ -\dfrac{\partial^2\tilde{G}_z^0}{\partial x^2}(0);\ \dfrac{\partial\tilde{G}_z^0}{\partial x}(0);\ \dfrac{\partial^4\tilde{G}_z^0}{\partial x^4}(L);\right. \\ \left. -\dfrac{\partial^3\tilde{G}_z^0}{\partial x^3}(L);\ \dfrac{\partial^2\tilde{G}_z^0}{\partial x^2}(L);\ -\dfrac{\partial\tilde{G}_z^0}{\partial x}(L)\right) \\ \left(-\dfrac{\partial^4\tilde{G}_z^L}{\partial x^4}(0);\ \dfrac{\partial^3\tilde{G}_z^L}{\partial x^3}(0);\ -\dfrac{\partial^2\tilde{G}_z^L}{\partial x^2}(0);\ \dfrac{\partial\tilde{G}_z^L}{\partial x}(0);\ \dfrac{\partial^4 G_z^L}{\partial x^4}(L);\right. \\ \left. -\dfrac{\partial^3\tilde{G}_z^L}{\partial x^3}(L) - \dfrac{1}{EI_y};\ \dfrac{\partial^2\tilde{G}_z^L}{\partial x^2}(L);\ -\dfrac{\partial\tilde{G}_z^L}{\partial x}(L)\right) \end{bmatrix},$$

$$[\mathbf{M}_2] = \begin{bmatrix} \left(\dfrac{\partial \tilde{G}_z^0}{\partial x}(0) - 1;\ -\tilde{G}_z^0(0);\ 0;\ 0;\ -\dfrac{\partial \tilde{G}_z^0}{\partial x}(L);\ \tilde{G}_z^0(L);\ 0;\ 0;\right) \\ \left(\dfrac{\partial \tilde{G}_z^L}{\partial x}(0);\ -\tilde{G}_z^L(0);\ 0;\ 0;\ -\dfrac{\partial \tilde{G}_z^L}{\partial x}(L) - 1;\ \tilde{G}_z^L(L);\ 0;\ 0;\right) \\ \left(-\dfrac{\partial^2 \tilde{G}_z^0}{\partial x^2}(0);\ \dfrac{\partial \tilde{G}_z^0}{\partial x}(0) - 1;\ 0;\ 0;\ \dfrac{\partial^2 \tilde{G}_z^0}{\partial x^2}(L);\ -\dfrac{\partial \tilde{G}_z^0}{\partial x}(L);\ 0;\ 0;\right) \\ \left(-\dfrac{\partial^2 \tilde{G}_z^L}{\partial x^2}(0);\ \dfrac{\partial \tilde{G}_z^L}{\partial x}(0);\ 0;\ 0;\ \dfrac{\partial^2 \tilde{G}_z^L}{\partial x^2}(L);\ -\dfrac{\partial \tilde{G}_z^L}{\partial x}(L) - 1;\ 0;\ 0;\right) \end{bmatrix},$$

$$[\mathbf{q}]^T = \left[U_z(0);\ \frac{\partial U_z}{\partial x}(0);\ \frac{\partial^2 U_z}{\partial x^2}(0);\ \frac{\partial^3 U_z}{\partial x^3}(0);\ U_z(L);\ \frac{\partial U_z}{\partial x}(L);\ \frac{\partial^2 U_z}{\partial x^2}(L);\ \frac{\partial^3 U_z}{\partial x^3}(L)\right],$$

$$[\mathbf{F}]^T = \left[\frac{\tilde{F}(\omega)\tilde{G}_z^0(x_2)}{EI_y};\ \frac{\tilde{F}(\omega)\tilde{G}_z^L(x_2)}{EI_y};\ -\frac{\tilde{F}(\omega)}{EI_y}\frac{\partial \tilde{G}_z^0(x_2)}{\partial x};\ -\frac{\tilde{F}(\omega)}{EI_y}\frac{\partial \tilde{G}_z^L(x_2)}{\partial x}\right].$$

## APPENDIX C: NOMENCLATURE

| | |
|---|---|
| $A_f$ | cross-sectional discharge area |
| $c$ | speed of sound in the fluid |
| $\tilde{G}_p$ | fundamental solution of the adjoint convective Helmholtz operator |
| $\tilde{G}_z$ | fundamental solution of the transverse motion operator |
| $k_p = \omega/c$ | acoustic wave number |
| $k_z$ | transverse wave number |
| $M_f$ | flexural moment |
| $N$ | normal stress resultant |
| $U_x(\omega)$ | the Fourier transform of the pipe's longitudinal displacement |
| $\bar{V}$ | mean velocity of fluid |
| $x_1$ | position of the localised force $F_x$ |
| $x_3$ | position of the restriction of the cross-sectional discharge area |
| $\Delta P$ | pressure drop in the pipe |
| $\rho_f$ | fluid density |
| $\Delta A_f$ | local restriction of the cross-sectional discharge area |
| $A_t$ | cross-sectional pipe wall area |
| $E$ | Young's modulus |
| $\tilde{G}_x$ | fundamental solution of the longitudinal motion operator |
| $I_y$ | the $y$-moment of inertia |
| $k_x$ | longitudinal wave number |
| $L$ | length of the pipe |
| $M = \bar{V}/c$ | Mach number |
| $p(x_0)$ | fluctuated pressure at $x_0$ |
| $U_z(\omega)$ | the Fourier transform of the pipe's transverse displacement |

| | |
|---|---|
| $x_0$ | a point on $[0, L]$ |
| $x_2$ | position of the localised force $F_z$ |
| $(x, y, z)$ | the longitudinal and the two lateral co-ordinates |
| $\omega$ | frequency |
| $\rho_t$ | pipe wall density |