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DYNAMIC ANALYSIS OF SHALLOW SHELLS OF RECTANGULAR BASE

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In this investigation a systematic analytic procedure for the dynamic analysis and response of thin shallow shells with a rectangular layout is presented. The shell types examined are the elliptic and hyperbolic paraboloid, the hypar, the conoidal parabolic and the soap-bubble shell, although in principle any shell geometry expressed by a continuous surface equation can be treated. The eigenvalue problem solution is based on the one hand on the consideration of the shell as a system of two interdependant plates whose boundary conditions comply with the prevailing bending and membrane boundary conditions of the shell, and on the other hand on the consistent use of beam eigenfunctions, in the context of a Galerkin solution procedure. The series solution obtained in this way converges rapidly and provides practically acceptable results even in cases with one or more free edges, where the boundary conditions cannot be strictly satisfied. The whole analysis is carried out on the basis of a few non-dimensionalized geometrical parameters, which are the only input required for the computer program specially written for that purpose.

1. INTRODUCTION

It is a recognized fact that the development of analytical methods for the evaluation of the dynamic behaviour of shallow shells for civil engineering applications, as for example shell roofs, has fallen well behind in comparison to their static analysis. The static analysis of shallow shells was originally initiated by Marguerre [1] and then taken over mainly by Vlasov [2]. In reference [3] one can find a brief layout of the main developments in this direction until the advent of the finite element era. Vlasov [2] has condensed the two equations of equilibrium of Marguerre, established in terms of the deflection function and a stress potential, by introducing a mixed potential which enabled him to present a single differential equation of eighth degree. On the basis of that equation he obtained also an exact solution for the eigenfrequencies for simply supported shells of rectangular base in the form of an elliptic, as well as of a hyperbolic paraboloid.

For the next two decades the analytical research on the dynamic problem of shells was mainly confined to close structural forms for the needs of the aerospace industry. It is well understandable that the development of the discretization

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methods together with the increased computer facilities has directed the main bulk of the engineering research towards a refinement of those methods either in discretizing the proper structural domain (finite elements, finite strips) or more recently in discretizing the boundary itself (boundary elements).

However, for the direct insight into the physical behaviour of a structure the analytical treatment has always been the most appropriate one. It was Leissa [4] who first provided an invaluable report on analytical treatment of the dynamic behaviour of shells. Moreover Leissa in reference [5] re-examined the dynamic problem of doubly curved shallow shells with boundary conditions other than simply supported. This is accomplished by using the Ritz method through minimization of the maximum total energy and for the first time a practical solution for cantilevered shallow shells of rectangular planform is given, with obtained frequencies very close to the exact values. This method is also applied by the same author in reference [6] to the investigation of the above type of shells considered in completely free form.

Undoubtedly the most powerful formulation for the analytical investigation of the dynamic behaviour of shallow shells has been the introduction by Lim and Liew of the so called pb-2 method [7], which was initially introduced very successfully by Liew, for the dynamic analysis of plates. This very efficient and highly accurate method is carried through a Ritz procedure, as it was done by Leissa, but with the special feature of the use of the so-called pb-2 shape function. This function consists of the product of (1) a complete set of two-dimensional orthogonal polynomial functions and (2) a basic function formed from the product of the equations of the boundaries, each raised to an appropriate power.

It is exactly these characteristics which enable the application of the pb-2 method to a broad range of boundary conditions on the one hand, and to various geometric configurations of the planform and of the shell itself on the other, as demonstrated by Liew and Lim in a sequence of papers [8–11]. However the majority of the listed papers deal with doubly curved shallow shells of rectangular planform.

In this investigation a systematic procedure is presented for the evaluation of the eigenvalues and eigenforms, as well as of the dynamic response of shallow shells over a rectangular base with various boundary conditions. The shells may have theoretically any geometry, but for practical purposes of civil engineering interest the elliptical or hyperbolic paraboloid, hypar, conoidal and soap-bubble forms are investigated. The analysis is based on the analytical treatment of the Marguerre equations with the inertia term added, which leads to the concept of two interconnected plate equations, each with its appropriate boundary conditions. The two plates are treated simultaneously through a Galerkin procedure on the basis of complete sets of beam eigenfunctions satisfying accurately, wherever possible, the corresponding boundary conditions. The procedure enables the evaluation of the dynamic characteristics of a shell on the basis of only a few non-dimensionalized parameters concerning the geometry of the structure, requiring at most 50 unknown coefficients, instead of the significantly more required by the pb-2 method. The results of the proposed method for some boundary conditions involving free edges can have an approximate character, but nevertheless they satisfy in a proper manner the requirements of preliminary design purposes.

2. GOVERNING EQUATIONS

A thin shallow shell of constant thickness h made of a homogeneous, isotropic, linearly elastic material is considered. The projection of the shell on the xy plane is a rectangle with dimensions a, b. The equation of the middle surface of the shell, referred to a system of orthogonal axes (x, y, z) may be expressed as

$$z = z(x, y). \tag{1}$$

The shell is subjected to distributed transverse forces $p(x, y, t)$ varying with time. It is assumed that its resulting deformation is within the limits of validity of the theory of small deformation.

A shell is characterized as shallow if any infinitesimal line element of its middle surface may be approximated by the length of its projection on the xy plane. This implies that

$$\left(\frac{\partial z}{\partial x}\right)^2 \ll 1 \qquad \left(\frac{\partial z}{\partial y}\right)^2 \ll 1 \qquad \left(\frac{\partial z}{\partial z}\right)\left(\frac{\partial z}{\partial y}\right) \ll 1. \tag{2}$$

Moreover, the lateral boundary of a shallow shell may be approximated by its projection on the xy plane in regard with its boundary conditions.

According to Vlasov [2] the above conditions are practically satisfied for shells with a rise-to-span ratio less than 1/5.

The equations of the Marguerre theory of thin shallow shells [1] after the addition of the inertia term, may be expressed as

$$\nabla^4 w(x, y, t) = \frac{12(1-\nu^2)}{Eh^3}\left[L(\phi) - \rho h \frac{\partial^2 w}{\partial t^2} + p(x, y, t)\right], \tag{3}$$

$$\nabla^4 \phi(x, y, t) = -EhL(w), \tag{4}$$

where w is the transverse component of displacement (deflection) of the middle surface of the shell, ϕ is the stress function, ν is the Poisson's ratio, ρ is the density and h is the thickness of the shell. Moreover, the operators ∇^4 and L are defined as

$$\nabla^4(\,) \equiv \frac{\partial^4(\,)}{\partial x^4} + 2\frac{\partial^4(\,)}{\partial x^2 \partial y^2} + \frac{\partial^4(\,)}{\partial y^4}, \tag{5}$$

$$L(\,) \equiv \frac{\partial^2 z}{\partial y^2}\frac{\partial^2(\,)}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2(\,)}{\partial x \partial y} + \frac{\partial^2 z}{\partial x^2}\frac{\partial^2(\,)}{\partial y^2}. \tag{6}$$

The stress resultants per unit length of shell section, may be obtained from the stress function ϕ and the deflection w on the basis of the following relations:

Membrane stress resultants

$$N_x = \frac{\partial^2 \phi}{\partial y^2} \qquad N_y = \frac{\partial^2 \phi}{\partial x^2} \qquad N_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}. \tag{7}$$

Bending stress resultants

$$M_x = \frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = \frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = \frac{Eh^3}{12(1+v)}\frac{\partial^2 w}{\partial x\, \partial y}, \tag{8}$$

$$Q_x = \frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\, \partial y^2}\right) \qquad Q_y = \frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2\, \partial y}\right). \tag{9}$$

In the above expressions N_x, N_y and N_{xy} are the normal force in the x and y directions and the tangential shear force, respectively, whereas M_x, M_y and M_{xy} are the bending moments about the y and x axes and the twisting moment, respectively. Q_x and Q_y are the respective shearing forces of the shell.

3. BOUNDARY CONDITIONS

The boundary conditions which have to be satisfied from the resulting stress resultants and the components of displacement are the following [3]:

Membrane boundary conditions

$$\text{(a)} \quad \text{either} \quad N_s = -\frac{\partial^2 \phi}{\partial s\, \partial v} = 0 \quad \text{or} \quad u_s = 0 \tag{10a}$$

and

$$\text{(b)} \quad \text{either} \quad N_v = \frac{\partial^2 \phi}{\partial s^2} = 0 \quad \text{or} \quad u_v = 0. \tag{10b}$$

Bending boundary conditions

$$\text{(a)} \quad \text{either} \quad M_s = \frac{Eh^3}{12(1-v^2)}\left(\frac{\partial^2 w}{\partial v^2} + v\frac{\partial^2 w}{\partial s^2}\right) = 0 \quad \text{or} \quad \frac{\partial w}{\partial v} = 0 \tag{11a}$$

and

$$\text{(b)} \quad \text{either} \quad Q^{eff} = Q + \frac{\partial M_v}{\partial s} = \frac{Eh^3}{12(1-v^2)}\left[\frac{\partial^3 w}{\partial v^3} + (2-v)\frac{\partial^3 w}{\partial v\, \partial s^2}\right] = 0 \quad \text{or} \quad w = 0. \tag{11b}$$

In the above relations u represents the horizontal components of displacement of a point of the middle surface and the co-ordinates s and v as applied to stress resultants or components of displacements, are tangential and normal to the boundary of the shell, respectively. The boundary conditions of practical interest are:

Membrane boundary condition

$$(S_M):\quad u_s = 0,\qquad N_v = 0,$$

$$(C_M):\quad u_s = 0,\qquad u_v = 0,$$

$$(F_M):\quad N_s = 0,\qquad N_v = 0. \tag{12}$$

Bending boundary conditions

$$(S_B):\quad w = 0,\qquad M_s = 0$$

$$(C_B):\quad w = 0,\qquad \frac{\partial w}{\partial v} = 0$$

$$(F_B):\quad M_s = 0,\qquad Q^{eff} = Q + \frac{\partial M_v}{\partial s} = 0. \tag{13}$$

The above basic boundary conditions can be satisfied by the following boundary conditions applied on w and ϕ ([3]).
Membrane boundary conditions (*conditions on* ϕ)

$$(S_M):\quad \phi = 0,\qquad \frac{\partial^2\phi}{\partial v^2} = 0, \tag{14}$$

$$(C_M):\quad \frac{\varphi^2\phi}{\partial v^2} = 0,\qquad \frac{\partial^3\phi}{\partial v^3} = 0. \tag{15}$$

The above conditions satisfy the membrane type of support (C_M) only on boundaries for which the shell curvature normal to their direction is zero and also by assuming the Poisson's ratio equal to zero.

$$(F_M):\quad \phi = 0,\qquad \frac{\partial\phi}{\partial v} = 0. \tag{16}$$

Bending boundary conditions (*conditions on* w)

$$(S_B):\quad w = 0,\qquad \frac{\partial^2 w}{\partial v^2} = 0, \tag{17}$$

$$(C_B):\quad w = 0,\qquad \frac{\partial w}{\partial v} = 0, \tag{18}$$

$$(F_B):\quad \frac{\partial^2 w}{\partial v^2} = 0,\qquad \frac{\partial^3 w}{\partial v^3} = 0. \tag{19}$$

This case can be satisfied by the above conditions only approximately. It should not be applied on two adjacent edges. The Poisson's ratio is assumed equal to zero.

However, the present investigation is limited to shells each of whose edges is supported in one of the following ways: simply supported: (S_B, S_M) or (S_B, C_M) or (S_B, F_M); fixed against rotation: (C_B, S_M) or (C_B, C_M) or (C_B, F_M); free: (F_B, F_M).

4. THE EIGENVALUE PROBLEM

It is assumed that $p(x, y, t) = 0$ and further, through separation of the time variable, that:

$$w(x, y, t) = W_i(x, y) * T(t), \qquad \phi(x, y, t) = \Phi_i(x, y) * T(t) \qquad (20, 21)$$

Substituting in equations (3) and (4) the following equations can be obtained

$$\frac{\frac{Eh^3}{12(1-\nu^2)} \nabla^4 W_i - L(\Phi_i)}{\rho h W_i} = \frac{\mathrm{d}^2 T/\mathrm{d}t^2}{T}, \qquad (22)$$

$$\nabla^4 \Phi_i + EhL(W_i) = 0. \qquad (23)$$

From equation (22) it can be deduced that both its members must be equal to a constant $-\omega_i^2$. Then the following equations are obtained:

$$\nabla^4 W_i(x, y) = q_w(x, y), \qquad \nabla^4 \Phi_i(x, y) = q_\phi(x, y), \qquad (24, 25)$$

where

$$q_w(x, y) = \frac{12(1-\nu^2)}{Eh^3}[L(\Phi_i) + \rho h \omega_i^2 W_i(x, y)] \qquad (26)$$

and

$$q_\phi(x, y) = -EhL(W_i). \qquad (27)$$

ω_i represents the ith eigenfrequency of the shell corresponding to the respective eigenform $W_i(x, y)$.

Equations (24) and (25) represent the bending of two plates with the same dimensions as the shell base, each with its own boundary conditions, namely those examined in the previous paragraph, which are called bending and membrane plates, respectively.

It is seen that each solution ω_i^2, $W_i(x, y)$, $\phi_i(x, y)$ of the eigenvalue problem has to satisfy the following two conditions: (a) the deflections of the bending plate under the loading $q_w(x, y)$ have to be identical with the eigenfunction $W_i(x, y)$; (b) the deflections of the membrane plate under the loading $q_\phi(x, y)$ have to be identical with the eigenfunction $\Phi_i(x, y)$.

5. METHOD OF SOLUTION

Using the principle of the Galerkin method of procedure, an approximate solution $W_i^{mn}(\xi, \eta)$ and $\Phi_i^{mn}(\xi, \eta)$ of equations (24) and (25) of the following form is postulated.

$$W_i^{mn}(\xi, \eta) = h \sum_{r=1}^{m} \sum_{s=1}^{n} C_{rs}^{W(i)} F_r(\xi) G_s(\eta), \qquad (28)$$

$$\Phi_i^{mn}(\xi, \eta) = Eh^3 \sum_{r=1}^{m} \sum_{s=1}^{n} C_{rs}^{F(i)} f_r(\xi) g_s(\eta), \tag{29}$$

where the non-dimensional co-ordinates ξ and η are defined as

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{b}. \tag{30}$$

In expressions (28) and (29), $C_{rs}^{W(i)}$ and $C_{rs}^{F(i)}$ $(r = 1, 2, 3, \ldots, m)$, $(s = 1, 2, 3, \ldots, n)$ are unknown coefficients. The functions $F_r(\xi)$, $G_s(\eta)$ and $f_r(\xi)$, $g_s(\eta)$ are beam eigenfunctions. Thus, if the beam functions F_r and G_s satisfy the bending boundary conditions for the "bending plate" and the beam functions f_r and g_s satisfy the membrane boundary conditions for the "membrane plate", then the approximate solution (28) and (29) converges to the exact solution as the respective number m and n of terms retained in the series expansions increases. The beam functions are defined as follows:

$$F_r(\xi) = k_{\xi r} \sinh(\alpha_r \xi) + l_{\xi r} \cosh(\alpha_r \xi) + m_{\xi r} \sin(\alpha_r \xi) + n_{\xi r} \cos(\alpha_r \xi),$$

$$G_s(\eta) = k_{\eta s} \sinh(\beta_s \eta) + l_{\eta s} \cosh(\beta_s \eta) + m_{\eta s} \sin(\beta_s \eta) + n_{\eta s} \cos(\beta_s \eta) \tag{31}$$

and

$$f_r(\xi) = \bar{k}_{\xi r} \sinh(\alpha_r \xi) + \bar{l}_{\xi r} \cosh(\alpha_r \xi) + \bar{m}_{\xi r} \sin(\alpha_r \xi) + \bar{n}_{\xi r} \cos(\alpha_r \xi),$$

$$g_s(\eta) = \bar{k}_{\eta s} \sinh(\beta_s \eta) + \bar{l}_{\eta s} \cosh(\beta_s \eta) + \bar{m}_{\eta s} \sin(\beta_s \eta) + \bar{n}_{\eta s} \cos(\beta_s \eta). \tag{32}$$

For any given set of boundary conditions at two opposite edges of a shell, the corresponding parameters in the above expressions can be established from Tables 1 and 2.

When the Galerkin method is applied to equations (24) and (25), the following $2mn$ equations are obtained

$$\int_0^1 \int_0^1 [\bar{\nabla}^4 W_i^{mn} - \bar{q}_w(\xi, \eta)] F_p(\xi) G_q(\eta) \, d\xi \, d\eta = 0, \tag{33}$$

$$\int_0^1 \int_0^1 [\bar{\nabla}^4 \Phi_i^{mn} - \bar{q}_\phi(\xi, \eta)] f_p(\xi) g_q(\eta) \, d\xi \, d\eta = 0, \tag{34}$$

with

$$\bar{q}_w(\xi, \eta) = \frac{12(1 - \nu^2)}{Eh^3} [\bar{L}(\Phi_i) + \rho h \omega_i^2 W_i] \tag{35}$$

and

$$\bar{q}_\phi(\xi, \eta) = -Eh\bar{L}(W_i), \tag{36}$$

where the modified operators $\bar{\nabla}^4$ and $\bar{L}$ are defined as:

$$\bar{\nabla}^4(\ ) \equiv \frac{1}{a^4}\frac{\partial^4(\ )}{\partial\xi^4} + \frac{2}{a^2b^2}\frac{\partial^4(\ )}{\partial\xi^2\,\partial\eta^2} + \frac{1}{b^4}\frac{\partial^4(\ )}{\partial\eta^4} \tag{37}$$

$$\bar{L}(\ ) \equiv \frac{1}{a^2b^2}\left[\frac{\partial^2 z}{\partial\eta^2}\frac{\partial^2(\ )}{\partial\xi^2} - 2\frac{\partial^2 z}{\partial\xi\,\partial\eta}\frac{\partial^2(\ )}{\partial\xi\,\partial\eta} + \frac{\partial^2 z}{\partial\xi^2}\frac{\partial^2(\ )}{\partial\eta^2}\right] \tag{38}$$

Substitution of the expressions (28) and (29) into the first part of equations (33) and (34) respectively, yields:

$$h\sum_{r=1}^{m}\sum_{s=1}^{n} C^w_{rs}\left\langle\left[\alpha_r^4\left(\frac{b}{a}\right)^2 + \beta_s^4\left(\frac{a}{b}\right)^2\right]\delta^F_{rp}\delta^G_{sq} + 2F(r,p)G(s,q)\right\rangle$$

$$= \frac{12(1-\nu^2)}{Eh^3}a^2b^2\int_0^1\int_0^1 \bar{L}(\Phi_i)F_p(\xi)G_q(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{12(1-\nu^2)}{Eh^3}a^2b^2$$

$$\times\int_0^1\int_0^1 \rho h\omega_i^2 W_i(\xi,\eta)F_p(\xi)G_q(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{39}$$

$$Eh^3\sum_{r=1}^{m}\sum_{s=1}^{n} C^F_{rs}\left\langle\left[\bar{\alpha}_r^4\left(\frac{b}{a}\right)^2 + \bar{\beta}_s^4\left(\frac{a}{b}\right)^2\right]\delta^f_{rp}\delta^g_{sq} + 2f(r,p)g(s,q)\right\rangle$$

$$= -Eh^3a^2b^2\int_0^1\int_0^1 \bar{L}(W_i)f_p(\xi)g_q(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{40}$$

where

$$\delta^F_{rp} = \int_0^1 F_r(\xi)F_p(\xi)\,\mathrm{d}\xi \qquad \delta^G_{sq} = \int_0^1 G_s(\eta)G_q(\eta)\,\mathrm{d}\eta, \tag{41}$$

and

$$F(r,p) = \int_0^1 F_r''(\xi)F_p(\xi)\,\mathrm{d}\xi \qquad G(s,q) = \int_0^1 G_s''(\eta)G_q(\eta)\,\mathrm{d}\eta. \tag{42}$$

The expressions δ^f_{rp}, δ^g_{sq} and $f(r,p)$, $g(s,q)$ in equation (40) are defined from the expressions (41) and (42) by interchanging F and G with f and g, respectively.

The right hand side of equations (39) and (40) depends on the geometry of the shell, on account of the operator $\bar{L}$.

TABLE 1

Beam eigenfunctions for bending boundary conditions

Boundary conditions of opposite edges	$F_n(\zeta)$ or $G_n(\zeta)$ $n = 1, 2, 3, \ldots$	α_n	A_n
$[S_B\text{–}S_B]$	$\sin(n\pi\zeta)$	–	–
$[C_B\text{–}C_B]$	$A_n(\sinh(\alpha_n\zeta) - \sin(\alpha_n\zeta)]$ $+\cosh(\alpha_n\zeta) - \cos(\alpha_n\zeta)$	$\cosh(\alpha_n)\cos(\alpha_n) = 1$	$\dfrac{\cos(\alpha_n) - \cosh(\alpha_n)}{\sinh(\alpha_n) - \sin(\alpha_n)}$
$[S_B\text{–}C_B]$	$A_n \sin(\alpha_n\zeta) + \sinh(\alpha_n\zeta)$	$\tanh(\alpha_n) - \tan(\alpha_n) = 0$	$-\dfrac{\sinh(\alpha_n)}{\sin(\alpha_n)}$
$[F_B\text{–}F_B]$	$F_1(\zeta) = G_1(\zeta) = 2$. For $n > 1$: $A_n[\sinh(\alpha_n\zeta) + \sin(\alpha_n\zeta)]$ $+ \cosh(\alpha_n\zeta) + \cos(\alpha_n\zeta)$	$\alpha_1 = 0$. For $n > 1$ same as $[C_B\text{–}C_B]$	Same as $[C_B\text{–}C_B]$
$[F_B\text{–}C_B]$	Same as $[F_B\text{–}F_B]$	$\cosh(\alpha_n)\cos(\alpha_n) = -1$	$-\dfrac{\cosh(\alpha_n) + \cos(\alpha_n)}{\sinh(\alpha_n) + \sin(\alpha_n)}$
$[F_B\text{–}S_B]$	Same as $[F_B\text{–}F_B]$	Same as $[S_B\text{–}C_B]$	Same as $[C_B\text{–}C_B]$

TABLE 2

Beam eigenfunctions for membrane boundary conditions

Boundary conditions of opposite edges	$f_n(\zeta)$ or $g_n(\zeta)$ $n = 1, 2, 3, \ldots$	α_n	A_n
$[S_M\text{–}S_M]$	$\sin(n\pi\zeta)$	–	–
$[C_M\text{–}C_M]$	$A_n[\sinh(\alpha_n\zeta) + \sin(\alpha_n\zeta)]$ $+ \cosh(\alpha_n\zeta) + \cos(\alpha_n\zeta)$	$\cosh(\alpha_n)\cos(\alpha_n) = 1$	$\dfrac{\cos(\alpha_n) - \cosh(\alpha_n)}{\sinh(\alpha_n) - \sin(\alpha_n)}$
$[S_M\text{–}C_M]$	$A_n \sin(\alpha_n\zeta) + \sinh(\alpha_n\zeta)$	$\tanh(\alpha_n) - \tan(\alpha_n) = 0$	$\dfrac{\sinh(\alpha_n)}{\sin(\alpha_n)}$
$[F_M\text{–}F_M]$	$A_n[\sinh(\alpha_n\zeta) - \sin(\alpha_n\zeta)]$ $+ \cosh(\alpha_n\zeta) - \cos(\alpha_n\zeta)$	Same as $[C_M\text{–}C_M]$	Same as $[C_M\text{–}C_M]$
$[F_M\text{–}C_M]$	Same as $[F_M\text{–}F_M]$	$\cosh(\alpha_n)\cos(\alpha_n) = -1$	$-\dfrac{\cosh(\alpha_n) + \cos(\alpha_n)}{\sinh(\alpha_n) + \sin(\alpha_n)}$
$[F_M\text{–}S_M]$	Same as $[F_M\text{–}F_M]$	Same as $[S_M\text{–}C_M]$	Same as $[C_M\text{–}C_M]$

A fairly general coverage of shell geometries is possible when it is assumed that

$$\frac{\partial^2 z}{\partial \xi^2} = X_1(\xi) + Y_1(\eta) + \bar{X}_1(\xi) \cdot \bar{Y}_1(\eta) + K_{11},$$

$$\frac{\partial^2 z}{\partial \xi\, \partial \eta} = X_2(\xi) + Y_2(\eta) + \bar{X}_2(\xi) \cdot \bar{Y}_2(\eta) + K_{12},$$

$$\frac{\partial^2 z}{\partial \eta^2} = X_3(\xi) + Y_3(\eta) + \bar{X}_3(\xi) \cdot \bar{Y}_3(\eta) + K_{22}. \tag{43}$$

The analytical treatment of this geometrical formulation can be followed in reference [3].

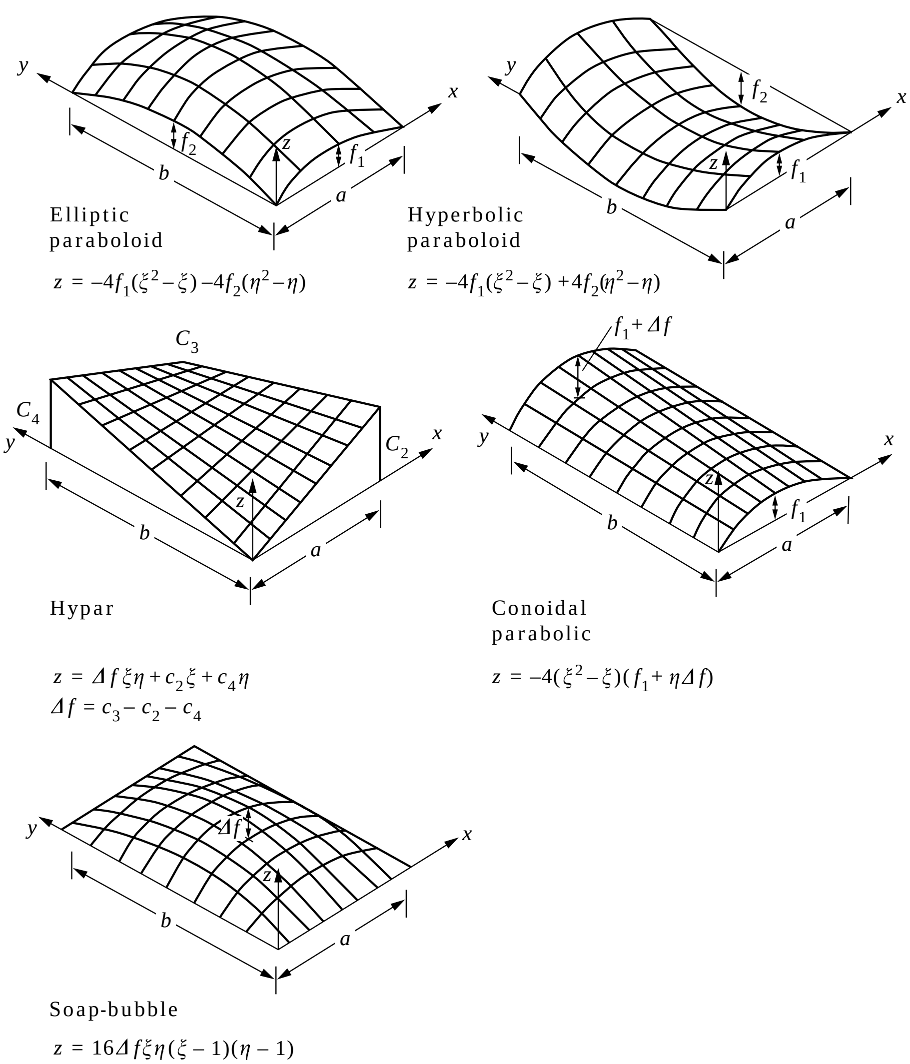

Figure 1. Types of shells. $\xi = x/a$; $\eta = y/b$.

However, in this investigation only five types of shells which are common in the civil engineering applications are considered, as Figure 1 shows. In this respect it is convenient to introduce the following non-dimensionalized parameters.

$$\gamma = \frac{a}{b}, \qquad \epsilon = \frac{ab}{h^2}, \qquad \lambda_1 = \frac{f_1}{a}, \qquad \lambda_2 = \frac{f_2}{b}, \qquad \lambda = \frac{\Delta f}{a}. \tag{44}$$

The parameters γ and ϵ are referred to as the aspect ratio and "thinness parameter", respectively, while the parameters λ_1, λ_2, and λ are referred to as the "shallowness parameters" of the shell.

The above parameters have to comply with the geometric restrictions introduced for shallow shells, so they are limited by the "shallowness" and "thinness" requirements i.e., the ratio of maximum rise to span length on the one side and the ratio of the thickness to the minimum radius of curvature on the other side, must be less than 1/5 and 1/20, respectively. The permissible ranges of these parameters appear in Table 3.

Substitution of the expressions (28) and (29), as well as the respective curvatures and twist $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$, $\partial^2 z/\partial x\,\partial y$ for the shells considered in terms of the above non-dimensionalized geometric parameters into the right hand side of equations (39) and (40), leads to the following matrix formulation:

$$\begin{bmatrix} [\mathbf{T}^W] - k\omega_i^2[\mathbf{I}] & [\mathbf{T}_0^F] \\ [\mathbf{T}_0^W] & [\mathbf{T}^F] \end{bmatrix} \begin{Bmatrix} \{\mathbf{C}_W\}^{(i)} \\ \{\mathbf{C}^F\}^{(i)} \end{Bmatrix} = \begin{Bmatrix} \{\mathbf{0}\} \\ \{\mathbf{0}\} \end{Bmatrix}. \tag{45}$$

In the above relations, $\{\mathbf{C}^W\}^{(i)}$ and $\{\mathbf{C}^F\}^{(i)}$ are $(mn \times 1)$ matrices whose elements are the coefficients $C_{rs}^{W,(i)}$ and $C_{rs}^{F,(i)}$, respectively. The subscripts r and s of the coefficient of the jth row of these matrices are equal to the elements of the jth row of the $(mn \times 2)$ matrix $[\zeta]$ defined as

$$[\zeta] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ * & * \\ 1 & n \\ 2 & 1 \\ 2 & 2 \\ * & * \\ 2 & n \\ * & * \\ * & * \\ m & 1 \\ m & 2 \\ * & * \\ m & n \end{bmatrix}. \tag{46}$$

Matrix $[\mathbf{I}]$ is the unit diagonal matrix of order $mn \times mn$.

TABLE 3

Ranges of geometric parameters

Type	Shallowness restriction	Thinness restriction
Elliptic paraboloid	$\lambda_1 + (\lambda_2/\gamma) < 0{\cdot}20$	$\sqrt{\varepsilon\gamma} > 160\lambda_1$ and $\sqrt{\varepsilon/\gamma} > 160\lambda_2$
Hyperbolic paraboloid	$\max(\lambda_1, \lambda_2) < 0{\cdot}20$	$\sqrt{\varepsilon\gamma} > 160\lambda_1$ and $\sqrt{\varepsilon/\gamma} > 160\lambda_2$
Hypar	$\lambda < 0{\cdot}20$	$\sqrt{\varepsilon/\gamma} > 20/\lambda$
Conoidal parabolic	$\lambda_1 + \lambda_2 < 0{\cdot}20$	$\sqrt{\varepsilon\gamma} > 160(\lambda_1 + \lambda)$
Soap-bubble	$\lambda < 0{\cdot}20$	$\sqrt{\varepsilon\gamma} > 160\lambda$

The matrices $[\mathbf{T}^w]$ and $[\mathbf{T}^F]$ are obtained from the left side of equations (39) and (40), respectively, and are independant of the shell type. Their elements of the ith row and jth column are:

$$T^W(i,j) = \frac{1}{12(1-\nu^2)}\left\{\left[\alpha_r^4\left(\frac{b}{a}\right)^2 + \beta_s^4\left(\frac{a}{b}\right)^2\right]\delta_{rp}^F\delta_{sq}^G + 2F(r,p)G(s,q)\right\}, \quad (47)$$

$$T^F(i,j) = \left\{\left[\bar{\alpha}_r^4\left(\frac{b}{a}\right)^2 + \bar{\beta}_s^4\left(\frac{a}{b}\right)^2\right]\delta_{rp}^f\delta_{sq}^g + 2f(r,p)g(s,q)\right\}, \quad (48)$$

where

$$r = \zeta(j,1), \qquad s = \zeta(j,2), \qquad p = \zeta(i,1), \qquad q = \zeta(i,2). \quad (49)$$

The matrices $[\mathbf{T}_0^w]$ and $[\mathbf{T}_0^F]$ are obtained from the right side of equations (33) and (34) and they depend on the shell geometry. Their elements of the ith row and jth column are:

(1) Elliptic/hyperbolic paraboloid

$$T_0^W(i,j) = -8\left[\pm\left(\lambda_2\sqrt{\frac{\varepsilon}{\gamma}}\right)F_2(r,p)G_0(s,q) + (\lambda_1\sqrt{\varepsilon\gamma})F_0(r,p)G_2(s,q)\right], \quad (50)$$

$$T_0^F(i,j) = 8\left[\pm\left(\lambda_2\sqrt{\frac{\varepsilon}{\gamma}}\right)f_2(r,p)g_0(s,q) + (\lambda_1\sqrt{\varepsilon\gamma})f_0(r,p)g_2(s,q)\right]. \quad (51)$$

(2) Hypar

$$T_0^W(i,j) = -2(\lambda_1\sqrt{\varepsilon\gamma})F_1(r,p)G_1(s,q), \quad (52)$$

$$T_0^F(i,j) = 2(\lambda_1\sqrt{\varepsilon\gamma})f_1(r,p)g_1(s,q). \quad (53)$$

(3) Conoidal parabolic

$$T_0^W(i,j) = -8(\lambda_1\sqrt{\varepsilon\gamma})F_0(r,p)G_2(s,q) - 8(\lambda\sqrt{\varepsilon\gamma})F_0(r,p)G_{12}(s,q) - 8(\lambda\sqrt{\varepsilon\gamma})F_1(r,p)G_1(s,q) + 16(\lambda\sqrt{\varepsilon\gamma})F_{11}(r,p)G_1(s,q), \tag{54}$$

$$T_0^F(i,j) = 8(\lambda_1\sqrt{\varepsilon\gamma})f_0(r,p)g_2(s,q) + 8(\lambda\sqrt{\varepsilon\gamma})f_0(r,p)g_{12}(s,q) + 8(\lambda\sqrt{\varepsilon\gamma})f_1(r,p)g_1(s,q) - 16(\lambda\sqrt{\varepsilon\gamma})f_{11}(r,p)g_1(s,q). \tag{55}$$

(4) Soap-bubble

$$T_0^W(i,j) = 32(\lambda\sqrt{\varepsilon\gamma})\left\{\begin{matrix} F_0(r,p)G_{22}(s,q) - F_0(r,p)G_{12}(s,q) + F_{22}(r,p)G_0(s,q) - F_{12}(r,p)G_0(s,q) \\ -2[-F_{11}(r,p)G_1(s,q) - F_1(r,p)G_{11}(s,q) + 2F_{11}(r,p)G_{11}(s,q) + \frac{1}{2}F_1(r,p)G_1(s,q)] \end{matrix}\right\}, \tag{56}$$

$$T_0^F(i,j) = -32(\lambda\sqrt{\varepsilon\gamma})\left\{\begin{matrix} f_0(r,p)g_{22}(s,q) - f_0(r,p)g_{12}(s,q) + f_{22}(r,p)g_0(s,q) - f_{12}(r,p)g_0(s,q) \\ -2[-f_{11}(r,p)g_1(s,q) - f_1(r,p)g_{11}(s,q) + 2f_{11}(r,p)g_{11}(s,q) + \frac{1}{2}f_1(r,p)g_1(s,q)] \end{matrix}\right\}. \tag{57}$$

In the above expressions:

$$F_i(r,p) = \int_0^1 \frac{\mathrm{d}^{(i)}F_r}{\mathrm{d}\xi^{(i)}} f_p(\xi)\,\mathrm{d}\xi, \qquad F_{jk}(r,p) = \int_0^1 \xi^j \frac{\mathrm{d}^{(k)}F_r}{\mathrm{d}\xi^{(k)}} f_p(\xi)\,\mathrm{d}\xi \qquad (58, 59)$$

and

$$G_i(s,q) = \int_0^1 \frac{\mathrm{d}^{(i)}G_s}{\mathrm{d}\eta^{(i)}} g_q(\eta)\,\mathrm{d}\eta, \qquad G_{jk}(s,q) = \int_0^1 \eta^j \frac{\mathrm{d}^{(k)}G_s}{\mathrm{d}\eta^{(k)}} g_q(\eta)\,\mathrm{d}\eta. \qquad (60, 61)$$

The matrices $[\mathbf{f}_i]$, $[\mathbf{g}_i]$, $[\mathbf{f}_{11}]$ and $[\mathbf{g}_{12}]$ ($i = 0, 1, 2$) are obtained from the expressions (58)–(61) by interchanging F, G with f, g, respectively. All the above integrals are evaluated in their algebraic form in reference [3].

Moreover in equation (45)

$$k = \left(\frac{ab}{h^2}\right)^2 \frac{\rho h^2}{E} \lambda_0, \qquad (62)$$

where

$$\lambda_0 = \delta^F_{rr}\delta^G_{ss} \qquad (63)$$

From equations (41) it can be deduced that λ_0 equals 1 or 0·5 or 0·25, according to whether the shell has none, one or both of its opposite boundaries of the bending type $[S_B–S_B]$.

The matrix formulation (45) leads to the following typical eigenvalue equation

$$[[\mathbf{S}] - k\omega_i^2[\mathbf{I}]]\{\mathbf{C}^W\}^{(i)} = \{\mathbf{0}\} \qquad (64)$$

where

$$[\mathbf{S}] = [\mathbf{T}^W] - [\mathbf{T}_0^F][\mathbf{T}^F]^{-1}[\mathbf{T}_0^W]. \qquad (65)$$

Also the following expression for $\{\mathbf{C}^F\}^{(i)}$ is accordingly derived

$$\{\mathbf{C}^F\}^{(i)} = -[\mathbf{T}^F]^{-1}[\mathbf{T}_0^W]\{\mathbf{C}^W\}^{(i)}. \qquad (66)$$

The matrix $[\mathbf{S}]$ is a $mn \times mn$ non-singular and non-symmetric matrix which depends on the geometry and the material of the shell, as well as on its boundary conditions. It plays the role of a "stiffness matrix" of the shell.

Equation (64) is the typical form of the eigenvalue equation in finite elements techniques and in spite of the absence of symmetry, the matrix $[\mathbf{S}]$ yields always (mn) real and positive eigenvalues Ω_i^2 with their respective eigenvectors $\{\mathbf{C}^W\}^{(i)}$ from which the eigenforms $W_i(\xi, \eta)$ are obtained through equation (28).

The eigenvalues Ω_i^2 of the matrix $[\mathbf{S}]$ lead to the non-dimensionalized eigenfrequencies of the shell, according to the relation

$$\omega_i^2 \frac{\rho h^2}{E} = \frac{\Omega_i^2}{\lambda_0 \varepsilon^2}. \qquad (67)$$

The eigenfunctions $\Phi_i(\xi, \eta)$ are also obtained through equation (29), after the determination of the respective eigenvectors $\{\mathbf{C}^F\}^{(i)}$ from equation (66).

All the above eigenvalue analysis results converge to the accurate ones, for sufficiently large values of m and n. In this respect, one should always keep in mind the approximate character of shallow shell theory as such, regarding the identification of the geometry of a shell element with its projection in the xy plane, an assumption which also applies inevitably to the vibrating masses too.

6. ANALYSIS OF FORCED VIBRATIONS

It is assumed that the vertical loading acting on the shallow shell can be expressed in the form

$$p(x, y, t) = p_0 q_1(\xi) q_2(\eta) \Omega(t), \tag{68}$$

where $\Omega(t)$ is the forcing function.

The unknown functions of the problem $w(x, y, t)$ and $\phi(x, y, t)$, are expressed as

$$w(x, y, t) = \sum_{i=1}^{(mn)} W_i(x, y) T_i(t), \qquad \phi(x, y, t) = \sum_{i=1}^{(mn)} \Phi_i(x, y) T_i(t), \quad (69, 70)$$

where $W_i(x, y)$ and $\Phi_i(x, y)$ are already known from the eigenvalue analysis and $T_i(t)$ are the (mn) unknown time functions of the problem.

Substitution of the expressions (69) and (70) into equation (3) and taking into account equation (24), yields:

$$\sum_{i=1}^{(mn)} W_i(x, y) \frac{\mathrm{d}^2 T_i(t)}{\mathrm{d}t^2} + \sum_{i=1}^{(mn)} \omega_i^2 W_i(x, y) T_i(t) = \frac{1}{\rho h} p(x, y, t). \tag{71}$$

As equation (25) is already valid, equation (4) is automatically satisfied.

Multiplying both sides of equation (69) by $W_k(x, y)$ and integrating over the domain of the entire orthogonal base of the shell gives:

$$[\mathbf{C}]\{\{\ddot{\mathbf{T}}(t)\} + [\boldsymbol{\omega}^2]\{\mathbf{T}(t)\}\} = \frac{p_0}{\rho h^2}\{\mathbf{P}\}\Omega(t), \tag{72}$$

where

$$[\mathbf{C}] = [\mathbf{C}^W]^T[\mathbf{C}^W]. \tag{73}$$

$[\mathbf{C}^W]$ is a $(mn \times mn)$ square matrix assembled as:

$$[\mathbf{C}^W] = [\{\mathbf{C}^W\}^{(1)} \quad \{\mathbf{C}^W\}^{(2)} \quad * \quad * \quad \{\mathbf{C}^W\}^{(mn)}]. \tag{74}$$

Moreover, $\{\mathbf{P}\}$ is a $(mn \times 1)$ column matrix according to the expression:

$$\{\mathbf{P}\} = [\mathbf{C}^W]^T\{\mathbf{B}\}, \tag{75}$$

with $\{\mathbf{B}\}$, a $(mn \times 1)$ column matrix whose ith element $B(i)$ is obtained from:

$$B(i) = \left(\int_0^1 F_r(\xi) q_1(\xi)\, \mathrm{d}\xi\right)\left(\int_0^1 G_s(\eta) q_2(\eta)\, \mathrm{d}\eta\right), \tag{76}$$

where

$$r = \zeta(i, 1), \qquad s = \zeta(i, 2). \tag{77}$$

Further, $[\boldsymbol{\omega}^2]$ is the $(mn \times mn)$ diagonal matrix of ω_i^2 and $\{\mathbf{T}(t)\}$ is the $(mn \times 1)$ column matrix of the unknown time functions $T_i(t)$.

From the linear differential system (72) is finally obtained:

$$\{\mathbf{T}(t)\} = p_0 \frac{\varepsilon\sqrt{\lambda_0}}{h\sqrt{\rho E}} [\mathbf{Y}(t)]\{\mathbf{D}\}, \tag{78}$$

where $[\mathbf{Y}(t)]$ is a $(mn \times mn)$ diagonal matrix consisting of the functions

$$Y_i(t) = \frac{1}{\Omega_i} \int_0^t \sin \omega_i (t - \tau) \Omega(t)\, \mathrm{d}\tau, \tag{79}$$

and $\{\mathbf{D}\}$ is a $(mn \times 1)$ column matrix according to the expression

$$\{\mathbf{D}\} = [\mathbf{C}^W]^{-1}\{\mathbf{B}\}. \tag{80}$$

Introducing now the $(mn \times 1)$ column matrix time function

$$\{\overset{*}{\mathbf{T}}(t)\} = [\mathbf{Y}(t)]\{\mathbf{D}\}, \tag{81}$$

and after substitution of equations (28) and (29) into the expressions (69) and (70), respectively, the final expressions of the functions $w(\xi, \eta, t)$ and $\phi(\xi, \eta, t)$ are derived:

$$w(\xi, \eta, t) = \frac{p_0}{\sqrt{\rho E}} \sum_{i=1}^{(mn)} \left[\sum_{r=1}^{m} \sum_{s=1}^{n} C_{rs}^{W(i)} F_r(\xi) G_s(\eta) \right] \cdot \overset{*}{T}_i(t), \tag{82}$$

$$\phi(\xi, \eta, t) = p_0 h^2 \sqrt{\frac{E}{\rho}} \sum_{i=1}^{(mn)} \left[\sum_{r=1}^{m} \sum_{s=1}^{n} C_{rs}^{F(i)} f_r(\xi) g_s(\eta) \right] \cdot \overset{*}{T}_i(t). \tag{83}$$

The stress resultants of the shell can be deduced through substitution into the expressions (7), (8) and (9).

7. COMPUTATIONAL PROCEDURE

According to the above analytical procedure a computer program has been compiled for the dynamic analysis of shallow shells having the geometry defined in Figure 1. The input data given are the type of shell geometry, the boundary conditions of the shell, the values of the parameters γ, ε and λ_1, λ_2 or λ, the Poisson's ratio v, the load intensity p_0 with its functions $q_1(\xi)$ and $q_2(\eta)$, the forcing

TABLE 4
Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for the elliptic paraboloid shells* ($\varepsilon = 10\,000$, $\nu = 0{\cdot}15$)

	S_BS_M–S_BS_M/S_BS_M–S_BS_M				C_BS_M–C_BS_M/C_BS_M–C_BS_M			
a/b	0·5		1·0		0·5		1·0	
$f_1/a = f_2/b$	0·05	0·10	0·05	0·10	0·05	0·10	0·05	0·10
1	34·697	64·711	40·413	80·207	36·840	69·176	41·355	80·682
2	38·689	68·263	42·516	81·287	46·235	70·541	45·365	82·786
3	43·964	73·746	46·168	83·255	47·885	79·357	50·953	85·943
4	44·525	78·851	49·299	85·032	53·605	86·000	55·442	88·689

function $\Omega(t)$ and the selected numbers of beam functions in each direction m and n.

For a set of values m and n, the program adheres to the following steps: (1) Establishes the parameters and coefficients of the beam functions according to Tables 1 and 2. (2) Computes the quantities $F(r,p)$, $F_i(r,p)$, $F_{11}(r,p)$, $G(s,q)$, $G_i(s,q)$, $G_{12}(s,q)$ ($i = 0, 1, 2$), as well as their "f" and "g" counterparts defined by equations (42) and (58)–(61). (3) Computes the matrices $[\mathbf{T}^W]$, $[\mathbf{T}^F]$ on the basis of equations (46), (47) and the matrices $[\mathbf{T}_0^W]$ and $[\mathbf{T}_0^F]$ on the basis of the appropriate set of equations (50)–(57). (4) Computes the matrix $[\mathbf{S}]$ from equation (65). (5) Computes the eigenvalues Ω_i^2, the eigenvectors $\{\mathbf{C}^W\}^{(i)}$ of the matrix $[\mathbf{S}]$ and the non-dimensionalized eigenfrequencies according to equation (67). (6) Computes the eigenvectors $\{\mathbf{C}^F\}^{(i)}$ from equation (66). (7) Computes the eigenforms $W_i(\xi, \eta)$, $\Phi_i(\xi, \eta)$ according to equations (28), (29). (8) Computes the matrices $[\mathbf{C}^W]$, $\{\mathbf{B}\}$, $[\mathbf{Y}(t)]$ and $\{\mathbf{D}\}$ according to equations (74), (76), (79) and (80), respectively. (9) Computes the matrix $\{\mathbf{T}^*(t)\}$ from equation (81). (10) Computes the deflection $w(\xi, \eta, t)$ and the function $\phi(\xi, \eta, t)$ from equations (82), (83) and finally the stress resultants according to equations (7)–(9).

8. PARAMETRIC INVESTIGATION

On the basis of the above established non-dimensionalised geometric parameters of the shallow shells examined, namely the aspect ratio γ, the thinness parameter

TABLE 5
Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for the hyperbolic paraboloid shells* ($\varepsilon = 10\,000$, $\nu = 0{\cdot}15$)

	S_BS_M–S_BS_M/S_BS_M–S_BS_M				C_BS_M–C_BS_M/C_BS_M–C_BS_M			
a/b	0·5		1·0		0·5		1·0	
$f_1/a = f_2/b$	0·05	0·10	0·05	0·10	0·05	0·10	0·05	0·10
1	13·413	23·747	5·763	5·763	18·838	27·854	11·928	13·509
2	18·245	30·543	23·054	23·054	23·984	34·731	32·194	33·453
3	30·959	36·305	27·993	48·479	40·316	48·065	32·459	52·576
4	33·802	36·639	40·498	50·116	43·277	49·756	49·815	58·522

TABLE 6

Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for the soap-bubble shells* ($\varepsilon = 10\,000$, $\nu = 0{\cdot}15$)

	S_BS_M–S_BS_M/S_BS_M–S_BS_M				C_BS_M–C_BS_M/C_BS_M–C_BS_M			
a/b	0·5		1·0		0·5		1·0	
$\Delta f/a$	0·10	0·20	0·10	0·20	0·10	0·20	0·10	0·20
1	23·510	34·841	29·230	39·787	33·776	53·372	45·433	66·249
2	32·654	46·288	41·804	58·981	45·534	61·942	57·716	86·360
3	35·489	53·859	49·358	68·673	47·963	74·921	63·880	92·170
4	40·408	57·154	58·579	78·279	53·780	75·624	65·104	97·860

ε and the shallowness parameters λ, a parametric study for each type of shell is made, in order to have an assessment of their dynamic characteristics and to also show the possibilities of the procedure presented. The results are presented in the form of tables concerning the five shell types examined before, namely the elliptic and hyperbolic parabolid, the soap-bubble, the hypar and the conoid shell.

In each case the four lowest non-dimensional frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ are presented, under certain boundary conditions. The above shells are used mainly as concrete roofs in civil engineering applications. Especially the last two have the essential constructional advantage that their formwork consists only of rectilinear elements (Figure 1).

A. Elliptic and hyperbolic paraboloids (Tables 4 and 5)

These shells are considered resting on vertical walls along their boundaries which restrain the displacements of the shell boundaries in their respective plane (i.e., vertically and tangentially), but are very flexible against transverse (i.e., horizontal) displacements. The aspect ratio γ takes the values 0·5 and 1·0, respectively and the thinness parameter ε is considered equal to 10 000, a rather representative value for concrete shell roofs. The shallowness parameters λ_1, λ_2 are considered to be equal to 0·05 and 0·10. Two cases are taken into account, according to whether all the boundaries are allowed to rotate freely or not. The Poissons' ratio is taken equal to 0·15.

In the case of free rotation of the boundaries the results obtained are in complete agreement with those obtained by the exact formula given in reference [1], which in the present paper according to the notations that were introduced, takes the following form.

$$\omega_{mn}^2 \frac{\rho}{E} = \frac{h^2\pi^4}{12(1-\nu^2)}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 + \frac{64\cdot\left[\frac{f_2}{b^2}\left(\frac{m}{a}\right)^2 + \frac{f_1}{a^2}\left(\frac{n}{b}\right)^2\right]^2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}.$$

TABLE 7

Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for hypar shells* $[S_BS_M\text{–}S_BS_M/S_BS_M\text{–}S_BS_M]$ $(a/b = 1{\cdot}0,\ \nu = 0{\cdot}15)$

ab/h^2	10 000				20 000			
$\Delta f/a$	0·05	0·10	0·15	0·20	0·05	0·10	0·15	0·20
1	14·518	14·833	15·316	15·921	14·625	15·223	16·080	17·065
2	23·447	24·588	26·380	28·692	23·835	26·031	29·269	29·602
3	28·843	28·919	29·043	29·218	28·867	29·020	29·316	31·474
4	28·918	29·212	29·676	30·278	29·017	29·586	30·440	33·364

TABLE 8

Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for hypar shells* $[C_BS_M\text{–}C_BS_M/C_BS_M\text{–}C_BS_M]$ $(a/b = 1{\cdot}0,\ \nu = 0{\cdot}15)$

ab/h^2	10 000				20 000			
$\Delta f/a$	0·05	0·10	0·15	0·20	0·05	0·10	0·15	0·20
1	21·552	21·903	22·461	23·193	21·671	22·352	23·392	24·683
2	31·882	32·675	33·954	35·667	32·147	33·702	36·141	39·298
3	38·460	38·565	38·737	38·976	38·495	38·703	39·044	39·512
4	38·689	38·935	39·336	39·877	38·772	39·256	40·029	41·041

TABLE 9

Frequency coefficients $(\omega_i(ab/h)\sqrt{\rho/E})$ *for conoidal parabolic shells* $(\varepsilon = 10\,000,$ $f_1/a = 0{\cdot}10)$

	$S_BS_M\text{–}S_BS_M/S_BS_M\text{–}S_BS_M$ $(\nu = 0{\cdot}15)$						$F_BF_M\text{–}F_BF_M/S_BS_M\text{–}S_BS_M$ $(\nu = 0)$			
a/b	0·2		0·5		0·8		0·2		0·5	
$\Delta f/a$	0·00	0·10	0·00	0·10	0·00	0·10	0·00	0·10	0·00	0·10
1	16·489	18·026	23·747	26·340	20·774	24·430	2·678	4·027	5·803	9·192
2	29·802	39·486	25·383	32·751	35·229	35·752	3·620	4·810	8·898	10·643
3	51·247	58·277	36·639	43·744	42·151	43·206	7·895	9·961	15·543	16·833
4	58·238	60·818	53·399	53·308	46·112	53·716	10·178	14·707	16·974	18·935

From Tables 4 and 5 it is seen that by restraining the boundary rotation, the eigenfrequencies in the case of the elliptic paraboloid are hardly increased. Moreover an increase in the shallowness parameter causes a significantly greater increase in the eigenfrequencies in the elliptic rather than in the hyperbolic paraboloid shell. It is also seen that for square planforms in simply supported hyperbolic shells the eigenfrequencies are independant of the shallowness parameters.

B. Soap-bubble (Table 6)

The geometry of this doubly curved shell can be depicted with a good approximation by the deformed shape which assumes a rectangular simply

TABLE 10

Frequency coefficients ($\omega_i(ab/h)\sqrt{\rho/E}$) *for conoidal parabolic shells* ($\varepsilon = 10\,000$, $f_1/a = 0{\cdot}0$, $v = 0{\cdot}15$)

	S_BS_M–S_BS_M/S_BS_M–S_BS_M				C_BS_M–C_BS_M/C_BS_M–S_BS_M			
a/b	0·5		1·0		0·5		1·0	
$\Delta f/a$	0·10	0·20	0·10	0·20	0·10	0·20	0·10	0·20
1	10·870	14·990	13·659	18·590	13·453	13·750	7·925	9·046
2	24·424	25·460	16·441	19·363	20·761	28·250	18·786	19·553
3	24·811	35·075	28·713	30·231	34·992	36·318	24·451	33·035
4	31·063	35·632	30·418	38·564	36·176	40·110	26·894	33·617

supported membrane if subjected to a uniformly distributed load. The shell is considered either simply supported or clamped all around. The aspect ratio takes the values 0·5 and 1·0, whereas the shallowness parameter $\Delta f/a$ takes the values 0·10 and 0·20. The thinness parameter ε is constant and equal to 10 000.

C. Hypar shells (Tables 7 and 8)

These shells are also considered resting on vertical walls or appropriate structural elements which restrain the displacements of the straight boundaries in the plane of the wall. Moreover the edge rotation may be restrained or not. The aspect ratio is considered constant to 1·0 and the thinness parameter ε takes the values 10 000 and 20 000, respectively. The shallowness parameter $\Delta f/a$ lies within the range of 0·05 to 0·20.

It is seen that the respective eigenfrequencies are hardly influenced by a change in the shell thickness.

D. Conoidal parabolic (Tables 9 and 10)

In this investigation two types of shells are examined according to whether the more shallow curved edge is actually curved or not.

In the first case the shallowness parameter f_1/a is equal to 0·10 and the supplementary shallowness parameter $\Delta f/a$ takes the values 0·00 (i.e., cylindrical panel) and 0·10 (Figure 1). The shell is either simply supported all around or simply supported on each curved edge and free on each straight edge. Although in this last case the boundary conditions are not exactly satisfied, a comparison with a finite element solution according to reference [12] shows that also for this "approximately approached" case the results are practically always valuable. However, as it is found, the accuracy of the results is significantly decreased if the aspect ratio γ exceeds the value 0·5.

In the second case the shallowness parameter f_1/a is equal to zero and the parameter $\Delta f/a$ takes the values 0·10 and 0·20. In this case the shell is considered either simply supported all around or clamped along its three straight edges and free along its curved edge.

9. CONCLUSIONS

According to the described procedure, the evaluation of a dynamic analysis for various types of shallow shells over a rectangular layout is made possible by considering, apart from the appropriate boundary conditions, only a few non-dimensionalized variables regarding the geometry of the shell. In some cases the prevailing boundary conditions may not be exactly satisfied but even in those cases the discrepancy of the results obtained by the procedure presented is held into practically admissible limits.

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