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LETTERS TO THE EDITOR

AN UPPER BOUND ON RESPONSES OF NON-CLASSICALLY DAMPED LINEAR SYSTEMS

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1. INTRODUCTION

The free vibration of an n-degree-of-freedom linear second order system is represented by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = \theta_n, \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \tag{1}$$

for all $t \geqslant 0$. In equation (1),

$$x(t) = [x_1(t) \quad x_2(t) \cdots x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n, \tag{2}$$

denotes the vector of displacements (v^{T} denotes the transpose of a vector v); the mass matrix M and the stiffness matrix K belong to $\mathbb{R}^{n \times n}$ and are symmetric and positive definite; the damping matrix C belongs to $\mathbb{R}^{n \times n}$ and is symmetric and positive semi-definite; $x_0 \in \mathbb{R}^n$ and $\dot{x}_0 \in \mathbb{R}^n$ are the vectors of initial displacements and velocities, respectively; θ_n denotes the zero vector in $\mathbb{R}^n$.

In this note, we plan to derive an *a priori* upper bound on the sizes (norms) of the displacements of the system (1) without solving it (numerically). In recent years, researchers have derived bounds on the sizes of displacements and velocities of free or forced vibratory systems; see, e.g., references [1, 2, 3 (p. 136), 4–8, 9 (pp. 177–178), 10]. Such bounds can be used in the design and analysis of systems.

Bounds on the sizes of displacements of the system (1) are useful when (i) they are easily computable; (ii) they are tight. If the bounds are not easily computable, then one might as well solve the system (1) (numerically) in order to obtain the exact (very accurate) values for the displacement peaks. If, on the other hand, the bounds are easily computable, but are conservatively large, then they furnish no useful information to be used in the system design and analysis. It appears that the two requirements of ease-of-computation and tightness of the upper bounds oppose each other: the less (more, respectively) computational effort, the more (less) conservative bounds on the sizes of displacements. Despite this fact, one should attempt to derive easy-to-compute and tight bounds.

Most available bounds in the literature are not easily computable, except those in references [1–3, 7, 8]. In reference [7], upper bounds on the sizes of displacements of the system (1) are computed as follows. Let

$$\|x_i\|_\infty := \max_{t \geqslant 0} |x_i(t)|, \tag{3}$$

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denote the L_∞-norm of the displacement $x_i(\cdot)$ for an $i = 1, 2, \ldots, n$. Let

$$E_0 := \tfrac{1}{2} x_0^T K x_0 + \tfrac{1}{2} \dot{x}_0^T M \dot{x}_0, \tag{4}$$

denote the initial energy of the system (1). According to reference [7], the norm $\|x_i\|_\infty$ for an $i = 1, 2, \ldots, n$ satisfies

$$\|x_i\|_\infty \leqslant [2(K^{-1})_{ii} E_0]^{1/2}, \tag{5}$$

where $(K^{-1})_{ii} > 0$ denotes the ith diagonal elements of the matrix K^{-1}.

The upper bounds in inequality (5) depend on the matrices M and K, where the dependence on M is through E_0. Computing K^{-1} in order to obtain $(K^{-1})_{ii}$ for an $i = 1, 2, \ldots, n$ requires some computational effort, because K is in general a full matrix. An interesting feature of the bounds in inequality (5) is that they do not depend on the damping matrix C.

In reference [8], a single upper bound on the norms of all displacements of the system (1) is obtained, when the system is assumed to be *classically* damped, i.e., when $CM^{-1}K = KM^{-1}C$. The bound is given by

$$\|x_i\|_\infty \leqslant ([\lambda_{max}(M)/\lambda_{min}(M)](x_0^T x_0 + \dot{x}_0^T \dot{x}_0/\omega_1^2))^{1/2}, \tag{6}$$

for all $i = 1, 2, \ldots, n$, where $\lambda_{max}(M)$ and $\lambda_{min}(M)$ are the largest and smallest eigenvalues of the mass matrix M, respectively, ω_1 is the lowest undamped natural frequency of the system, and x_0 and $\dot{x}_0$ are the vectors of initial displacements and velocities, respectively.

The bound in inequality (6) is a single upper bound on the norms of all displacements of the system (1). Therefore, it is computed only once. Recall that there are n upper bounds in inequality (5), and so are there n times of computation. Computing the bound in inequality (6) is an easy task. In computing this bound some computational effort is required to compute the square of the lowest undamped natural frequency, ω_1^2. It is straightforward to compute ω_1^2, since there are certain numerical methods by which ω_1^2 is readily computed. Some of such numerical methods are the power method, Given's method, QR method, inverse iteration method, and Rayleigh's quotient iteration method, which primarily compute the smallest eigenvalue of an eigenvalue problem (see, e.g., references [11 (Chapter 6), 12]). It should be pointed out that the lowest natural frequency, ω_1, is an important piece of information for vibratory systems. Therefore, computing ω_1—to be used either in inequality (6) or in the system design and analysis—is well worth the effort. Note that there is no need to compute ω_1^2, when the initial velocities are zero.

Our goal in this note is to relax the assumption in reference [8] that the system (1) is classically damped, and to obtain an easy-to-compute bound on the norms of displacements of the system (1). It turns out that inequality (6) provides such an upper bound, even when the system is not classically damped.

2. AN UPPER BOUND ON DISPLACEMENTS

We first obtain the normalized representation of the system (1). Such representation is obtained via a linear change of co-ordinates applied to the system (1). The change of co-ordinates is

$$x(t) = Uq(t), \tag{7}$$

for all $t \geqslant 0$, where $U \in \mathbb{R}^{n \times n}$ is the (non-singular) modal matrix corresponding to the system (1) (see, e.g., references [9 (pp. 173–175), 11 (pp. 178–181)]) and

$$q(t) = [q_1(t) \quad q_2(t) \cdots q_n(t)]^{\mathrm{T}} \in \mathbb{R}^n, \tag{8}$$

is the vector of normalized displacements. The columns of the modal matrix are the eigenvectors of the symmetric generalized eigenvalue problem

$$Ku^{(i)} = \omega_i^2 Mu^{(i)}, \tag{9}$$

where $\omega_i^2 > 0$ and $u^{(i)} \in \mathbb{R}^n$ are an eigenvalue (undamped natural frequency squared) and the corresponding eigenvector for an $i = 1, 2, \ldots, n$, respectively. The modal matrix is commonly orthonormalized according to

$$U^{\mathrm{T}}MU = I_n, \tag{10}$$

where U^{T} denotes the transpose of the matrix U and I_n denotes the $n \times n$ identity matrix. Since equation (10) holds, the matrix K satisfies

$$U^{\mathrm{T}}KU = \mathrm{diag}\,[\omega_1^2, \omega_2^2, \ldots, \omega_n^2] =: \Omega^2, \tag{11}$$

where, without the loss of generality, the natural frequencies are ordered as $\omega_1 \leqslant \omega_2 \leqslant, \ldots, \leqslant \omega_n$.

Using equations (10) and (11), the system (1) under the change of co-ordinates in equation (7) is represented as

$$I_n\ddot{q}(t) + \tilde{C}\dot{q}(t) + \Omega^2 q(t) = \theta_n, \tag{12a}$$

for all $t \geqslant 0$, with the initial conditions

$$q_0 := q(0) = U^{-1}x_0 = U^{\mathrm{T}}Mx_0, \qquad \dot{q}_0 := \dot{q}(0) = U^{-1}\dot{x}_0 = U^{\mathrm{T}}M\dot{x}_0. \tag{12b}$$

In equation (12a),

$$\tilde{C} := U^{\mathrm{T}}CU \in \mathbb{R}^{n \times n}, \tag{13}$$

is known as the *normalized* damping matrix. The system (1) is not classically damped in general. That is, $CM^{-1}K \neq KM^{-1}C$ (see, e.g., references [3 (pp. 144–145), 13]). Therefore, the matrix $\tilde{C}$ is not necessarily diagonal, due to which the system (12a) is a set of coupled differential equations and not a set of n scalar second order linear systems.

We rewrite the system (12a) as

$$\Omega^{-2}\ddot{q}(t) + \Omega^{-2}\tilde{C}\dot{q}(t) + q(t) = \theta_n, \tag{14}$$

for all $t \geqslant 0$, with the initial conditions in equation (12b). We will use equation (14) to obtain an upper bound on the norms of displacements of the system (1). We first establish a preliminary result.

Lemma 2.1. Consider the coefficient matrices M, C, and K of the system (1), the matrix Ω^2 in equation (11), and the normalized damping matrix $\tilde{C}$ in equation (13). The matrix $\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2}$ is positive semi-definite if and only if the matrix $CM^{-1}K + KM^{-1}C$ is positive semi-definite.

Proof. Let $v \in \mathbb{R}^n$ be a non-zero vector. We can write

$$v^{\mathrm{T}}(\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2})v = v^{\mathrm{T}}\Omega^{-2}(\tilde{C}\Omega^2 + \Omega^2\tilde{C})\Omega^{-2}v = w^{\mathrm{T}}(\tilde{C}\Omega^2 + \Omega^2\tilde{C})w, \quad (15)$$

where $w := \Omega^{-2}v \in \mathbb{R}^n$ and the last identity follows due to the symmetry of Ω^{-2}. From equations (10) and (11), we obtain

$$UU^{\mathrm{T}} = M^{-1}, \qquad \Omega^2 = U^{-1}M^{-1}KU. \quad (16a, b)$$

Using equations (13) and (16b) in $\tilde{C}\Omega^2$ and equations (11) and (13) in $\Omega\tilde{C}$, we obtain

$$w^{\mathrm{T}}(\tilde{C}\Omega^2 + \Omega^2\tilde{C})w = w^{\mathrm{T}}(U^{\mathrm{T}}CM^{-1}KU + U^{\mathrm{T}}KUU^{\mathrm{T}}CU)w. \quad (17)$$

Using equation (16a) in the right-hand side of equation (17), we obtain

$$w^{\mathrm{T}}(\tilde{C}\Omega^2 + \Omega^2\tilde{C})w = (Uw)^{\mathrm{T}}(CM^{-1}K + KM^{-1}C)(Uw). \quad (18)$$

Comparing equations (15) and (18), we conclude the equivalence of the positive semi-definiteness of $\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2}$, $\tilde{C}\Omega^2 + \Omega^2\tilde{C}$, and $CM^{-1}K + KM^{-1}C$. □

We now present an upper bound on the displacements of the system (1).

Theorem 2.2. Consider the system (1) and let the matrix $CM^{-1}K + KM^{-1}C$ be positive semi-definite. The L_∞-norm of the displacement $x_i(\cdot)$ satisfies

$$\|x_i\|_\infty \leqslant ([\lambda_{max}(M)/\lambda_{min}(M)](x_0^{\mathrm{T}}x_0 + \dot{x}_0^{\mathrm{T}}\dot{x}_0/\omega_1^2))^{1/2}, \quad (19)$$

for all $i = 1, 2, \ldots, n$, where $\lambda_{max}(M)$ and $\lambda_{min}(M)$ are the largest and smallest eigenvalues of the mass matrix M, respectively, ω_1 is the lowest undamped natural frequency of the system, and x_0 and $\dot{x}_0$ are the vectors of initial displacements and velocities, respectively.

Proof. For the system (14) consider the function

$$E(t) = \tfrac{1}{2}q^{\mathrm{T}}(t)q(t) + \tfrac{1}{2}\dot{q}^{\mathrm{T}}(t)\Omega^{-2}\dot{q}(t), \quad (20)$$

for all $t \geqslant 0$, where at $t = 0$,

$$E_0 := E(0) = \tfrac{1}{2}q_0^{\mathrm{T}}q_0 + \tfrac{1}{2}\dot{q}_0^{\mathrm{T}}\Omega^{-2}\dot{q}_0. \quad (21)$$

The derivative of $E(\cdot)$ along the solution of the system (14) satisfies

$$\dot{E}(t) = -\dot{q}^{\mathrm{T}}(t)\Omega^{-2}\tilde{C}\dot{q}(t), \quad (22)$$

for all $t \geqslant 0$. Writing

$$\Omega^{-2}\tilde{C} = (\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2})/2 + (\Omega^{-2}\tilde{C} - \tilde{C}\Omega^{-2})/2, \quad (23)$$

in equation (22), we obtain

$$\dot{E}(t) = -\tfrac{1}{2}\dot{q}^{\mathrm{T}}(t)(\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2})\dot{q}(t), \quad (24)$$

for all $t \geqslant 0$. Since $CM^{-1}K + KM^{-1}C$ is positive semi-definite, by *Lemma* 2.1, the matrix $\Omega^{-2}\tilde{C} + \tilde{C}\Omega^{-2}$ in equation (24) is positive semi-definite. Thus, $\dot{E}(t) \leqslant 0$ for all $t \geqslant 0$. Therefore, the function $E(\cdot)$ is non-increasing. That is, $E(t) \leqslant E_0$ for all $t \geqslant 0$, which can be written as

$$q^{\mathrm{T}}(t)q(t) + \dot{q}^{\mathrm{T}}(t)\Omega^{-2}\dot{q}(t) \leqslant q_0^{\mathrm{T}}q_0 + \dot{q}_0^{\mathrm{T}}\Omega^{-2}\dot{q}_0, \tag{25}$$

for all $t \geqslant 0$. From inequality (25), we obtain

$$q^{\mathrm{T}}(t)q(t) \leqslant q_0^{\mathrm{T}}q_0 + \dot{q}_0^{\mathrm{T}}\Omega^{-2}\dot{q}_0, \tag{26}$$

for all $t \geqslant 0$. Using equations (7) and (12b) in inequality (26), we obtain

$$\begin{aligned} x^{\mathrm{T}}(t)U^{-\mathrm{T}}U^{-1}x(t) &\leqslant x_0^{\mathrm{T}}U^{-\mathrm{T}}U^{-1}x_0 + \dot{x}_0^{\mathrm{T}}U^{-\mathrm{T}}\Omega^{-2}U^{-1}\dot{x}_0 \\ &\leqslant x_0^{\mathrm{T}}U^{-\mathrm{T}}U^{-1}x_0 + \lambda_{max}(\Omega^{-2})\dot{x}_0^{\mathrm{T}}U^{-\mathrm{T}}U^{-1}\dot{x}_0, \end{aligned} \tag{27}$$

for all $t \geqslant 0$, where the last inequality follows from the definition of Rayleigh's quotient (see, e.g., references [11 (pp. 237–243), 14 (pp. 176–181)]). From equation (10), we have $U^{-\mathrm{T}}\ U^{-1} = M$. Using this fact and that $\lambda_{max}(\Omega^{-2}) = 1/\omega_1^2$, we can write inequality (27) as

$$x^{\mathrm{T}}(t)Mx(t) \leqslant x_0^{\mathrm{T}}Mx_0 + \dot{x}_0^{\mathrm{T}}M\dot{x}_0/\omega_1^2, \tag{28}$$

for all $t \geqslant 0$. Using the definition of Rayleigh's quotient on both sides of inequality (28), we obtain

$$\lambda_{min}(M)x^{\mathrm{T}}(t)x(t) \leqslant \lambda_{max}(M)(x_0^{\mathrm{T}}x_0 + \dot{x}_0^{\mathrm{T}}\dot{x}_0/\omega_1^2), \tag{29}$$

for all $t \geqslant 0$. Finally, using $x^{\mathrm{T}}(\cdot)x(\cdot)$ from inequality (29) in

$$\|x_i\|_\infty = \max_{t \geqslant 0} |x_i(t)| \leqslant \max_{t \geqslant 0} [x^{\mathrm{T}}(t)x(t)]^{1/2}, \tag{30}$$

we establish the bound in inequality (19). □

Remarks. (1) In reference [8], it is shown that if the system (1) is classically damped, i.e., if $CM^{-1}K = KM^{-1}C$, then the norms of all displacements of the system satisfy inequality (6). In this note, it is not assumed that the system (1) is classically damped. However, in *Theorem* 2.2, it is assumed that $C\,M^{-1}K + K\,M^{-1}C$ is positive semi-definite. Under this assumption, it is shown that the norms of all displacements of the system (1) satisfy inequality (19), which is the same as inequality (6).

It should be remarked that the identity $CM^{-1}K = KM^{-1}C$ implies that $CM^{-1}K + KM^{-1}C$ is positive semi-definite. The reason is as follows. If $CM^{-1}K = KM^{-1}C$, then the normalized damping matrix is diagonal (see, e.g., references [3 (pp. 144–145), 13]). That is,

$$\tilde{C} = U^{\mathrm{T}}CU = \mathrm{diag}\,[\tilde{c}_{11}, \tilde{c}_{22}, \ldots, \tilde{c}_{nn}], \tag{31}$$

where the diagonal elements of $\tilde{C}$ are non-negative real numbers due to the positive semi-definiteness of C. Let $y \in \mathbb{R}^n$ be a non-zero vector. Then,

$$\begin{aligned} y^{\mathrm{T}}CM^{-1}Ky &= y^{\mathrm{T}}(U^{-\mathrm{T}}U^{\mathrm{T}})C(UU^{-1})M^{-1}K(UU^{-1})y \\ &= (U^{-1}y)^{\mathrm{T}}(U^{\mathrm{T}}CU)(U^{-1}M^{-1}KU)(U^{-1}y) \\ &= (U^{-1}y)^{\mathrm{T}}\,\mathrm{diag}\,[\tilde{c}_{11}, \tilde{c}_{22}, \ldots, \tilde{c}_{nn}]\Omega^2(U^{-1}y), \end{aligned} \tag{32}$$

where equations (16b) and (31) were used in deriving the last identity in equation (32). The positive semi-definiteness of $CM^{-1}K$ follows from equation (32). Thus, $CM^{-1}K + KM^{-1}C = 2CM^{-1}K$ is positive semi-definite. Note, however, that when $CM^{-1}K + KM^{-1}C$ is positive semi-definite, it is not necessarily true that $CM^{-1}K = KM^{-1}C$. Consider, for instance, the following matrices:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0{\cdot}175 & -0{\cdot}075 & 0 \\ -0{\cdot}075 & 0{\cdot}125 & -0{\cdot}05 \\ 0 & -0{\cdot}05 & 0{\cdot}05 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{33}$$

It can be easily verified that the matrices M, C, and K are positive definite. The matrix

$$CM^{-1}K + KM^{-1}C = \begin{bmatrix} 0{\cdot}85 & -0{\cdot}6 & 0{\cdot}125 \\ -0{\cdot}6 & 0{\cdot}75 & -0{\cdot}325 \\ 0{\cdot}125 & -0{\cdot}325 & 0{\cdot}2 \end{bmatrix}, \tag{34}$$

has eigenvalues 0·0059, 0·3148, 1·4793, and hence is positive definite. However, it can be easily verified that $CM^{-1}K \neq KM^{-1}C$.

It is thus concluded that the class of systems, for which $CM^{-1}K = KM^{-1}C$ (classically damped systems), is a subclass of the systems for which $CM^{-1}K + KM^{-1}C$ is positive semi-definite. It happens that for systems encountered in practice, the matrix $CM^{-1}K + KM^{-1}C$ is usually positive semi-definite, where as the identity $CM^{-1}K = KM^{-1}C$ rarely holds.

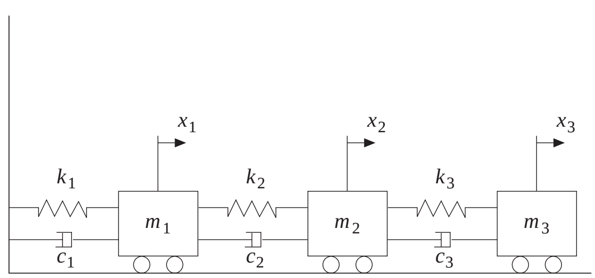

Figure 1. A system with three degrees of freedom, where $m_i = 1$ and $k_i = 1$ for all $i = 1, 2, 3$, and $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}075$, $c_3 = 0{\cdot}05$.

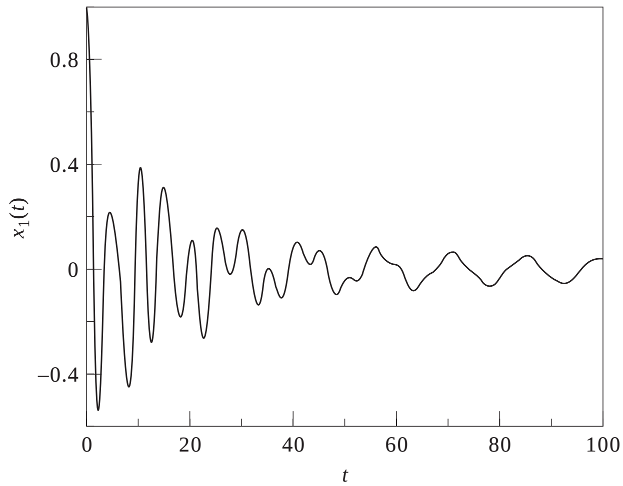

Figure 2. The time history of $x_1(\cdot)$ of the system (36).

(2) Let the mass matrix be $M = \text{diag}\,[m_1, m_2, \ldots, m_n]$, where, without the loss of generality, the diagonal elements are ordered as $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$. In this case, inequality (19) simplifies to

$$\|x_i\|_\infty \leqslant [(m_n/m_1)(x_0^{\mathrm{T}} x_0 + \dot{x}_0^{\mathrm{T}} \dot{x}_0/\omega_1^2)]^{1/2}, \tag{35}$$

for all $i = 1, 2, \ldots, n$.

(3) The bound in inequality (19) (respectively, inequality (35)) is tighter when the ratio $\lambda_{max}(M)/\lambda_{min}(M)(m_n/m_1$ for a diagonal $M)$ is not much larger than one. □

3. EXAMPLE

In this section, we give an example to study the bounds presented in this note. Consider the system in Figure 1 and let $m_i = 1$ and $k_i = 1$ for all $i = 1, 2, 3$, and $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}075$, $c_3 = 0{\cdot}05$. The free vibration of this system is represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}175 & -0{\cdot}075 & 0 \\ -0{\cdot}075 & 0{\cdot}125 & -0{\cdot}05 \\ 0 & -0{\cdot}05 & 0{\cdot}05 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \theta_3, \tag{36a}$$

for all $t \geqslant 0$, with the initial conditions

$$x_0 = [1 \quad 0 \quad 0]^{\mathrm{T}}, \qquad \dot{x}_0 = \theta_3. \tag{36b}$$

Identifying the matrices M and K in equation (36a), we can obtain

$$E_0 = \tfrac{1}{2} x_0^{\mathrm{T}} K x_0 + \tfrac{1}{2} \dot{x}_0^{\mathrm{T}} M \dot{x}_0 = 1, \qquad K^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{37a, b}$$

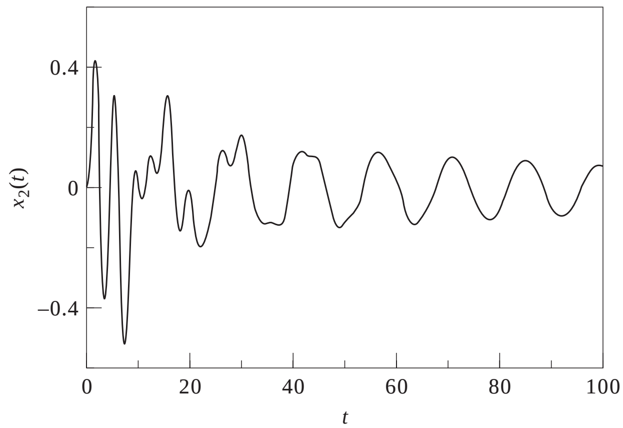

Figure 3. The time history of $x_2(\cdot)$ of the system (36).

Therefore, by inequality (5), we obtain

$$\|x_1\|_\infty \leqslant 1\cdot4142, \qquad \|x_2\|_\infty \leqslant 2, \qquad \|x_3\|_\infty \leqslant 2\cdot4495. \tag{38}$$

The coefficient matrices of the system (36a) are those in equatin (33). Therefore, the system (36) is not classically damped. The matrix $CM^{-1}K + KM^{-1}C$, however, is positive definite. Therefore, the upper bound in inequality (19) can be used, by which

$$\|x_1\|_\infty \leqslant 1, \qquad \|x_2\|_\infty \leqslant 1, \qquad \|x_3\|_\infty \leqslant 1. \tag{39}$$

By the numerical integration, we obtain responses of the system (36) as depicted in Figures 2, 3, and 4, From these figures, we obtain

$$\|x_1\|_\infty = 1, \qquad \|x_2\|_\infty = 0\cdot516, \qquad \|x_3\|_\infty = 0\cdot575. \tag{40}$$

Comparing the exact values of $\|x_i\|_\infty$ for all $i = 1, 2, 3$ in equation (40) and their corresponding upper bounds in inequalities (38) and (39), we conclude that the bounds in inequalities (39) are tighter than those in inequalities (38). We,

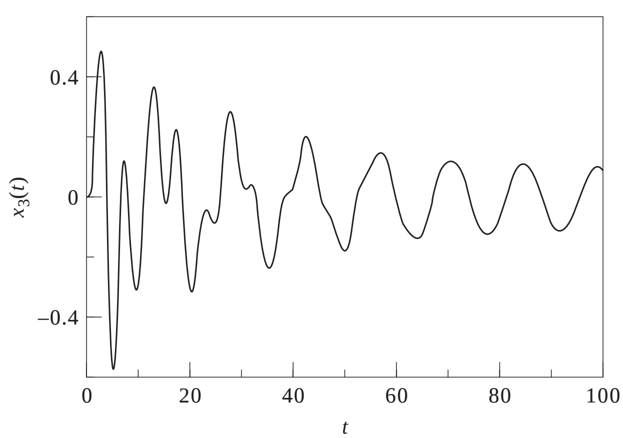

Figure 4. The time history of $x_3(\cdot)$ of the system (36).

however, point out that inequality (19) does not always result in bounds tighter than those obtained by inequality (5). There can be systems and initial conditions for which the bounds obtained by inequality (19) are more conservative than the corresponding bounds obtained by inequality (5).

4. CONCLUSIONS

In this note, the free vibration of n-degree-of-freedom linear non-classically damped second order systems is considered. A single and easy-to-compute upper bound on the norms of displacements of such systems is derived. The upper bound depends on the ratio of the largest eigenvalue to the smallest eigenvalue of the mass matrix of the system, the lowest undamped natural frequency of the system, and the vectors of initial displacements and velocities. The upper bound is independent of the lowest natural frequency when the initial velocities are zero.

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