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COMMENTS ON "ON THE PHYSICAL INTERPRETATION OF PROPER ORTHOGONAL MODES IN VIBRATIONS"

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An account of the relationship between proper orthogonal decomposition (POD) and normal modes of vibration is presented by the authors of reference [1]. The authors also outlined an interesting historical review of POD and indicated that Kosambi was probably the originator. A geometric interpretation along the lines of singular systems analysis of state–space reconstruction is also provided. The objective of this letter is to add to the discussion initiated by the authors and to point out the potential problems with POD in order estimation, needed in state–space reconstruction of vibrating systems using time delay embedding. It may be noted that POD (or Karhunen–Loeve (KL) expansion) is closely related to the principal component analysis (PCA) and singular value decomposition (SVD). The use of KL expansion in stochastic structural dynamics can be found in reference [2]. Let $(X(t)\colon \alpha \leqslant t \leqslant \beta)$, α, $\beta \in R$, be a mean-square continuous time series and $Var\, X(t) \leqslant \infty$, $\forall t$. Then the generalised spectral representation of the time series $X(t)$ is referred to as the KL expansion. It is interesting to note that when t is a discrete variable, one has a finite collection of random variables $(X(1), \ldots, X(n))$ and the KL expansion reduces to the principal component analysis introduced by Hotelling in 1933 in his study of educational psychology [3]. The relation of the PCA to the algebraic eigenvalue problems encountered in vibrating systems has also been well-known to the structural dynamics community for quite some time [4]. Mees *et al*. [5] pointed out that the connection between POD and PCA was first noticed by Watanabe in 1965. Mees *et al*. also gave a lucid interpretation of PCA in terms of the SVD. It is interesting to observe that SVD was introduced by Beltrami in his attempt to teach bilinear forms to Italian Mathematics students [6] and subsequently Jordan [7], Sylvester [8], Schmidt [9] and Weyl [10] developed the method further. SVD has also been widely used in the context of substructuring problems in structural dynamics and Modal analysis [11]. In these cases SVD is found to provide an efficient and reliable tool in solving rank deficiency problems and model reduction. SVD is a frequently used tool to compute the number of active degrees of freedom or order of the model in system identification [12].

In chaotic dynamical systems, the state–space reconstruction based on Taken's theorem is often used. The advantage of this approach is that one can reconstruct the dynamics from the measurements of a single state variable. The use of SVD in selecting the best choice of embedding parameters of systems exhibiting chaotic

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attractors and the potential pitfalls in such attempts have been considered by physicists in the case of difference equations such as logistic map, Henon map and autonomous differential equations such as the Lorenz system. In all these cases, Mees *et al*. [5] have shown that while SVD is a useful tool for noise reduction of the time series, its use in choosing the number of active degrees of freedom is limited. Vibrating systems encountered in structural dynamics usually have time-dependent inputs and hence are non-autonomous. In the author's previous works the use of SVD in the state–space reconstruction of a chaotic time series obtained from the first mode response of a buckled beam and Duffing–Ueda systems is examined [12, 13]. The effect of passive non-linear damping inherent in structural elements on the singular value spectrum is also considered. Based on this work the problems with SVD in the context of state–space reconstruction of non-linear vibrating systems is illustrated with an example [4]:

Consider the Duffing–Holmes oscillator,

$$x'' + 2\zeta x' - x + x^3 = f\cos(\omega t). \tag{1}$$

Here ζ refers to the damping coefficient, f and ω refer to the amplitude and frequency of the external excitation. The above equation is integrated numerically using the usual Runge–Kutta method for selected parameter values ($\zeta = 0{\cdot}125$, $\omega = 1{\cdot}0, f = 0{\cdot}4$). It is known that for this set of parameter values a chaotic time series is obtained through the period-doubling route. This time series data set consisting of 20 000 points is obtained after ignoring the transients. The state–space reconstruction procedure is carried out using the time delay method. The space in which the dynamics is reconstructed is called the embedding space and its dimension is called the embedding dimension n. Normally, one chooses an *a priori* embedding dimension l and then varies l till consistent values of the embedding dimension n are obtained.

From the time series of the chaotic set one forms an embedding matrix **E** using the method of delays. Now one can compute the spectrum of singular values of **E** using the SVD. The rank of **E** is indicated by the decrease of the singular values. The regime where the singular values reach a constant value is termed as the noise floor. The number of singular values in the regime before approaching the noise floor (n') changes with the change in *a priori* embedding dimension l. It is conjectured in the literature that n' approaches a constant value as l is increased further and hence provides a rational basis for the choice of the embedding dimension n. It has also been pointed out in the literature that the existence of the noise floor in the singular value spectrum can also be used as a noise reduction strategy. If one knows the noise level *a priori*, then one can discard that subspace of the embedding matrix which corresponds to singular values below the noise threshold.

The SVD of the embedding matrix **E** is obtained for the time series data collected by integration of equation (1). The value of l is selected as 30. This value is in accordance with the Taken's theorem which states that the embedding dimension should be at least $(2D + 1)$ or higher where D is the dimension of the attractor. The fact that the selected value is much higher than the dimension of the system given by equation (1) ensures that Taken's theorem is satisfied. The obtained singular value spectrum is shown in Figure 1. It can be seen that there is an abrupt decrease in the values of singular spectrum indicating the horizontal noise floor. Only the first 20 singular values are shown in these figures. The remaining ones are found to lie on the noise floor and hence are omitted. One observes that the existence of horizontal floor establishes that SVD can be potentially used for noise reduction of the data obtained. The number of singular values (n') in the regime prior to the noise floor increases with increase in l and no asymptotic behaviour is observed. This is at variance with the conjecture suggested in the literature that n' is insensitive to the choice of *a priori* embedding dimension, for a high value of l. Hence, one can conclude that though there exists a noise floor in the singular spectrum (indicating that SVD can be used for noise reduction), its potential use in selecting the number of active degrees of freedom or order is limited in some systems exhibiting chaotic behaviour. For autonomous systems and difference equations similar conclusions were obtained by Mees *et al*. [5].

A potential problem with SVD is that it cannot distinguish two time series having the same covariance structure but differing in higher order structure [14]. For a chaotic time series, the embedding dimension n is equal to the *a priori* embedding dimension l, however large l might be. The main reason for this is that the co-ordinates in SVD become the Fourier coefficients in the limit of large l. In the case of a chaotic time series, it is possible that all the Fourier coefficients are non-vanishing. Hence, the use of SVD in obtaining "order" of a chaotic time series needs to be carefully examined. It is possible in the case of hyperbolic systems to map the Taken's state–space reconstruction to a non-linear AR process with appropriate assumptions. There exist order estimation schemes for such non-linear AR processes based on SVD. A comprehensive account of order estimation

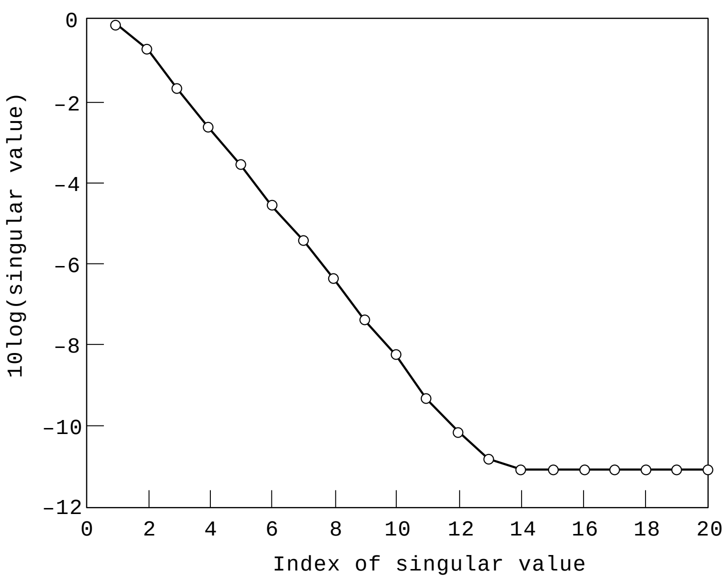

Figure 1

of non-linear vibrating systems and its relation to the non-linear analogs of Akaike's Information Criterion will be presented in a future work.

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