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FREE VIBRATION ANALYSIS OF CURVED AND TWISTED CYLINDRICAL THIN PANELS

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A turbomachinery blade is treated as a cylindrical thin panel with curvature and twist. The non-linear strain–displacement relations of the model are derived based on the general thin shell theory, and a numerical method for analysing the free vibrations of curved and twisted cylindrical thin panels is presented by means of the principle of virtual work for the free vibration using the Rayleigh–Ritz method, assuming two dimensional polynomial functions as displacement functions. It is shown that the method is effective in solving free vibration problems for cylindrical thin panels with curvature and twist by comparing the numerical results with previous results. The effects of curvature and twist on the frequency parameters and mode shapes are also discussed.

1. INTRODUCTION

Blades are often part of machinery rotating at high speed, so it is very important to ensure safety while rotating. The configuration of turbomachinery blades is complex, usually thin with a small aspect ratio, twisted in the lengthwise direction and cambered in the chordwise direction. That is the reason why so many researchers have studied them for the past few decades.

There are hundreds of references related to the vibration problems of turbomachinery blades in published literature. It is known that the important things are what kind of a model is used for representing the blades and how the governing equations of the model are established. In the early researches, the vibration analysis of the blades was based on beam theory, which has some limitations, and is inadequate for evaluating the higher vibration frequencies and modes of a thin blade with a small aspect ratio. Two-dimensional models were then introduced to study the vibration of the blades.

A comprehensive review of the research advances in the vibration of shallow shells since the 1970s was presented by Liew *et al*. [1]. A series of studies using the Rayleigh–Ritz method was begun in the early 1980s. More than a decade ago, Leissa and his co-workers presented a doubly-curved shallow shell model having rectangular planform [2–5] which can be used for analysing blades having relatively small double curvatures very well, but is inadequate for blades having

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large curvatures and large twist. Liew *et al*. presented shallow shell models [1], such as the shallow conical shell model, the shallow cylindrical model with variable thickness, and analysed the vibration characteristics of these models. Leissa, Liew and their co-workers have done a lot of work on the vibrations of the turbomachinery blades and the effects of geometric parameters on the vibration characteristics based on shallow shell theory.

On the other hand, Tsuiji *et al*. proposed that a blade be treated as a thin plate having curvature and twist [6, 7], in which they studied the effects of curvature, twist and other geometric parameters on the vibration characteristics of the model in detail using general thin shell theory. In particular, they presented a model of a cylindrical thin panel having twist which represented turbomachinery blades [8]. The Rayleigh–Ritz method was used for the numerical procedure of vibration analysis of the model. Compared with other methods, it was verified that the procedure is effective and accurate for dealing with a cylindrical thin panel having large twist and chordwise curvature. But, the vibrations of shell models having both spanwise and chordwise curvatures, as well as twist, have not been previously studied.

It was pointed out that the non-linear strain–displacement relations are necessary for analysing the vibration of rotating blades [6]. In the present paper, turbomachinery blades are treated as cylindrical thin panels having twist, chordwise and spanwise curvatures. The non-linear strain–displacement relationships of the model are derived and then the principle of virtual work for free vibration is formulated. The Rayleigh–Ritz method, assuming two-dimensional algebraic polynomials as displacement functions, is used for analysing the vibration of the model. The effectiveness of this method is verified by comparing the present results with the available results [5]. Finally, the effects of the geometric parameters of the model on the vibration characteristics are studied.

2. STRAIN–DISPLACEMENT RELATIONSHIPS

A model of turbomachinery blades shown in Figure 1 is considered, which is a cylindrical thin panel having twist, chordwise and spanwise curvatures. There are two co-ordinate systems in Figure 1. One is a right-hand co-ordinate system

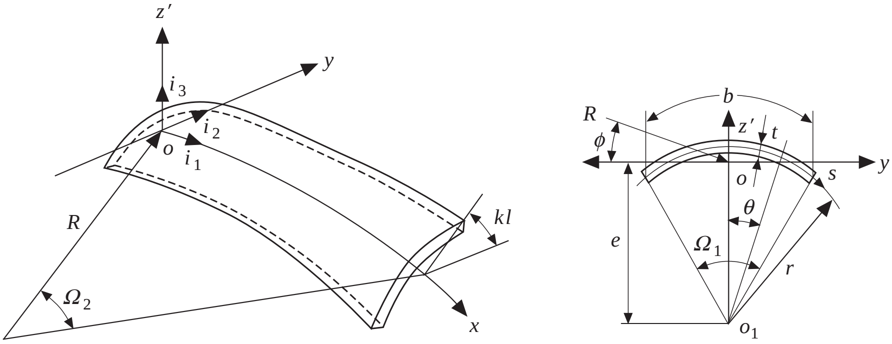

Figure 1. A curved and twisted cylindrical thin panel.

(x, y, z') with the origin O, where x is a curvilinear axis in the lengthwise direction, which is considered as a twisting center axis, the z'-axis takes in the radial direction where the cylindrical arc is equally divided into two parts, and O is a point on the z'-axis. Another is a cylindrical co-ordinate system with the origin O_1 at the center of the cylinder, where θ is an angle measured from the z'-axis, and s-axis takes in the circumferential direction on the middle surface. Ω_1, r and b are the central angle, the average radius and the arc length of the cylindrical panel respectively. $1/R$ is the curvature of the x-axis, l is the length of the panel along the x-axis, Ω_2 is the central angle ($\Omega_2 = l/R$), k is the twist angle per unit length, e is the distance between the two points O and O_1, and the z-axis is normal to the middle surface with the outward direction considered positive. ϕ is the angle between the y-axis and the radial direction of the x-axis in the fixed end of the panel. kl is the twist angle at the free end of the panel.

$\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are the unit vectors of the co-ordinate system (x, y, z'). A position vector of a point on the middle surface before deformation is defined by $\mathbf{r}_c^{(0)}$

$$\mathbf{r}_c^{(0)} = \boldsymbol{\rho}_x + r \sin\theta\, \mathbf{i}_2 + (r\cos\theta - e)\mathbf{i}_3, \tag{1}$$

where $\boldsymbol{\rho}_x$ is the position vector of the point on the x-axis.

The partial derivatives of $\mathbf{r}_c^{(0)}$ with respect to x and s $(r\theta)$ are defined as the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ on the middle surface, respectively,

$$\mathbf{a}_1 = \frac{\partial \mathbf{r}_c^{(0)}}{\partial x} = A\mathbf{i}_1 + k(e - r\cos\theta)\mathbf{i}_2 + kr\sin\theta\mathbf{i}_3, \tag{2}$$

$$\mathbf{a}_2 = \frac{\partial \mathbf{r}_c^{(0)}}{r\partial\theta} = \cos\theta\mathbf{i}_2 - \sin\theta\mathbf{i}_3. \tag{3}$$

A unit vector perpendicular to the middle surface is denoted by $\mathbf{a}_3$, which is chosen so that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ form a right-hand orthogonal system,

$$\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2/|\mathbf{a}_1 \times \mathbf{a}_2| = (1/B)(-ek\sin\theta\mathbf{i}_1 + A\sin\theta\mathbf{i}_2 + A\cos\theta\mathbf{i}_3), \tag{4}$$

where A and B are defined in Appendix, and other coefficients used in the following equations are also defined in Appendix.

An arbitrary point P outside the middle surface of the panel is considered and its position vector $\mathbf{r}^{(0)}$ before deformation is given by

$$\mathbf{r}^{(0)} = \mathbf{r}_c^{(0)} + z\mathbf{a}_3, \tag{5}$$

where z is the distance of the point P from the middle surface in the direction of $\mathbf{a}_3$. The partial derivatives of $\mathbf{r}^{(0)}$ with respect to x, s and z are defined as base vectors $\mathbf{g}_i$ $(i = 1, 2, 3)$ respectively. They are given by

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}^{(0)}}{\partial x} = (A + zc_{11})\mathbf{i}_1 + \{k(e - r\cos\theta) + zc_{12}\}\mathbf{i}_2 + (kr\sin\theta + zc_{13})\mathbf{i}_3,$$

$$\mathbf{g}_2 = \frac{\partial \mathbf{r}^{(0)}}{r\partial\theta} = zc_{21}\mathbf{i}_1 + (\cos\theta + zc_{22})\mathbf{i}_2 - (\sin\theta - zc_{23})\mathbf{i}_3,$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{r}^{(0)}}{\partial z} = -(ek/B)\sin\theta\mathbf{i}_1 + (A/B)\sin\theta\mathbf{i}_2 + (A/B)\cos\theta\mathbf{i}_3. \tag{6}$$

The displacement vector $\mathbf{U}$ of the point P is defined by

$$\mathbf{U} = U(x, s, z)\mathbf{a}_1 + V(x, s, z)\mathbf{a}_2 + W(x, s, z)\mathbf{a}_3, \tag{7}$$

where $U(x, s, z)$, $V(x, s, z)$ and $W(x, s, z)$ are the components in the directions of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively.

The position vector $\mathbf{r}$ of the point P after deformation is given by

$$\begin{aligned}\mathbf{r} = \mathbf{r}^{(0)} + \mathbf{U} = \mathbf{r}^{(0)} &+ \{AU - (ek/B)\sin\theta W\}\mathbf{i}_1 \\ &+ \{k(e - r\cos\theta)U + \cos\theta V + (A/B)\sin\theta W\}\mathbf{i}_2 \\ &+ \{kr\sin\theta U - \sin\theta V + (A/B)\cos\theta W\}\mathbf{i}_3.\end{aligned} \tag{8}$$

The partial derivatives of $\mathbf{r}$ with respect to x, s and z are defined as base vectors $\mathbf{G}_i$ $(i = 1, 2, 3)$ respectively. They are

$$\mathbf{G}_1 = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{g}_1 + G_{11}\mathbf{i}_1 + G_{12}\mathbf{i}_2 + G_{13}\mathbf{i}_3, \qquad \mathbf{G}_2 = \frac{\partial \mathbf{r}}{r\,\partial\theta} = \mathbf{g}_2 + G_{21}\mathbf{i}_1 + G_{22}\mathbf{i}_2 + G_{23}\mathbf{i}_3,$$

$$\mathbf{G}_3 = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{g}_3 + G_{31}\mathbf{i}_1 + G_{32}\mathbf{i}_2 + G_{33}\mathbf{i}_3, \tag{9}$$

where G_{ij} $(i, j = 1, 2, 3)$ are expressed as follows:

$$G_{11} = A\frac{\partial U}{\partial x} - \frac{ek}{B}\sin\theta\frac{\partial W}{\partial x} + \frac{2kp}{R}U + \frac{1}{R}\cos(kx + \phi - \theta)V + c_{11}W,$$

$$G_{12} = ke_1\frac{\partial U}{\partial x} + \cos\theta\frac{\partial V}{\partial x} + \frac{A}{B}\sin\theta\frac{\partial W}{\partial x} - \left\{\frac{A}{R}\cos(kx + \phi) + k^2 r\sin\theta\right\}U$$
$$+ k\sin\theta V + c_{12}W,$$

$$G_{13} = kr\sin\theta\frac{\partial U}{\partial x} - \sin\theta\frac{\partial V}{\partial x} + \frac{A}{B}\cos\theta\frac{\partial W}{\partial x} + \left\{\frac{A}{R}\sin(kx + \phi) + k^2 e_1\right\}U$$
$$+ k\cos\theta V + c_{13}W,$$

$$G_{21} = \frac{1}{r}\left\{A\frac{\partial U}{\partial\theta} - \frac{ek}{B}\sin\theta\frac{\partial W}{\partial\theta} + \frac{r}{R}\cos(kx + \phi - \theta)U + rc_{21}W\right\},$$

$$G_{22} = \frac{1}{r}\left(ke_1\frac{\partial U}{\partial\theta} + \cos\theta\frac{\partial V}{\partial\theta} + \frac{A}{B}\sin\theta\frac{\partial W}{\partial\theta} + kr\sin\theta U - \sin\theta V + rc_{22}W\right),$$

$$G_{23} = \frac{1}{r}\left(kr\sin\theta\frac{\partial U}{\partial\theta} - \sin\theta\frac{\partial V}{\partial\theta} + \frac{A}{B}\cos\theta\frac{\partial W}{\partial\theta} + kr\cos\theta U - \cos\theta V + rc_{23}W\right),$$

$$G_{31} = A\frac{\partial U}{\partial z} - \frac{ek}{B}\sin\theta\frac{\partial W}{\partial z},$$

$$G_{32} = ke_1\frac{\partial U}{\partial z} + \cos\theta\frac{\partial V}{\partial z} + \frac{A}{B}\sin\theta\frac{\partial W}{\partial z},$$

$$G_{33} = kr\sin\theta\frac{\partial U}{\partial z} - \sin\theta\frac{\partial V}{\partial z} + \frac{A}{B}\cos\theta\frac{\partial W}{\partial z}. \tag{10}$$

By the use of equations (6) and (9), the Green strain tensors f_{ij} $(i, j = 1, 2, 3)$ can be obtained from the following equation

$$2f_{ij} = \mathbf{G}_i\mathbf{G}_j - \mathbf{g}_i\mathbf{g}_j \quad (i, j = 1, 2, 3), \tag{11}$$

namely,

$$\begin{aligned}
2f_{11} = {} & 2\{B^2 + k^2e_2^2 + z(c_{11}A + c_{12}ke_1 + c_{13}kr\sin\theta)\}\frac{\partial U}{\partial x} \\
& + 2\{ke_2 + z(c_{12}\cos\theta - c_{13}\sin\theta)\}\frac{\partial V}{\partial x} \\
& + 2\left[\frac{Ak}{R}p + z\left\{c_{11}\frac{2k}{R}p - c_{12}\left(\frac{A}{R}\cos(kx+\phi) + k^2r\sin\theta\right)\right.\right. \\
& \left.\left. + c_{13}\left(\frac{A}{R}\sin(kx+\phi) + k^2e_1\right)\right\}\right]U \\
& + 2\left[ek^2\sin\theta + \frac{A}{R}\cos(kx+\phi-\theta) + z\left\{c_{11}\frac{1}{R}\cos(kx+\phi-\theta)\right.\right. \\
& \left.\left. + c_{12}k\sin\theta + c_{13}k\cos\theta\right\}\right]V \\
& + 2\{c_{11}A + c_{12}ke_1 + c_{13}kr\sin\theta + z(c_{11}^2 + c_{12}^2 + c_{13}^2)\}W + G_{11}^2 \\
& + G_{12}^2 + G_{13}^2,
\end{aligned}$$

$$\begin{aligned}
2f_{12} = {} & \frac{1}{r}\{B^2 + k^2e_2^2 + z(c_{11}A + c_{12}ke_1 + c_{13}kr\sin\theta)\}\frac{\partial U}{\partial \theta} \\
& + \frac{1}{r}\{ke_2 + z(c_{12}\cos\theta - c_{13}\sin\theta)\}\frac{\partial V}{\partial \theta} \\
& + \{ke_2 + z(c_{21}A + c_{22}ke_1 + c_{23}kr\sin\theta)\}\frac{\partial U}{\partial x} \\
& + \{1 + z(c_{22}\cos\theta - c_{23}\sin\theta)\}\frac{\partial V}{\partial x} \\
& + z\left[\left\{c_{11}\frac{1}{R}\cos(kx+\phi-\theta) + c_{12}k\sin\theta + c_{13}k\cos\theta\right\}\right. \\
& + c_{21}\frac{2k}{R}p - c_{22}\left\{\frac{A}{R}\cos(kx+\phi) + k^2r\sin\theta\right\} \\
& \left. + c_{23}\left\{\frac{A}{R}\sin(kx+\phi) + k^2e_1\right\}\right]U
\end{aligned}$$

$$+\left[-\frac{ek}{r}\sin\theta+z\left\{-\frac{1}{r}(c_{12}\sin\theta+c_{13}\cos\theta)\right.\right.$$

$$\left.\left.+c_{21}\frac{1}{R}\cos(kx+\phi-\theta)+c_{22}k\sin\theta+c_{23}k\cos\theta\right\}\right]V$$

$$+2\left\{\frac{ek}{BR}\sin\theta\cos(kx+\phi-\theta)-\frac{Ak}{B}+z(c_{11}c_{21}+c_{12}c_{22}+c_{13}c_{23})\right\}W$$

$$+G_{11}G_{21}+G_{12}G_{22}+G_{13}G_{23},$$

$$2f_{13}=\{B^2+k^2e_2^2+z(c_{11}A+c_{12}ke_1+c_{13}kr\sin\theta)\}\frac{\partial U}{\partial z}$$

$$+\{ke_2+z(c_{12}\cos\theta-c_{13}\sin\theta)\}\frac{\partial V}{\partial z}$$

$$+\frac{\partial W}{\partial x}+\left\{\frac{A^2}{BR}\sin(kx+\phi-\theta)-\frac{2ek^2}{BR}p\sin\theta+\frac{Ak^2}{B}e_2\right\}U$$

$$+\left\{-\frac{ek}{BR}\sin\theta\cos(kx+\phi-\theta)+\frac{Ak}{B}\right\}V$$

$$+G_{11}G_{31}+G_{12}G_{32}+G_{13}G_{33},$$

$$2f_{22}=\frac{2}{r}\{ke_2+z(c_{21}A+c_{22}ke_1+c_{23}kr\sin\theta)\}\frac{\partial U}{\partial\theta}$$

$$+\frac{2}{r}\{1+z(c_{22}\cos\theta-c_{23}\sin\theta)\}\frac{\partial V}{\partial\theta}$$

$$+2z\left\{c_{21}\frac{1}{R}\cos(kx+\phi-\theta)+c_{22}k\sin\theta+c_{23}k\cos\theta\right\}U$$

$$-z\frac{2}{r}(c_{22}\sin\theta+c_{23}\cos\theta)V$$

$$+\frac{2}{r}\left\{\frac{A}{B}+zr(c_{21}^2+c_{22}^2+c_{23}^2)\right\}W+G_{21}^2+G_{22}^2+G_{23}^2,$$

$$2f_{23}=\frac{1}{r}\frac{\partial W}{\partial\theta}+\{ke_2+z(c_{21}A+c_{22}ke_1+c_{23}kr\sin\theta)\}\frac{dU}{\partial z}$$

$$+\{1+z(c_{22}\cos\theta-c_{23}\sin\theta)\}\frac{\partial V}{\partial z}$$

$$+\left\{\frac{Ak}{B}-\frac{ek}{BR}\sin\theta\cos(kx+\phi-\theta)\right\}U$$

$$-\frac{A}{Br}V+G_{21}G_{31}+G_{22}G_{32}+G_{23}G_{33},$$

$$2f_{33}=2\frac{\partial W}{\partial z}+G_{31}^2+G_{32}^2+G_{33}^2. \tag{12}$$

A local orthogonal co-ordinate system (ξ, η, ζ) issuing from the point P is introduced, and the strains ϵ_{ij} $(i, j = 1, 2, 3)$ of the point P with respect to this co-ordinate system can be obtained from the following equation (reference [9]):

$$2\epsilon_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \alpha_k}{\partial \beta_i} \frac{\partial \alpha_l}{\partial \beta_j} 2f_{kl} \quad (i, j = 1, 2, 3). \tag{13}$$

If $(\alpha_1, \alpha_2, \alpha_3)$ are used instead of (ξ, η, ζ) and $(\beta_1, \beta_2, \beta_3)$ in place of (x, s, z), the coefficients $\partial \alpha_k / \partial \beta_i$ are given by

$$\frac{\partial x}{\partial \xi} = \frac{1}{F}\left(1 + z\frac{A}{Br}\right), \qquad \frac{\partial x}{\partial \eta} = z\frac{1}{F}\frac{ek}{B^2 r} h_3, \qquad \frac{\partial x}{\partial \zeta} = 0,$$

$$\frac{\partial s}{\partial \xi} = \frac{1}{F}\left\{k(r - e\cos\theta) + z\frac{k}{B}h_1\right\},$$

$$\frac{\partial s}{\partial \eta} = \frac{1}{F}(B + zh_4), \qquad \frac{\partial s}{\partial \zeta} = 0, \qquad \frac{\partial z}{\partial \xi} = 0, \qquad \frac{\partial z}{\partial \eta} = 0, \qquad \frac{\partial z}{\partial \zeta} = 1,$$

$$F = B(1 + zp_1) \tag{14}$$

The non-linear strains e_{ij} $(i, j = \xi, \eta, \zeta)$ with respect to the local co-ordinate system can be written as

$$e_{\xi\xi} = \epsilon_{11} = U_\xi + \tfrac{1}{2}U_\xi^2 + \tfrac{1}{2}V_\xi^2 + \underline{\tfrac{1}{2}W_\xi^2}, \qquad e_{\eta\eta} = \epsilon_{22} = V_\eta + \tfrac{1}{2}U_\eta^2 + \tfrac{1}{2}V_\eta^2 + \underline{\tfrac{1}{2}W_\eta^2},$$

$$e_{\zeta\zeta} = \epsilon_{33} = W_\zeta + \tfrac{1}{2}U_\zeta^2 + \tfrac{1}{2}V_\zeta^2 + \underline{\tfrac{1}{2}W_\zeta^2},$$

$$e_{\xi\eta} = 2\epsilon_{12} = U_\eta + V_\xi + U_\xi U_\eta + V_\xi U_\eta + \underline{W_\xi W_\eta},$$

$$e_{\xi\zeta} = 2\epsilon_{13} = U_\zeta + W_\xi + U_\xi U_\zeta + V_\xi V_\zeta + \underline{W_\xi W_\zeta},$$

$$e_{\eta\zeta} = 2\epsilon_{23} = V_\zeta + W_\eta + U_\eta U_\zeta + V_\eta V_\zeta + \underline{W_\eta W_\zeta}, \tag{15}$$

where

$$U_\xi = \frac{1}{F}\left[B\left(1 + z\frac{A}{Br}\right)\frac{\partial U}{\partial x} + \frac{k}{r}(-Be_2 + zh_1)\frac{\partial U}{\partial \theta}\right.$$

$$+ \left\{\frac{Ak}{BR}p + z\frac{1}{B^2 r}\left(\frac{A^2 k}{R}p + ekh_3 h_2\right)\right\}U$$

$$+ \left\{\frac{1}{Br}q + z\frac{1}{Rr}\cos(kx + \phi - \theta)\right\}V$$

$$\left. + \left\{h_4 + \frac{ek^2}{B^2 r}h_3 e_2 + z\frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1 h_3\right)\right\}W\right],$$

$$V_\xi = \frac{1}{F}\left[ke_2\left(1 + z\frac{A}{Br}\right)\frac{\partial U}{\partial x} + \left(1 + z\frac{A}{Br}\right)\frac{\partial V}{\partial x} - \frac{k^2}{r}e_2\left(e_2 - z\frac{1}{B}h_1\right)\frac{\partial U}{\partial \theta}\right.$$

$$\left. + \frac{k}{r}\left(-e_2 + z\frac{1}{B}h_1\right)\frac{\partial V}{\partial \theta} - \left(1 + z\frac{A}{Br}\right)h_2 U - \frac{ek}{Br}h_3 W\right],$$

$$W_\xi = \frac{1}{F}\left[\left\{-Bh_4 + z\frac{1}{r}\left(\frac{ek^2}{B^2}h_1 h_3 - Ah_4\right)\right\}U + \frac{ek}{Br}h_3 V + \left(1 + z\frac{A}{Br}\right)\frac{\partial W}{\partial x}\right.$$

$$\left. + \frac{k}{r}\left(-e_2 + z\frac{1}{B}h_1\right)\frac{\partial W}{\partial \theta}\right],$$

$$U_\eta = \frac{1}{F}\left[z\frac{ek}{Br}h_3\frac{\partial U}{\partial x} + \left\{h_2 + z\frac{1}{B}\left(h_2 h_4 + \frac{Aek^2}{B^2 Rr}ph_3 + \frac{ek^2}{B^2 r}(e\cos\theta - r)h_2 h_3\right)\right\}U\right.$$

$$\left. - \frac{ek}{r}\left\{\sin\theta + z\frac{1}{B}\left(h_4\sin\theta - \frac{1}{B^2}h_2 h_3\right)\right\}V - \frac{ek}{Br}h_3 W + \frac{B}{r}(B + zh_4)\frac{\partial U}{\partial \theta}\right],$$

$$V_\eta = \frac{1}{F}\left[\frac{k}{r}e_2(B + zh_4)\frac{\partial U}{\partial \theta} + \frac{1}{r}(B + zh_4)\frac{\partial V}{\partial \theta} + z\frac{ek^2}{B^2 r}h_3 e_2\frac{\partial U}{\partial x}\right.$$

$$\left. + z\frac{ek}{B^2 r}h_3\frac{\partial V}{\partial x} - z\frac{ek}{B^2 r}h_3 h_2 U + \left\{\frac{A}{r} + z\frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1 h_3\right)\right\}W\right],$$

$$W_\eta = \frac{1}{F}\left[\left\{kh_1 - z\frac{k}{Br}e_2\left(Ah_4 - \frac{ek^2}{B^2}h_1 h_3\right)\right\}U - \left\{\frac{A}{r} + z\frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1 h_3\right)\right\}V\right.$$

$$\left. + \frac{1}{r}(B + zh_4)\frac{\partial W}{\partial \theta} + z\frac{ek}{B^2 r}h_3\frac{\partial W}{\partial x}\right],$$

$$U_\zeta = B\frac{\partial U}{\partial z}, \qquad V_\zeta = ke_2\frac{\partial U}{\partial z} + \partial V/\partial z, \qquad W_\zeta = \frac{\partial W}{\partial z}. \tag{16}$$

The non-linear strains are significant in the analysis of the vibrations of curved and twisted thin cylindrical panels rotating at high speed. In this paper, the components of the non-linear strains, which also appear in the non-linear strains of plates in the same manner, are considered, and the underlined terms in equation (14) are adopted.

For thin panels, it can be assumed that the cross-sections of the panels which are perpendicular to the middle surface before deformation remain plane and perpendicular to the deformed middle surface and suffer no strains in their plane. A unit vector $\mathbf{n}$ normal to the deformed middle surface is defined as

$$\mathbf{n} = (\mathbf{G}_1 \times \mathbf{G}_2)_{z=0}/|(\mathbf{G}_1 \times \mathbf{G}_2)|_{z=0} = \Gamma_1\mathbf{a}_1 + \Gamma_2\mathbf{a}_2 + \Gamma_3\mathbf{a}_3, \tag{17}$$

where Γ_1, Γ_2 and Γ_3 are the components in the directions of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. From equations (9) and (16), Γ_i ($i = 1, 2, 3$) can be given as follows:

$$\Gamma_1 = \frac{1}{B}\left(h_4 u - \frac{ek}{B^2 r} h_3 v - \frac{1}{B}\frac{\partial w}{\partial x} + \frac{k}{Br} e_2 \frac{\partial w}{\partial \theta}\right),$$

$$\Gamma_2 = -\frac{k}{B}\left(1 + \frac{ek^2}{B^2 R} p \sin\theta e_2\right) u + \frac{1}{Br}\left(\frac{ek^2}{B^2} h_3 e_2 + A\right) v$$

$$+ \frac{k}{B^2} e_2 \frac{\partial w}{\partial x} - \frac{1}{r}\left(1 + \frac{k^2}{B^2} e_2^2\right)\frac{\partial w}{\partial \theta},$$

$$\Gamma_3 = 1 + \frac{Ak}{B^2 R} pu + \frac{1}{B^2 r} qv + \frac{1}{B}\left(\frac{A}{r} + \frac{ek^2}{B^2 r} h_3 e_2 + h_4\right) w + \frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial v}{\partial \theta} \doteq 1, \quad (18)$$

where u, v and w are the displacements of the point on the middle surface in $\mathbf{a}_i$ ($i = 1, 2, 3$) directions and two-dimensional functions with respect to x and θ. By the use of equation (17), for an arbitrary point in a curved and twisted cylindrical thin panel, the displacements U, V and W can be expressed by

$$U = u + z\Gamma_1 = \left(1 + z\frac{1}{B} h_4\right) u - z\frac{ek}{B^3 r} h_3 v - z\frac{1}{B^2}\frac{\partial w}{\partial x} + z\frac{k}{B^2 r} e_2 \frac{\partial w}{\partial \theta},$$

$$V = -z\frac{k}{B}\left(1 + \frac{ek^2}{B^2 R} p \sin\theta e_2\right) u + \left\{1 + z\frac{1}{Br}\left(A + \frac{ek^2}{B^2} h_3 e_2\right)\right\} v + z\frac{k}{B^2} e_2 \frac{\partial w}{\partial x}$$

$$-z\frac{1}{r}\left(1 + \frac{k^2}{B^2} e_2^2\right)\frac{\partial w}{\partial \theta}, \qquad W = w + z(\Gamma_3 - 1) = w. \quad (19)$$

Substituting equation (19) into equation (15), the non-linear strain–displacement relations of a curved and twisted cylindrical thin panel can be rewritten as follows:

$$e_{\xi\xi} = \frac{1}{F}(\Omega_\xi + z\Gamma_\xi + z^2\Phi_\xi) + \tfrac{1}{2}\Psi_\xi^2, \qquad e_{\eta\eta} = \frac{1}{F}(\Omega_\eta + z\Gamma_\eta + z^2\Phi_\eta) + \tfrac{1}{2}\Psi_\eta^2,$$

$$e_{\zeta\zeta} = 0, \qquad e_{\xi\eta} = \frac{1}{F}(\Omega_{\xi\eta} + z\Gamma_{\xi\eta} + z^2\Phi_{\xi\eta}) + \Psi_\xi\Psi_\eta, \qquad e_{\xi\zeta} = 0, \qquad e_{\eta\zeta} = 0, \quad (20)$$

where

$$\Omega_\xi = B\frac{\partial u}{\partial x} - \frac{Bk}{r}e_2\frac{\partial u}{\partial \theta} + \frac{Ak}{BR}pu + \frac{1}{Br}qv + \left(h_4 + \frac{ek^2}{B^2r}h_3e_2\right)w,$$

$$\Gamma_\xi = \left(\frac{A}{r} + h_4\right)\frac{\partial u}{\partial x} + \frac{k}{r}(h_1 - h_4e_2)\frac{\partial u}{\partial \theta} + \frac{k}{R}\left\{-\frac{ek^2}{B^2r}p\cos\theta e_2 - \frac{2Aek^2}{B^4R}p^2\sin\theta\right.$$

$$+\frac{2ek^2}{B^4r}pq\sin\theta e_2 - \cos(kx+\phi-\theta) - \frac{e^2k^2}{B^2}\sin^2\theta\cos(kx+\phi-\theta)$$

$$\left.+\frac{A^2}{B^2r}p\right\}u - \frac{ek}{B^2r}h_3\frac{\partial v}{\partial x} + \frac{ek^2}{B^2r^2}h_3e_2\frac{\partial v}{\partial \theta}$$

$$+\frac{1}{r}\left\{\frac{A}{B^2}h_2 - \frac{ek^2}{B^2R}e_2\cos\theta\cos(kx+\phi-\theta) + \frac{2Aek^2}{B^4R}ph_3 - \frac{2ek^2}{B^4r}h_3qe_2\right.$$

$$\left.+\frac{1}{R}\cos(kx+\phi-\theta)\right\}v$$

$$-\frac{1}{B}\frac{\partial^2 w}{\partial x^2} - \frac{k^2}{Br^2}e_2^2\frac{\partial^2 w}{\partial \theta^2} + \frac{2k}{Br}e_2\frac{\partial^2 w}{\partial x\,\partial \theta} + \frac{k}{B^3}\left(\frac{A}{R}p - \frac{1}{r}qe_2\right)\frac{\partial w}{\partial x}$$

$$-\frac{1}{Br}\left\{-\frac{ek^2}{r}\sin\theta e_2 + \frac{Ak^2}{B^2R}pe_2 + \frac{1}{B^2r}q(B^2 - k^2e_2^2)\right\}\frac{\partial w}{\partial \theta}$$

$$+\frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2}h_1h_3\right)w,$$

$$\Phi_\xi = -\frac{Aek}{B^3r^2}h_3\frac{\partial v}{\partial x} - \frac{ek^2}{B^3r^2}h_1h_3\frac{\partial v}{\partial \theta} - \frac{A}{B^2r}\frac{\partial^2 w}{\partial x^2} + \frac{k^2}{B^2r^2}h_1e_2\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{k}{B^2r^2}(Ae_2 - rh_1)\frac{\partial^2 w}{\partial x\,\partial \theta} + \frac{k}{B^4r}\left(\frac{A^2}{R}p + h_1q\right)\frac{\partial w}{\partial x}$$

$$-\frac{1}{r^2}\left\{\frac{A^2k^2}{B^4R}pe_2 + \frac{1}{R}\cos(kx+\phi-\theta) + \frac{ek^2}{B^2}h_1\sin\theta + \frac{k^2}{B^4}h_1qe_2\right\}\frac{\partial w}{\partial \theta},$$

$$\Psi_\xi = -h_4 u + \frac{ek}{B^2 r} h_3 v + \frac{1}{B}\frac{\partial w}{\partial x} - \frac{k}{Br} e_2 \frac{\partial w}{\partial \theta}, \qquad \Omega_\eta = \frac{Bk}{r} e_2 \frac{\partial u}{\partial \theta} + \frac{B}{r}\frac{\partial v}{\partial \theta} + \frac{A}{r} w,$$

$$\Gamma_\eta = \frac{ek^2}{B^2 r} h_3 e_2 \frac{\partial u}{\partial x} + \left(\frac{k}{r} h_4 e_2 - \frac{k}{r} h_1\right)\frac{\partial u}{\partial \theta} + \frac{e^2 k^3}{B^2 R r} p \sin^2\theta u + \frac{ek}{B^2 r} h_3 \frac{\partial v}{\partial x}$$

$$+\frac{1}{r}\left(h_4 + \frac{A}{r}\right)\frac{\partial v}{\partial \theta} - \frac{2e^2k^2}{B^2 r^2} h_3 \sin\theta v - \frac{B}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{ek}{Br}\sin\theta \frac{\partial w}{\partial x}$$

$$+\frac{ek^2}{Br^2} e_2 \sin\theta \frac{\partial w}{\partial \theta} + \frac{1}{Br}\left(Ah_4 - \frac{ek^2}{B^2} h_1 h_3\right) w,$$

$$\Phi_\eta = -\frac{ek^2}{B^3 r} h_1 h_3 \frac{\partial u}{\partial x} + \frac{ek^3}{B^3 R r}\left\{eph_4 \sin^2\theta - h_3 e_2 \cos(kx + \phi - \theta) + \frac{A}{B^2} p h_1 h_3\right\} u$$

$$+\frac{Aek}{B^3 r^2} h_3 \frac{\partial v}{\partial x} + \frac{e^2k^2}{B^3 r^2} h_3 \left\{\frac{1}{B^2} h_2 h_3 - h_4 \sin\theta + \frac{1}{R}\sin\theta \sin(kx + \phi - \theta)\right\} v$$

$$-\frac{1}{r^2} h_4 \frac{\partial^2 w}{\partial \theta^2} - \frac{ek}{B^2 r^2} h_3 \frac{\partial^2 w}{\partial x \, \partial \theta} + \frac{ek}{B^4 r}(h_2 h_3 - B^2 h_4 \sin\theta)\frac{\partial w}{\partial x}$$

$$+\frac{ek^2}{B^4 r^2}(e\cos\theta - r)(B^2 h_4 \sin\theta - h_2 h_3)\frac{\partial w}{\partial \theta},$$

$$\Psi_\eta = \frac{k}{B} h_1 u - \frac{A}{Br} v + \frac{1}{r}\frac{\partial w}{\partial \theta},$$

$$\Omega_{\xi\eta} = ke_2 \frac{\partial u}{\partial x} + \frac{1}{r}(B^2 - k^2 e_2^2)\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - \frac{k}{r} e_2 \frac{\partial v}{\partial \theta} - \frac{ek}{r}\sin\theta v - \frac{2ek}{Br} h_3 w,$$

$$\Gamma_{\xi\eta} = \frac{2Ak}{Br} e_2 \frac{\partial u}{\partial x} + \frac{2}{Br}(B^2 h_4 + k^2 h_1 e_2)\frac{\partial u}{\partial \theta} + \frac{2ek^2}{BRr}\left\{\frac{2A}{B^2} p h_3 - e_2 \cos(kx + \phi)\right\} u$$

$$+\frac{2A}{Br}\frac{\partial v}{\partial x} - \frac{2Ak}{Br^2} e_2 \frac{\partial v}{\partial \theta} + \frac{2ek}{Br}\left\{\frac{1}{R}\sin\theta \sin(kx + \phi - \theta) + \frac{2}{B^2 r} h_3 q\right\} v$$

$$+\frac{2k}{r^2} e_2 \frac{\partial^2 w}{\partial \theta^2} - \frac{2}{r}\frac{\partial^2 w}{\partial x \, \partial \theta} + \frac{2}{B^2 r} q \frac{\partial w}{\partial x} - \frac{2k}{B^2 r^2} q e_2 \frac{\partial w}{\partial \theta},$$

$$\Phi_{\xi\eta} = \frac{k^2}{B^2}\left\{-\frac{e}{R^2 r}\, e_2 \sin\theta \cos^2(kx+\phi-\theta) + \frac{e^2k^2}{B^2 r}(1-A)h_3 \sin\theta\right.$$

$$-\frac{2Ae^2k^2}{B^4R^2r}\,p^2 h_3 \sin\theta + \frac{1}{Rr}\,ph_1$$

$$-\frac{e}{Rr}\,h_3 \cos(kx+\phi-\theta) + \frac{e}{Rr}\,ph_4 \cos\theta - \frac{e}{R}\,h_4 \sin\theta \sin(kx+\phi-\theta)$$

$$\left.-\frac{2e}{B^2Rr}\,ph_4 q \sin\theta - \frac{1}{Rr}\,h_1 e_2 \cos(kx+\phi-\theta)\right\}u$$

$$+\frac{1}{Br}\left\{\frac{1}{R}\cos(kx+\phi-\theta) + \frac{1}{B^2}\,h_4 q + \frac{Aek^2}{B^4R}\,ph_3\right\}\frac{\partial w}{\partial x}$$

$$-\frac{k}{Br}\left\{\frac{1}{Rr}\,e_2\cos(kx+\phi-\theta) + \frac{e}{B^2r}\,h_2h_3 + \frac{ek^2}{B^2Rr}\,pe_2\cos\theta\right.$$

$$\left.-\frac{1}{B^2Rr}\,qe_2\sin(kx+\phi-\theta)\right\}\frac{\partial w}{\partial\theta} - \frac{ek}{B^3r}\,h_3\frac{\partial^2 w}{\partial x^2} + \frac{k}{Br^2}(h_4e_2-h_1)\frac{\partial^2 w}{\partial\theta^2}$$

$$+\frac{1}{Br}\left(\frac{ek^2}{B^2r}\,h_3e_2 - \frac{A}{r} - h_4\right)\frac{\partial^2 w}{\partial x\,\partial\theta}. \tag{21}$$

3. THE METHOD OF VIBRATION ANALYSIS

For the free vibrations of thin panels, the principle of virtual work can be written as

$$\iiint_{volume}(\sigma_{\xi\xi}\delta e_{\xi\xi} + \sigma_{\eta\eta}\delta e_{\eta\eta} + \tau_{\xi\eta}\delta e_{\xi\eta})F\,\mathrm{d}x\,r\,\mathrm{d}\theta\,\mathrm{d}z$$

$$-\iiint_{volume}\rho\omega^2(U\mathbf{a}_1 + V\mathbf{a}_2 + W\mathbf{a}_3)\delta(U\mathbf{a}_1 + V\mathbf{a}_2 + W\mathbf{a}_3)F\,\mathrm{d}x\,r\,\mathrm{d}\theta\,\mathrm{d}z = 0. \tag{22}$$

Substituting equations (19) and (20) into equation (22), integrating the equation with respect to z, and neglecting the rotating inertia and the terms having z^i whose power i is greater than three, the principle of virtual work for the free vibrations of curved and twisted cylindrical thin panels can be rewritten as:

$$\iint_{area} \frac{H}{B}\left\{\Omega_\xi\,\delta\Omega_\xi + \Omega_\eta\,\delta\Omega_\eta + v(\Omega_\xi\,\delta\Omega_\eta + \Omega_\eta\,\delta\Omega_\xi) + \frac{1-v}{2}\,\Omega_{\xi\eta}\,\delta\Omega_{\xi\eta}\right\} \mathrm{d}x\, r\, \mathrm{d}\theta$$

$$+\iint_{area} \frac{D}{B}\Big[\Gamma_\xi\,\delta\Gamma_\xi + \Omega_\xi\,\delta\Phi_\xi + \Phi_\xi\,\delta\Omega_\xi + \Gamma_\eta\,\delta\Gamma_\eta + \Omega_\eta\,\delta\Phi_\eta + \Phi_\eta\,\delta\Omega_\eta$$

$$+v(\Gamma_\xi\,\delta\Gamma_\eta + \Gamma_\eta\,\delta\Gamma_\xi + \Phi_\xi\,\delta\Omega_\eta + \Omega_\eta\,\delta\Phi_\xi + \Phi_\eta\,\delta\Omega_\xi + \Omega_\xi\,\delta\Phi_\eta)$$

$$+\frac{1-v}{2}(\Gamma_{\xi\eta}\,\delta\Gamma_{\xi\eta} + \Omega_{\xi\eta}\,\delta\Phi_{\xi\eta} + \Phi_{\xi\eta}\,\delta\Omega_{\xi\eta})$$

$$-p_1\Big\{\Omega_\xi\,\delta\Gamma_\xi + \Gamma_\xi\,\delta\Omega_\xi + \Omega_\eta\,\delta\Gamma_\eta$$

$$+\Gamma_\eta\,\delta\Omega_\eta + v(\Omega_\xi\,\delta\Gamma_\eta + \Gamma_\eta\,\delta\Omega_\xi + \Omega_\eta\,\delta\Gamma_\xi + \Gamma_\xi\,\delta\Omega_\eta)$$

$$+\frac{1-v}{2}(\Omega_{\xi\eta}\,\delta\Gamma_{\xi\eta} + \Gamma_{\xi\eta}\,\delta\Omega_{\xi\eta})\Big\}\Big]\mathrm{d}x\, r\, \mathrm{d}\theta - \iint_{area} \rho\omega^2 tB[\{B^2 + k^2e_2^2\}u\,\delta u$$

$$+ke_2(v\,\delta u + u\,\delta v) + v\,\delta v + w\,\delta w]\,\mathrm{d}x\, r\, \mathrm{d}\theta = 0, \tag{23}$$

where $H = Et/(1-v^2)$, $D = Et^3/\{12(1-v^2)\}$, E is Young's modulus, v Poisson's ratio, ρ material density material and ω angular frequency.

In order to generalize equations independently of the dimensions of panels, the following non-dimensional variables and parameters are introduced,

$$\bar{x} = x/l, \quad \bar{u} = u/l, \quad \bar{v} = v/l, \quad \bar{w} = w/l, \quad \overline{R} = R/l, \quad \bar{r} = r/l,$$
$$\bar{e} = e/r, \quad \overline{k} = kl. \tag{24}$$

The Rayleigh–Ritz method is used for the free vibration analysis of the panels, so the non-dimensional displacements $\bar{u}$, $\bar{v}$ and $\bar{w}$ are assumed as two dimensional polynomial functions with respect to $\bar{x}$ and θ, which should satisfy the boundary conditions at $\bar{x} = 0$, namely,

$$\bar{u} = 0, \quad \bar{v} = 0, \quad \bar{w} = 0, \quad \frac{\partial \bar{w}}{\partial \bar{x}} = 0, \tag{25}$$

and given as

$$\bar{u} = \sum_{i=1}^{N_{\bar{u}}}\sum_{j=0}^{M_{\bar{u}}} a_{ij}\bar{x}^i\theta^j, \quad \bar{v} = \sum_{k=1}^{N_{\bar{v}}}\sum_{l=0}^{M_{\bar{v}}} b_{kl}\bar{x}^k\theta^l, \quad \bar{w} = \sum_{m=2}^{N_{\bar{w}}}\sum_{n=0}^{M_{\bar{w}}} c_{mn}\bar{x}^m\theta^n, \tag{26}$$

where a_{ij}, b_{kl} and c_{mn} are unknown coefficients, N_i and M_i ($i = \bar{u}, \bar{v}, \bar{w}$) are the maximum power of $\bar{x}$ and θ in the displacement functions respectively.

TABLE 1

Convergence of λ versus the number of integrating points

No.	Points 10	12	16
1	6·6501960	6·6501953	6·6501955
2	22·122679	22·122679	22·122680
3	37·098692	37·098692	37·098693
4	51·700550	51·700547	51·700550
5	64·579485	64·579482	64·579484
6	91·069888	91·069888	91·069888
7	118·27498	118·27498	118·27498
8	121·25889	121·25888	121·25889
9	129·76028	129·76024	129·76024
10	184·76464	184·76466	184·76466

Substituting equations (24) and (26) into equation (23), and integrating the equation with respect to $\bar{x}$ and θ, the vibration equation is presented in a matrix form as:

$$(\mathbf{A} - \lambda^2\mathbf{B})\mathbf{q} = 0, \tag{27}$$

where **A** and **B** are stiffness and mass matrices respectively, and **q** is a vector consisting of unknown coefficients a_{ij}, b_{kl} and c_{mn}. λ is a non-dimensional frequency parameter, defined as

$$\lambda^2 = \rho t \omega^2 l^4 / D \tag{28}$$

TABLE 2

Convergence of λ versus the number of terms in the displacement functions

$N_{\bar{u}}, M_{\bar{u}}$	6,6	6,7	7,6	7,7	7,8
$N_{\bar{v}}, M_{\bar{v}}$	6,6	6,7	7,6	7,7	7,8
$N_{\bar{w}}, M_{\bar{w}}$	7,7	7,8	8,7	8,8	8,9
	Terms				
No.	42/42/48	48/48/54	49/49/56	56/56/63	63/63/70
1	6·6770	6·6590	6·6706	6·6522	6·6501
2	22·172	22·155	22·144	22·126	22·122
3	37·207	37·190	37·120	37·104	37·099
4	52·074	52·046	51·735	51·707	51·700
5	64·869	64·826	64·631	64·585	64·579
6	92·875	92·841	91·103	91·074	91·069
7	120·62	120·46	118·42	118·28	118·27
8	122·22	122·08	121·44	121·28	121·26
9	137·47	137·44	129·82	129·76	129·76
10	186·29	186·13	184·96	184·79	184·76

From equation (27), the eigenvalues and corresponding eigenvectors can be evaluated by common methods.

4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, several models are studied by the method presented in this paper. The first ten frequency parameters λ and some corresponding mode shapes are shown to verify the effectiveness of the method and to explain the effects of geometric parameters on the frequency parameters and the mode shapes of vibration. Poisson's ratio ν is 0·3. The origin O of the co-ordinate system (x, y, z') is considered as the center of gravity of the cross-section of the panel at the fixed end, so the parameter e can be given by

$$e = r \sin(\Omega_1/2)/\Omega_1/2 \tag{29}$$

4.1. CONVERGENCE STUDIES

The Rayleigh–Ritz method with two-dimensional polynomial functions to approximate the displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$, and the Gauss–Legendre integration method are used for analysing the vibration problems of cylindrical panels, so that it is necessary to check the effects of the number of terms in the displacement functions and the number of points adopted in integration on the convergence of frequency parameters. A model with a severe configuration is taken to be considered, namely, $\Omega_1 = 90^\circ$, $\Omega_2 = 90^\circ$, $\overline{k} = 60^\circ$, $\phi = 0^\circ$, an aspect ratio $l/b = 2$ and a thickness ratio $b/t = 20$.

When the displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ having 63, 63 of 70 terms are considered, the convergence is checked for 10, 12 and 16 Gauss–Legendre integrating points, respectively, and the results are shown in Table 1. It is shown that the convergence of the first ten frequency parameters λ $(\omega l^2\sqrt{\rho t/D})$ is very good. So 12 integrating points are used for the following numerical analysis.

Table 2 shows the effects of the number of terms in the displacement functions on the first ten frequency parameters λ. In Table 2, N_i and M_i $(i = \bar{u}, \bar{v}, \bar{w})$ represent the maximum power of $\bar{x}$ and θ in the displacement functions i, the "Terms" represents the number of terms existing in each of the displacement functions. In this paper, the maximum power in the displacement functions are assumed as follows:

$$N_{\bar{v}} = N_{\bar{u}}, \qquad N_{\bar{w}} = N_{\bar{u}} + 1, \qquad M_{\bar{v}} = M_{\bar{u}}, \qquad M_{\bar{w}} = M_{\bar{u}} + 1. \tag{30}$$

According to equation (26), for the case of $N_{\bar{u}} = N$ and $M_{\bar{u}} = M$, it can be shown that the displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ have $N \times (M+1)$, $N \times (M+1)$ and $N \times (M+2)$ terms respectively. The method has a good rate of convergence for the first ten frequency parameters when $N_{\bar{u}}$ and $M_{\bar{u}}$ are 7 and 8 respectively.

4.2. COMPARISON WITH AVAILABLE RESULTS

Leissa *et al*. studied the vibration characteristics of doubly-curved cantilevered shallow shells having rectangular planform [5], in which the shells having a square planform $a'/b' = 1$, a thickness ratio $b'/h = 100$, Poisson's ratio $\nu = 0{\cdot}3$, a ratio

TABLE 3

Comparison of frequency parameters λ with reference [5]

Ω_1	$\Omega_2(l/R)$	Method	Symmetric mode				Antisymmetric mode			
			1	2	3	4	1	2	3	4
5·7320°	−5·7392°	Leissa	4·9440	25·795	32·758	56·767	8·5961	31·512	68·129	71·904
		Present	4·9498	25·802	32·817	56·798	8·6107	31·481	68·241	71·879
	0·0°	Leissa	5·2174	24·778	28·230	55·156	8·6138	31·565	64·325	72·042
		Present	5·2189	24·767	28·215	55·121	8·6099	31·555	64·267	72·004
	5·7392°	Leissa	4·8282	22·694	32·687	61·282	8·6090	31·385	66·366	73·486
		Present	4·8299	22·708	32·740	61·355	8·6220	31·422	66·485	73·550
11·4783°	−11·5370°	Leissa	6·5038	29·931	46·125	65·010	8·8019	32·683	73·991	77·842
		Present	6·5319	30·033	46·441	65·328	8·8662	32·579	73·846	78·530
	0·0°	Leissa	8·3683	26·828	35·120	58·726	8·9063	33·249	64·646	75·030
		Present	8·3916	26·738	35·139	58·627	8·8964	33·230	64·414	74·928
	11·5370°	Leissa	6·5854	24·893	38·933	72·267	8·8553	32·161	68·789	80·677
		Present	6·6167	25·044	39·252	72·456	8·9142	32·341	69·201	81·238
28·9550°	−30·0°	Leissa	8·2429	36·503	72·331	99·151	9·4214	36·294	82·126	107·57
		Present	8·4091	37·367	73·683	102·77	9·8746	35·713	82·720	113·00
	0·0°	Leissa	16·991	30·650	47·704	90·217	10·595	42·234	65·585	90·029
		Present	17·183	30·332	47·507	90·154	10·613	42·391	64·160	89·739
	30·0°	Leissa	9·0327	30·476	49·237	89·361	9·7809	33·998	72·253	96·738
		Present	9·1706	32·188	51·760	92·936	10·198	35·280	75·319	101·85

TABLE 4

Frequency parameters λ ($\Omega_1 = 60^\circ$, $\phi = 0^\circ$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	9·2209	8·2068	6·9984	6·0511	5·3700	9·1744	7·9943	6·8077	5·9159	5·2873	9·2109	7·9755	6·8309	5·9962	5·4166
2	15·375	16·894	18·989	20·119	19·334	16·135	17·829	19·675	20·559	19·708	17·228	18·998	20·496	21·056	20·148
3	42·572	45·819	41·251	37·149	36·241	37·719	41·898	41·663	36·399	33·775	34·910	39·028	40·956	35·273	31·161
4	56·693	50·512	52·097	54·437	54·827	54·718	51·536	48·321	50·850	52·179	52·071	50·690	45·797	47·167	48·742
5	58·528	61·506	66·328	71·504	76·278	67·408	62·821	64·581	67·587	70·142	64·785	61·389	60·394	62·674	64·191
6	86·884	90·666	92·468	93·457	94·183	76·546	82·271	88·765	94·247	96·352	80·450	78·827	84·279	89·656	91·121
7	92·335	92·373	95·769	99·775	102·71	95·601	97·546	101·83	106·52	110·70	100·20	104·07	108·04	107·41	109·40
8	122·35	120·52	115·44	113·15	115·25	122·77	123·72	118·26	113·22	115·02	107·36	109·02	113·47	118·91	123·96
9	129·49	127·15	129·63	133·07	136·60	135·52	129·43	128·47	131·72	135·30	143·38	135·81	126·04	125·20	129·72
10	145·40	146·76	148·54	150·27	151·90	146·08	146·69	150·76	155·61	160·24	162·23	161·62	165·80	170·74	174·80

Table 5

Frequency parameters λ ($\Omega_1 = 60^\circ$, $\phi = -45^\circ$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	8·8411	8·0326	6·9496	6·0490	5·3763	8·0167	7·3403	6·5008	5·7661	5·1947	7·2373	6·7295	6·1213	5·5627	5·1130
2	14·230	15·699	17·770	19·070	18·774	13·184	14·966	17·065	18·405	18·345	12·280	14·215	16·305	17·640	17·757
3	40·608	42·075	41·721	40·263	39·974	34·357	36·208	37·842	39·169	40·247	30·784	33·092	35·399	37·307	38·727
4	49·015	49·693	50·466	52·316	53·821	45·557	47·857	50·964	54·440	56·769	42·324	44·691	48·677	54·357	59·532
5	69·571	67·513	68·654	71·657	75·061	76·113	75·555	72·887	70·691	70·634	70·644	73·274	76·597	74·885	68·916
6	87·027	89·123	90·394	90·602	90·655	84·810	83·694	85·288	88·476	91·902	89·571	86·558	82·229	82·933	89·048
7	99·666	97·718	97·792	99·837	102·60	110·88	105·24	100·92	98·564	98·972	123·68	116·07	108·47	103·80	103·61
8	118·17	117·55	114·66	111·49	110·37	133·81	133·38	129·74	122·10	116·47	127·47	125·01	122·01	120·30	119·29
9	136·83	135·47	136·80	140·04	143·65	139·68	137·28	133·16	132·93	134·08	163·05	157·17	144·79	132·55	124·86
10	153·65	152·00	151·22	151·29	152·26	160·45	155·50	155·85	158·19	160·75	186·70	179·62	173·98	169·50	167·63

TABLE 6

Frequency parameters λ ($\Omega_1 = 60^\circ$, $\phi = -90^\circ$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	8·5766	8·0277	7·0517	6·1853	5·5127	7·4761	7·1509	6·5034	5·8600	5·3128	6·6630	6·4627	6·0256	5·5484	5·1112
2	13·544	14·449	16·259	17·746	17·981	11·580	12·180	13·558	15·049	15·937	10·018	10·414	11·414	12·669	13·717
3	41·378	40·448	38·437	36·704	36·499	34·047	33·766	33·107	32·611	33·059	28·534	28·429	28·238	28·342	29·352
4	42·851	43·431	44·888	46·894	48·823	37·169	37·586	38·797	40·684	42·984	34·177	34·558	35·649	37·282	39·163
5	74·915	75·501	76·974	78·755	80·214	82·940	81·811	80·264	78·953	77·785	76·747	75·386	73·000	71·003	69·805
6	88·847	88·509	87·532	86·235	85·570	84·077	84·771	85·075	84·566	83·826	83·435	84·599	86·275	86·885	86·085
7	106·90	106·91	106·58	105·43	103·76	119·26	118·32	115·84	112·41	108·58	123·73	123·08	121·15	118·03	114·00
8	111·14	111·19	111·75	113·42	116·06	127·02	127·18	127·43	127·56	127·71	143·08	141·69	138·34	134·46	131·15
9	142·79	143·40	145·24	148·10	151·24	167·63	165·81	161·45	156·58	152·12	170·38	169·30	166·36	162·07	156·90
10	171·14	168·63	165·08	162·02	159·81	179·15	177·22	176·54	177·34	178·57	195·18	196·68	199·72	201·83	201·05

TABLE 7

Frequency parameters λ ($\Omega_1 = 90^\circ$, $\phi = 0^\circ$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	12·642	10·932	9·3035	7·9978	6·9946	12·016	10·318	8·8022	7·6017	6·6889	11·517	9·9634	8·5784	7·4831	6·6522
2	15·915	17·940	19·977	21·197	21·001	17·068	19·199	21·023	21·958	21·519	18·578	20·726	22·256	22·823	22·126
3	48·491	51·607	48·937	44·838	42·664	43·320	46·952	47·485	43·208	39·979	39·924	43·487	45·124	41·212	37·104
4	55·855	53·455	54·364	56·295	56·978	52·064	50·550	50·483	53·280	54·699	48·135	46·775	46·999	49·990	51·707
5	71·934	70·923	72·677	75·801	79·073	79·648	73·952	69·495	69·240	71·274	72·715	72·075	65·928	63·375	64·585
6	88·890	89·042	89·887	90·778	91·332	86·144	87·889	90·846	93·359	94·550	92·016	85·928	86·706	89·839	91·074
7	97·879	99·324	100·66	101·67	103·06	95·635	98·947	104·23	108·59	111·20	100·96	103·00	108·73	114·06	118·28
8	126·23	127·55	129·24	130·91	132·81	131·27	130·38	127·59	124·87	125·84	118·80	120·86	123·99	123·48	121·28
9	147·59	143·73	141·54	142·15	144·51	142·66	141·08	139·79	139·72	139·41	145·99	140·71	131·59	127·38	129·76
10	161·23	160·45	158·46	155·30	151·80	171·69	168·03	166·32	166·62	167·65	186·53	187·25	184·96	185·09	184·79

TABLE 8

Frequency parameters λ ($\Omega_1 = 90°$, $\phi = -45°$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	12·321	10·790	9·2967	8·0494	7·0549	10·840	9·6650	8·5245	7·5140	6·6721	9·5187	8·7005	7·8681	7·0892	6·4158
2	14·554	16·559	18·625	20·065	20·350	13·943	16·107	18·253	19·800	20·243	13·451	15·689	17·934	19·570	20·117
3	46·293	47·341	48·176	47·989	46·800	40·176	41·215	42·084	42·828	43·412	36·759	38·059	38·980	39·619	40·223
4	60·138	61·793	58·978	56·799	56·791	55·579	58·886	62·285	65·426	65·533	50·800	53·777	57·047	60·373	62·981
5	71·105	68·636	70·845	73·936	76·491	83·227	79·269	74·772	70·515	69·646	83·212	85·965	84·644	80·485	74·772
6	90·892	90·335	89·851	89·470	89·135	89·064	89·898	90·400	90·718	91·343	92·098	88·515	88·732	89·711	90·631
7	106·17	105·42	105·30	105·95	107·44	113·86	109·23	106·81	107·29	110·07	127·34	119·32	111·96	109·83	113·34
8	131·71	135·33	133·42	128·61	125·18	147·62	143·82	138·73	131·22	123·67	140·89	135·19	130·87	127·75	123·82
9	143·67	138·82	140·26	145·72	151·12	154·37	157·82	151·47	147·88	146·91	177·39	172·64	159·89	148·05	141·09
10	170·71	168·04	164·72	161·17	157·79	167·65	160·39	162·92	165·35	167·20	186·56	185·21	181·86	177·32	174·54

TABLE 9

Frequency parameters λ ($\Omega_1 = 90^\circ$, $\phi = -90^\circ$)

	Ω_2 (°)														
	$\bar{k}$ (°) 30					$\bar{k}$ (°) 60					$\bar{k}$ (°) 90				
No.	0	15	30	45	60	0	15	30	45	60	0	15	30	45	60
1	12·237	11·034	9·5705	8·3321	7·3254	10·747	9·8533	8·7590	7·7911	6·9592	9·4678	8·8272	8·0043	7·2476	6·5707
2	13·640	15·018	16·891	18·436	19·169	11·737	12·801	14·356	15·895	17·059	10·282	11·066	12·306	13·675	14·933
3	46·105	46·358	47·101	47·892	45·632	40·757	40·951	41·517	42·245	42·087	37·763	37·946	37·857	37·797	38·256
4	54·775	53·671	50·994	48·361	49·750	45·431	45·134	44·368	43·664	44·558	38·210	38·132	38·587	39·434	40·478
5	77·422	77·919	79·017	80·114	80·893	90·792	89·650	87·730	85·956	84·563	89·844	87·719	84·733	82·171	80·452
6	92·065	92·095	91·984	91·355	90·210	92·652	93·083	92·996	91·790	89·616	92·276	93·761	94·818	94·237	91·795
7	114·79	114·12	112·97	112·48	112·69	128·97	127·54	124·23	119·85	114·97	130·72	130·05	127·78	123·58	117·97
8	126·56	126·69	126·10	124·12	121·72	134·33	134·65	134·48	133·13	131·11	154·16	152·83	149·31	144·62	139·70
9	144·64	145·38	147·91	152·08	156·60	173·07	173·43	174·93	174·66	169·47	197·34	193·46	186·69	179·75	172·89
10	186·34	183·97	178·78	173·34	169·17	194·11	189·29	182·15	179·09	183·16	210·48	211·63	212·22	214·02	211·26

($b'/R_y = 0{\cdot}1, 0{\cdot}2, 0{\cdot}5$) and a curvature ratio ($R_y/R_x = -1{\cdot}0, 0, 1{\cdot}0$) are studied, and eight frequency parameters ($\omega a'^2\sqrt{\rho h/D}$) for the first four symmetric and antisymmetric modes are presented as shown in Table 3. The same models having an aspect ratio l/b and a thickness ratio b/t which are expressed as

$$l/b = (a'/b')2\Omega_2 \sin(\Omega_1/2)/\Omega_1 \sin(\Omega_2), \qquad b/t = (b'/h)\Omega_1/2\sin(\Omega_1/2), \quad (30)$$

and Ω_1 ($5{\cdot}7320°$, $11{\cdot}4783°$, $28{\cdot}9550°$), Ω_2 ($\pm 5{\cdot}7392°$, $\pm 11{\cdot}5370°$, $0°$, $\pm 30°$) and $\phi = \pm 90°$ are analyzed by the present method and the frequency parameters obtained ($\omega l^2\sqrt{\rho t/D}$) are also shown in Table 3. It can be seen that the results obtained by the two methods agree well. For the cases in which Ω_1 and Ω_2 are less than $11{\cdot}4783°$ and $11{\cdot}5370°$, the difference is less than 1%, and the maximum

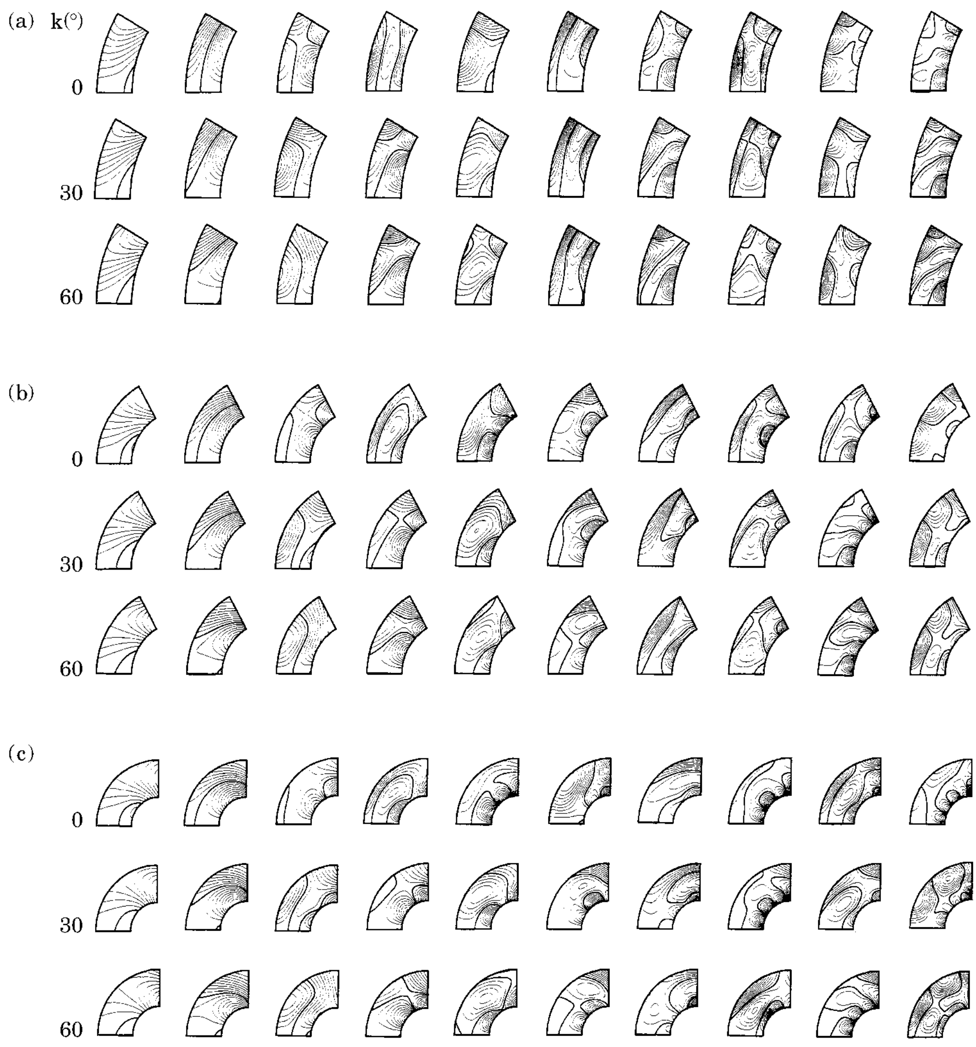

Figure 2. (a) Vibration modes ($\Omega_1 = 90°$, $\Omega_2 = 30°$); (b) Vibration modes ($\Omega_1 = 90°$, $\Omega_2 = 60°$); (c) Vibration modes ($\Omega_1 = 90°$, $\Omega_2 = 90°$).

difference of as much as 5·1% occurs for the fourth antisymmetric mode when Ω_1 and Ω_2 are 28·9550° and 30·0°, but it is only 2% for the case of $\Omega_1 = 28{\cdot}9550^\circ$ and $\Omega_2 = 0^\circ$. These mean that the difference will become larger with the increases of chordwise and spanwise curvatures. Also it can be seen that the difference caused by Ω_2 is greater than that caused by Ω_1. Leissa *et al.* studied the vibration problems based on the shallow shell theory, so that the method is capable of representing the vibration characteristics accurately for shells having relatively small curvatures and it is inadequate for shells having large curvatures. But the present method has no limitations.

4.3. RESULTS OF CURVED AND TWISTED CYLINDRICAL THIN PANELS

In this section, several models are studied by the present method in order to ascertain the effects of geometric parameters on the vibration characteristics. The numerical results for λ are shown in Tables 4–9 for the models having a different Ω_1 (60°, 90°), Ω_2 (30°, 60°, 90°), $\bar{k}$ (0°–60°), ϕ (0°, −45°, −90°), an aspect ratio $l/b = 2$ and a thickness ratio $b/t = 20$.

A set of vibration modes for the models having $\Omega_1 = 90^\circ$, Ω_2 (30°, 60°, 90°) and $\bar{k}$ (0°, 30°, 60°) are plotted in Figures 2(a–c). Modes 1–10 are arranged from left to right. The displacement $\bar{w}$ is only considered in these contour plots and the heavier lines are nodal lines ($\bar{w} = 0$).

From the tables and Figure 2, it can be observed that the first frequency parameter λ decreases monotonically with the increase of the twist angle $\bar{k}$, and the shape of the first vibration mode is similar to the first spanwise bending mode, which changes slightly while the twist angle increases. The second mode is the first torsional mode for the case of $\bar{k} = 0^\circ$, and the mode shape changes greatly when the twist angle increases from 0°–60°. The first torsional mode trends toward the second spanwise bending mode and the corresponding frequency parameter λ increases with the increase of the twist angle, but it begins to decrease when the twist angle reaches about 50°.

In a word, the effects caused by the twist angle are different for lower and higher frequency parameters, which is larger for lower frequency parameters than higher ones. It can be seen from the tables that the maximum variation can reach 45% for the first frequency parameter and about 10% for the tenth one.

Considering the effects of Ω_1, the frequency parameters increase with the increasing Ω_1 in any case with $\bar{k}$, Ω_2 and ϕ investigated in this paper. This means that the stiffness of the model increases when Ω_1 increases. The effects caused by Ω_2 are complex. In the case of $\phi = -90^\circ$, the first four frequency parameters increase, but the last four ones decrease while Ω_2 increases. For the case of $\phi = 0^\circ$, the first frequency parameter decreases and the second one increases while Ω_2 increases. In the case of $\phi = -45^\circ$, the first and the second frequency parameters decrease while Ω_2 increases. The others either increase or decrease with the increase of Ω_2.

The mode shapes shown in Figure 2 correspond to the models with large curvatures, so the symmetric modes and antisymmetric modes cannot be seen. But it will be seen when $\phi = -90^\circ$. From the mode shapes, the qualitative changes for a different twist angle $\bar{k}$, Ω_1 and Ω_2 can be seen.

5. CONCLUSIONS

In this paper, turbomachinery blades are considered as a model of cylindrical thin panel having twist, chordwise and spanwise curvatures. Based on the general thin shell theory, the non-linear strain–displacement relations are established. A numerical method for analysing the free vibration characteristics of the models is presented by using the principle of virtual work for the free vibration and the Rayleigh–Ritz method. It is shown that the method is effective for analysing the vibrations of the turbomachinery blades, and can provide accurate results when the proper number of integrating points and terms of displacement functions are adopted.

The effects of curvature and twist on the vibrations are studied. It is known that the stiffness of the model increases when the curvature of the s-axis increases. The frequency parameters may increase or decrease when the curvature of the x-axis increases. With the increase of the twist angle, the first frequency parameter increases, and the second one increases and begins to decrease when Ω_2 reaches about 50°, and the change in the second mode shape is obvious. The changes in the other mode shapes can be seen from Figure 2. The effects of the angle ϕ can also be seen in the tables.

The method presented in this paper is adequate for evaluating the vibration characteristics even if the models have large curvatures and large twists, and the angle ϕ can be changed freely in these models, which supplies a different model of turbomachinery blades for analysing the vibrations of the blades.

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APPENDIX

The coefficients adopted in this paper are defined as

$$A = 1 - (r/R)\sin(kx+\phi-\theta) + (e/R)\sin(kx+\phi), \qquad B = \sqrt{e^2k^2\sin^2\theta + A^2}$$

$$p = -r\cos(kx+\phi-\theta) + e\cos(kx+\phi),$$

$$q = (Ar/R)\cos(kx+\phi-\theta) + e^2k^2\sin\theta\cos\theta,$$

$$c_{11} = -\frac{A}{BR}\sin(kx+\phi-\theta) + \frac{Aek^2}{B^3R}p\sin\theta,$$

$$c_{12} = \frac{ek}{BR}\sin\theta\cos(kx+\phi) - \frac{Ak}{B}\cos\theta + \frac{c^2k^3}{B^3R}p\sin^3\theta,$$

$$c_{13} = \frac{k}{B}\sin\theta\left\{-\frac{e}{R}\sin(\mathrm{kx}+\phi) + A + \frac{e^2k^2}{B^2R}p\sin\theta\cos\theta\right\},$$

$$c_{21} = \frac{ek}{Br}\left(-\cos\theta + \frac{1}{B^2}q\sin\theta\right),$$

$$c_{22} = (1/\mathrm{Br})\left\{\frac{\mathrm{r}}{\mathrm{R}}\cos(kx+\theta-\phi)\sin\theta + A\cos\theta - (A/B^2\,q\sin\theta\right\},$$

$$c_{23} = (1/\mathrm{Br})\left\{\frac{\mathrm{r}}{\mathrm{R}}\cos(kx+\theta-\phi)\cos\theta + A\sin\theta - (A/B^2\,q\cos\theta\right\},$$

$$e_1 = e - r\cos\theta, \qquad e_2 = e\cos\theta - r,$$

$$h_1 = A - (e/R)\sin\theta\cos(kx+\phi-\theta),$$

$$h_2 = (A/R)\cos(kx+\phi-\theta) + ek^2\sin\theta,$$

$$h_3 = A\cos\theta - (r/R)\sin\theta\cos(kx+\phi-\theta),$$

$$h_4 = (ek^2/B^2R)\,p\sin\theta - (1/R)\sin(kx+\phi-\theta),$$

$$p_1 = (1/B)(A/r + h_4 + (ek^2/B^2r)\,h_3e_2). \tag{A.1}$$