*Journal of Sound and Vibration* (1999) **219**(2), 207–222
*Article No*. jsvi.1998.1879, available online at http://www.idealibrary.com.on IDEAL

# SOLUTION OF EIGENVALUE PROBLEMS FOR NON-CLASSICALLY DAMPED SYSTEMS WITH MULTIPLE FREQUENCIES

M. C. Kim, H. J. Jung and I. W. Lee

*Department of Civil Engineering*,
*Korea Advanced Institute of Science and Technology*, *Taejon* 305-701, *Korea*

(*Received* 24 *July* 1997, *and in final form* 7 *July* 1998)

An efficient solution method is presented to solve the eigenvalue problem arising in the dynamic analysis of non-classically damped structural systems with multiple eigenvalues. The proposed method is obtained by applying the modified Newton–Raphson technique and the orthonormal condition of the eigenvectors to the linear eigenproblem through matrix augmentation of the quadratic eigenvalue problem. In the iteration methods, such as the inverse iteration method and the subspace iteration method, singularity may occur during the factorizing process when the shift value is close to an eigenvalue of the system. However, even though the shift value is an eigenvalue of the system, the proposed method provides non-singularity, and that is analytically proved. Since the modified Newton–Raphson technique is adapted to the proposed method, initial values are needed. Because the Lanczos method effectively produces better initial values than other methods, the results of the Lanczos method are taken as the initial values of the proposed method. Two numerical examples are presented to demonstrate the effectiveness of the proposed method and the results are compared with those of the well-known subspace iteration method and the Lanczos method.



## 1. INTRODUCTION

In the analysis of dynamic response of structural systems, the equation of motion of a damped system can be expressed as

$$M\ddot{\mathbf{u}}(t) + C\dot{\mathbf{u}}(t) + K\mathbf{u}(t) = f(t), \tag{1}$$

where $M$, $K$ and $C$ are the $n$ by $n$ mass, stiffness and non-classical damping matrices [1], respectively, and $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$ and $\mathbf{u}(t)$ are the $n$ by 1 acceleration, velocity and displacement vectors, respectively. To find the free vibration solution of the system, first solve equation (1) for the homogeneous solution, which is of the form

$$\mathbf{u}(t) = \boldsymbol{\phi}\, \mathrm{e}^{\lambda t}. \tag{2}$$

Substituting equation (2) into equation (1) yields the quadratic eigenproblem such as

$$\lambda^2 M\boldsymbol{\phi} + \lambda C\boldsymbol{\phi} + K\boldsymbol{\phi} = 0, \tag{3}$$

0022–460X/99/020207 + 16 $30.00/0 

in which $\lambda$ and $\boldsymbol{\phi}$ are the eigenvalue and the corresponding eigenvector of the system. There are $2n$ eigenvalues for the system with $n$ degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or underdamped modes.

The common practice is to reformulate the quadratic system of equation to a linear one by doubling the order of the system such as

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix} = \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix}, \tag{4}$$

which may be rewritten as

$$A\boldsymbol{\psi} = \lambda B\boldsymbol{\psi} \tag{5}$$

with

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\psi} = \begin{Bmatrix} \boldsymbol{\phi} \\ \lambda\boldsymbol{\phi} \end{Bmatrix}. \tag{6}$$

Transformation methods such as QR [2], LZ [3] or Jacobi [4] determine all the eigenvalues and the associated eigenvectors in an arbitrary sequence. This is not very efficient in situations where only the lowest frequencies are of interest and there is a large number of degrees of freedom. Also transformation methods by their nature modify the initial matrices during the solution process and can not take full advantage of the sparseness of these matrices. The perturbation method [5–9] is used for the eigenvalue problem of lightly damped systems. Since weak damping implies that the eigensolution of the damped system will differ only slightly from that of the corresponding undamped system, it is to set the eigensolution of the undamped system as the zero order approximation of that of the damped system and let the higher order terms account for the slightly damped effect.

The classical inverse iteration method [10–12] is commonly used to solve for only a small number of desired modes. However, the method requires a great deal of complex arithmetic operations for each eigenvalue sought. The subspace iteration method [13, 14] is a more efficient alternative than the inverse iteration method. It yields all modes requested simultaneously and does not have the drawback that the higher modes are less accurate than the lower modes because it avoids the round-off errors of the inverse iteration method due to the deflation process. However, as in the inverse iteration method, a large number of complex arithmetic manipulations are required in the iteration process for general structural systems. Furthermore, when the shift value becomes close to an eigenvalue of the system, singularity may be encountered during the triangularization process.

In recent years there has been considerable interest in the Lanczos algorithm and its applications. The Lanczos algorithm for the computation of eigenvalues and eigenvectors of a real symmetric matrix was presented in reference [15] and improved in references [16–20]. The Lanczos algorithm to solve the eigenvalue problem of a non-classically damped system is dealt with in references [21–26]. The

two-sided Lanczos algorithm [21–24] requires the generation of two sets of Lanczos vectors, left and right, and the symmetric Lanczos algorithm [25, 26] uses a set of Lanczos vectors to reduce a large eigenvalue problem in a much smaller one. Although only real arithmetic is solved during the Lanczos recursive process, in contrast to the case of real symmetric eigenproblems, there will be a possibility of serious breakdown and the accuracy of the solutions obtained is low [27].

Although the authors presented the solution method for an eigenvalue problem with distinct eigenvalues [28], the method has the demerit that singularity will occur if the eigenvalue desired is multiple. Therefore, in the present paper we develop the method to solve an eigenproblem with guaranteed non-singularity for a damped structural system with multiple eigenvalues as well as distinct ones.

In the second section, the basic concept of the proposed method, which applies the modified Newton–Raphson technique to a linear eigenproblem, and the analytical proof of its non-singularity are presented. In the third section, two numerical examples are presented to identify the efficiency of the proposed method and the results of the proposed method are compared with those of the well-known subspace iteration method [14] and the Lanczos method [25].

## 2. METHOD OF ANALYSIS

### 2.1. PROBLEM DEFINITION

In this paper, an eigenproblem is considered of which the eigenvalue $\lambda_i$ has multiplicity $m$. For simplicity, assume that the first $m$ eigenvalues are equal

$$\lambda \equiv \lambda_1 = \lambda_2 = \cdots = \lambda_m. \tag{7}$$

Then equation (5) can be presented in matrix form for the $m$ multiple eigenvalues as follows

$$A\boldsymbol{\Psi} = B\boldsymbol{\Psi}\Lambda, \tag{8}$$

where $\Lambda = \text{diag}\,(\lambda_1, \ldots, \lambda_m) = \lambda I_m$ and $\boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \cdots \boldsymbol{\psi}_m]$ is an $n$ by $m$ matrix satisfying the orthonormal condition with respect to matrix $B$ suich as

$$\boldsymbol{\Psi}^{\mathrm{T}} B \boldsymbol{\Psi} = I_m, \tag{9}$$

where $I_m$ is a unitary matrix of order $m$.

The objective is to develop an efficient solution method with guaranteed non-singularity for an eigenproblem described by equations (8) and (9).

### 2.2. PROPOSED METHOD

Assume that initial approximate solutions of equation (8) $\Lambda^{(0)}$ and $\boldsymbol{\Psi}^{(0)}$ are known. Denoting the approximate eigenvalues and the associated eigenvectors after $k$ iterations by $\Lambda^{(k)}$ and $\boldsymbol{\Psi}^{(k)}$, the residual matrix becomes as follows:

$$R^{(k)} = A\boldsymbol{\Psi}^{(k)} - B\boldsymbol{\Psi}^{(k)}\Lambda^{(k)} \tag{10}$$

and

$$(\boldsymbol{\Psi}^{(k)})^{\mathrm{T}} B \boldsymbol{\Psi}^{(k)} = I_m, \tag{11}$$

where the residual matrix $R^{(k)} = [r_1^{(k)} \cdots r_m^{(k)}]$ denotes the error for each eigenpair, and is not generally zero because of substitution of approximate values in equation (8).

In order to get the solutions converged to the multiple eigenvalues and the associated eigenvectors of the system, the residual vectors should be removed. For the purpose of that, the Newton–Raphson technique is applied such as

$$\begin{aligned} R^{(k+1)} &= A\Psi^{(k+1)} - B\Psi^{(k+1)}\Lambda^{(k+1)} \\ &= 0 \end{aligned} \tag{12}$$

and

$$(\Psi^{(k+1)})^{\mathrm{T}} B \Psi^{(k+1)} = I_m, \tag{13}$$

where

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \Delta\Lambda^{(k)} \quad \text{and} \quad \Psi^{(k+1)} = \Psi^{(k)} + \Delta\Psi^{(k)}. \tag{14, 15}$$

Substituting equations (14) and (15) into equations (12) and (13) and neglecting the non-linear terms $B\Delta^{(k)}\Psi\Delta^{(k)}$ and $(\Delta\Psi^{(k)})^{\mathrm{T}} B \Delta\Psi^{(k)}$ yield the linear simultaneous equations for unknown incremental values $\Delta\Lambda^{(k)}$ and $\Delta\Psi^{(k)}$ as follows:

$$A\Delta\Psi^{(k)} - B\Delta\Psi^{(k)}\Lambda^{(k)} - B\Psi^{(k)}\Delta\Lambda^{(k)} = -R^{(k)} \tag{16}$$

and

$$(\Psi^{(k)})^{\mathrm{T}} B \Delta\Psi^{(k)} = 0. \tag{17}$$

Since the eigenvalue is multiple, the offdiagonal elements of $\Lambda^{(k)}$ are zero or very small compared with its diagonal elements at the $k$th iteration step, and the diagonal elements very close. Thus, the second term on the right side of equation (16) may be approximated by $\lambda_1^{(k)} B\Delta\Psi^{(k)}$, which yields

$$A\Delta\Psi^{(k)} - \lambda_1^{(k)} B\Delta\Psi^{(k)} - B\Psi^{(k)}\Delta\Lambda^{(k)} = -R^{(k)}. \tag{18}$$

The matrix form of equations (18) and (17) can be written such as

$$\begin{bmatrix} (A - \lambda_1^{(k)} B) & -B\Psi^{(k)} \\ (-B\Psi^{(k)})^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = -\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}. \tag{19}$$

Because the new coefficient matrix should be reformed and refactorized in each iteration step, the proposed method, despite its rapid convergence, is not efficient.

These blemishes may be overcome by applying the modified Newton–Raphson technique to equation (19) such as

$$\begin{bmatrix} (A - \lambda_1^{(0)} B) & -\mathrm{B}\Psi^{(k)} \\ (-B\Psi^{(k)})^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\Psi^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = -\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}. \tag{20}$$

The symmetric coefficient matrix of equation (20) is of order $(2n + m)$. While singularity occurs in the factorization process of the iteration methods such as the inverse iteration method [10–12] and the subspace iteration method [13, 14] when

the shift is close to an eigenvalue of the system, non-singularity of the proposed method is always guaranteed by means of including a side condition $(\boldsymbol{\Psi}^{(k)})^{\mathrm{T}} B \Delta \boldsymbol{\Psi}^{(k)} = 0$, as shown in equation (20). This is the main difference compared with the iteration method with shift. The complete procedure of the proposed method for calculating the eigenpairs is summarized in Table 1.

### 2.3. NON-SINGULARITY OF THE PROPOSED METHOD

In the iteration methods, such as the inverse and the subspace iteration methods, the shifting algorithm is adopted to improve the convergence. However, singularity may occur during the factorizing process when the shift value is close to an eigenvalue of the system. One of the characteristics of the proposed method is that its non-singularity is also guaranteed in this situation. If the proposed method is non-singular when the shift values is an eigenvalue itself, the coefficient matrix encountered in the iteration process must necessarily be non-singular. Therefore, the non-singularity of the proposed method is proved by introducing the new eigenproblem of the resulting matrix such as

$$E\mathbf{u}_i = \gamma_i F \mathbf{u}_i, \qquad i = 1, \ldots, 2n + m, \tag{21}$$

where $\gamma_i$ and $\mathbf{u}_i$ are the $i$th eigenvalue and the associated eigenvector of the new eigenproblem, respectively, and

$$E = \begin{bmatrix} A - \lambda_1 B & -B\boldsymbol{\Psi} \\ (-B\boldsymbol{\Psi})^{\mathrm{T}} & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} C & M & 0 \\ M & 0 & 0 \\ 0 & 0 & I_m \end{bmatrix} \tag{22, 23}$$

TABLE 1

*The algorithm of the proposed method*

1. Calculate initial values $\Lambda^{(0)} = \mathrm{diag}\,(\lambda_1^{(0)} \cdots \lambda_m^{(0)})$ and $\boldsymbol{\Psi} = [\psi_1^{(0)} \cdots \psi_m^{(0)}]$.

   (a) For $k = 0$

   (b) Define $\begin{bmatrix} A - \lambda_1^{(0)} B & -B\boldsymbol{\Psi}^{(k)} \\ -(B\boldsymbol{\Psi}^{(k)})^T & 0 \end{bmatrix}$.

   (c) Compute $-\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}$,

   where $R^{(k)} = [r_1^{(k)} \quad \cdots \quad r_m^{(k)}] = A\boldsymbol{\Psi}^{(k)} - B\boldsymbol{\Psi}^{(k)}\Lambda^{(k)}$.

   (d) Compute $\begin{bmatrix} (A - \lambda_1^{(0)} B) & -B\boldsymbol{\Psi}^{(k)} \\ (-B\boldsymbol{\Psi}^{(k)})^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\Psi}^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix} = -\begin{bmatrix} R^{(k)} \\ 0 \end{bmatrix}$ for $\begin{bmatrix} \Delta\boldsymbol{\Psi}^{(k)} \\ \Delta\Lambda^{(k)} \end{bmatrix}$.

   (e) Compute $\Lambda^{(k+1)} = \Lambda^{(k)} + \Delta\Lambda^{(k)}$ and $\boldsymbol{\Psi}^{(k+1)} = \boldsymbol{\Psi}^{(k)} + \Delta\boldsymbol{\Psi}^{(k)}$.

   (f) If the norm of the residual vector does not satisfy the predetermined error limit, then go to (b) with $k = k + 1$, otherwise stop.

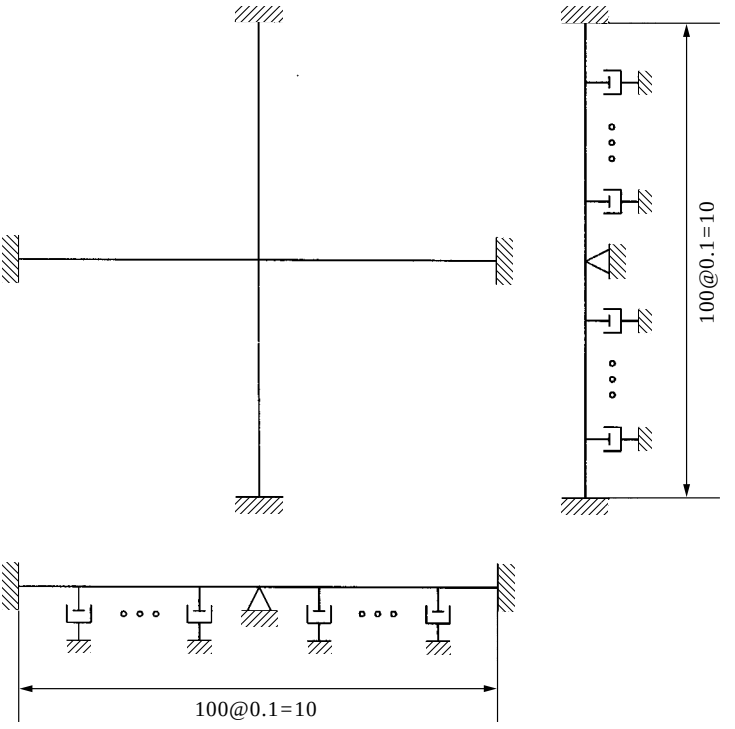


Figure 1. Plane frame structure with lumped dampers. Young's modulus: 1·0; mass density: 1·0; cross-section inertia: 1·0; cross-section area: 1·0; damping coefficients: $\alpha = 0{\cdot}001$, $\beta = 0{\cdot}001$; concentrated damping: 0·3.

or collectively

$$E\mathbf{U} = F\mathbf{U}\Gamma, \tag{24}$$

where $\Gamma = \text{diag}\,(\gamma_1 \cdots \gamma_{2n+m})$ and $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_{2n+m}]$.

The eigenpairs of equation (24) are as follows:

Eigenvalue $\gamma_i$:

$$\begin{cases} -1 & :ms \\ 1 & :ms \\ (\lambda_1 - \lambda_k) \quad k = m+1, \ldots, 2n & :(2n-m)s \end{cases}. \tag{25}$$

Eigenvector $\mathbf{u}_i$:

$$\begin{Bmatrix} \boldsymbol{\psi}_j \\ \mathbf{e}_j \end{Bmatrix}, \qquad \begin{Bmatrix} \boldsymbol{\psi}_j \\ -\mathbf{e}_j \end{Bmatrix}, \qquad \begin{Bmatrix} \boldsymbol{\psi}_k \\ 0 \end{Bmatrix} \qquad \begin{matrix} j = 1, \ldots, m \\ k = m+1, \ldots, 2n, \end{matrix} \tag{26}$$

TABLE 2

*Eigenvalues of the plane frame structure with multi-lumped dampers*

| Mode number | Eigenvalues |
|---|---|
| 1 | −0·09590 + j 8·6679 |
| 2 | −0·09590 + j 8·6679 |
| 3 | −0·09590 − j 8·6679 |
| 4 | −0·09590 − j 8·6679 |
| 5 | −0·60556 + j 15·5371 |
| 6 | −0·60556 + j 15·5371 |
| 7 | −0·60556 − j 15·5371 |
| 8 | −0·60556 − j 15·5371 |
| 9 | −0·57725 + j 20·7299 |
| 10 | −0·57725 + j 20·7299 |
| 11 | −0·57725 − j 20·7299 |
| 12 | −0·57725 − j 20·7299 |

where $\mathbf{e}_j^{\mathrm{T}}$ is a unit vector of order $m$ such as

$$\mathbf{e}_j^{\mathrm{T}} = \langle 0 \cdots 0 \quad 1 \quad 0 \cdots 0 \rangle$$
$$\mapsto j\text{th location} \tag{27}$$

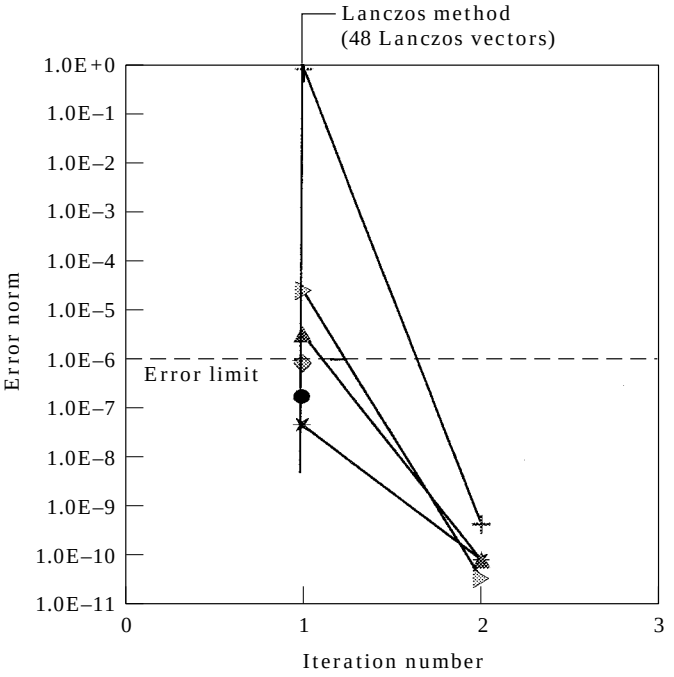


Figure 2. Variation of the error norm of the plane frame structure by the proposed method. ×, 1st, 3rd eigenpairs; ▲, 2nd, 4th eigenpairs; ●, 5th, 7th eigenpairs; ◆, 6th, 8th eigenpairs; ▷, 9th, 11th eigenpairs; +, 10th, 12th eigenpairs.

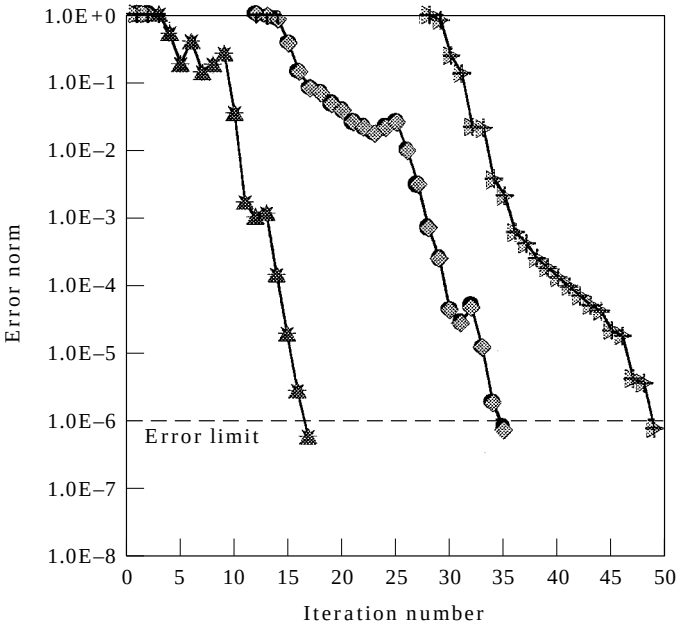


Figure 3. Variation of the error norm of the plane frame structure by the subspace iteration method. Key as for Figure 2.

Considering the determinant of equation (24),

$$\det [E] \det [\mathbf{U}] = \det [F] \det [\mathbf{U}] \det [\Gamma]$$

or

$$\begin{aligned} \det [E] &= \det [F] \det [\Gamma] \\ &= (-1)^m \det [F] \prod_{k=m+1}^{2n} (\lambda_1 - \lambda_k) \\ &\neq 0 \end{aligned} \tag{28}$$

because of

$$\begin{aligned} \det [F] &= \begin{vmatrix} C & M & 0 \\ M & 0 & 0 \\ 0 & 0 & I_m \end{vmatrix} = \begin{vmatrix} C & M \\ M & 0 \end{vmatrix} \\ &= (-1)^n \det [M] \det [M] \\ &\neq 0 \end{aligned} \tag{29}$$

The determinant of $E$ is not equal to zero because $\det [M] \neq 0$ by definition. The non-singularity of the proposed method is proved analytically.

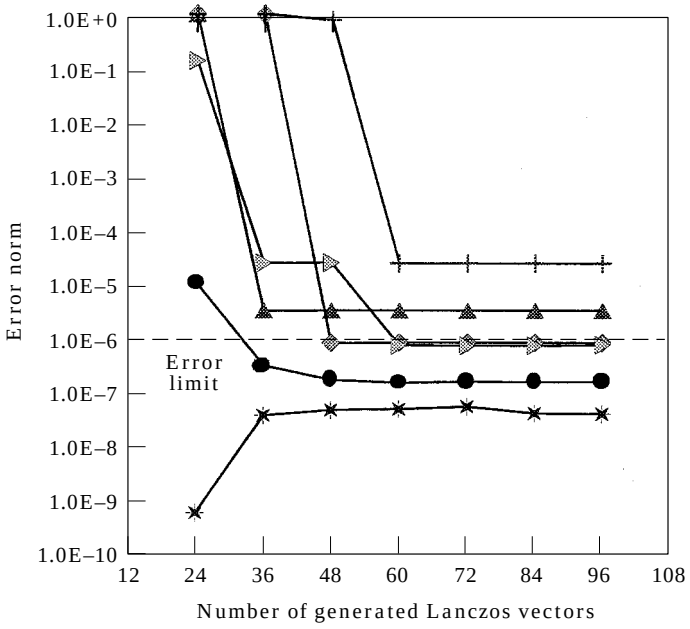


Figure 4. Variation of the error norm of the plane frame structure by the Lanczos method. Key as for Figure 2.

### 2.4. STARTING VALUES OF THE PROPOSED METHOD

Initial values of the proposed method can be obtained as the intermediate results of the iteration methods [11–14] or the results of the approximate methods [21–26]. In this paper, the starting values are taken as the results of the symmetric Lanczos method [25] with selectively reorthogonalization process because the method does not need complex arithmetic in the Lanczos recursive process, and because the multiplicity of the desired eigenvalues can be checked by the results of the $4p$ Lanczos vectors ($p$: the number of desired eigenvalues). In the Lanczos method, the initial Lanczos vector is set equal to $A^{-1}\langle 1 \cdots 1\rangle^{\mathrm{T}}$ and then is normalized with respect to matrix $B$.

## 3. NUMERICAL EXAMPLES

In this section two test problems with multiple eigenvalues are used to assess the performance of the proposed method for generalized eigenproblems. The CPU time spent for the first 12 eigenvalues and the associated eigenvectors and the

TABLE 3

*CPU time spent for the first* 12 *eigenvalues of the plane frame structure with multi-lumped dampers*

| Methods | CPU time in seconds (ratio) |
|---|---|
| Proposed method | 872·69 (1·00) |
| Subspace iteration method | 3096·62 (3·55) |

TABLE 4

*CPU time for the Lanczos method versus the number of generated Lanczos vectors of the plane frame structure*

| The number of generated Lanczos vectors | CPU time in seconds |
|---|---|
| 24 | 116·20 |
| 36 | 185·54 |
| 48 | 260·37 |
| 60 | 332·90 |
| 72 | 408·63 |
| 84 | 492·83 |
| 96 | 664·27 |

variation of the error norm to each iteration step of the proposed method are compared with those of the subspace iteration method [14]. The least subspace dimension to effectively calculate the required eigenpairs is $2p(=24)$. Each method is stopped when the error norms are reduced by a factor of $10^{-6}$, which yields a stable eigensolution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis [30]. The error norm [30] is defined as

$$\epsilon_i^{(k)} = \frac{\|r_i^{(k)}\|_2}{\|A\psi_i^{(k)}\|_2}, \tag{30}$$

where

$$\begin{aligned} R^{(k)} &= [r_1^{(k)} \cdots r_m^{(k)}] \\ &= A\Psi^{(k)} - B\Psi^{(k)}\Lambda^{(k)}. \end{aligned} \tag{31}$$

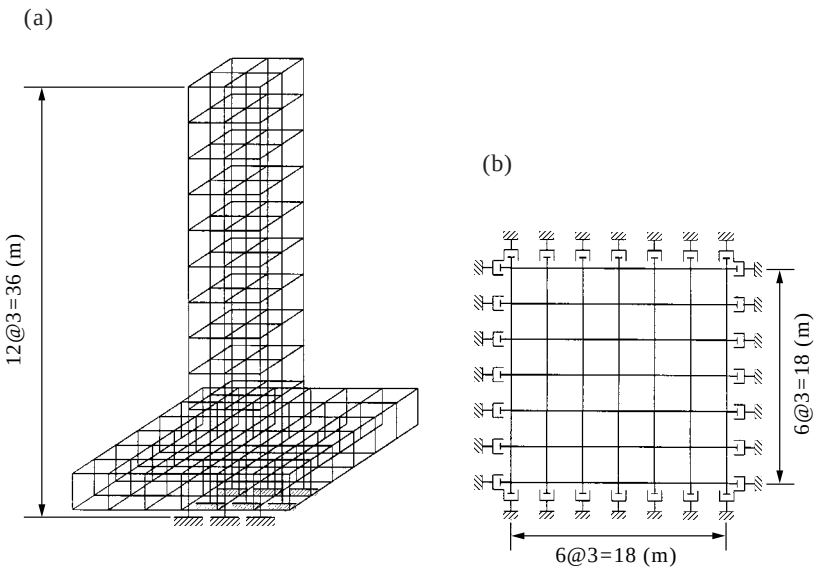


Figure 5. (a) Three-dimensional building structure. (b) Damping from two-layer foundation. Young's modulus (N/m$^2$): 2·1 E + 11; mass density (kg/m$^3$): 7850; cross-section inertia (m$^4$); 0·833 E-05; cross-section area (m$^2$): 0·01; proportional damping coefficients: $\alpha = -0{\cdot}1755$, $\beta = 0{\cdot}02005$; concentrated damping $C$ (N/m/s): 12 000.

TABLE 5

*Eigenvalues of the three-dimensional building structure with concentrated dampers*

| Mode number | Eigenvalues |
|---|---|
| 1 | −0·13811 + j 3·09308 |
| 2 | −0·13811 + j 3·09308 |
| 3 | −0·13811 − j 3·09308 |
| 4 | −0·13811 − j 3·09308 |
| 5 | −3·53017 + j 2·20867 |
| 6 | −3·53017 − j 2·20867 |
| 7 | −0·24297 + j 4·16980 |
| 8 | −0·24297 − j 4·16980 |
| 9 | −1·65509 + j 7·04244 |
| 10 | −1·65509 + j 7·04244 |
| 11 | −1·65509 − j 7·04244 |
| 12 | −1·65509 − j 7·04244 |

All executions are done on the CONVEX C3420 with 100 MIPS and 200 MFLOPS.

### 3.1. PLANE FRAME STRUCTURE WITH LUMPED DAMPERS

The finite element model of a plane frame is used as the first example. The dimensionless values of the geometric configuration and material properties are

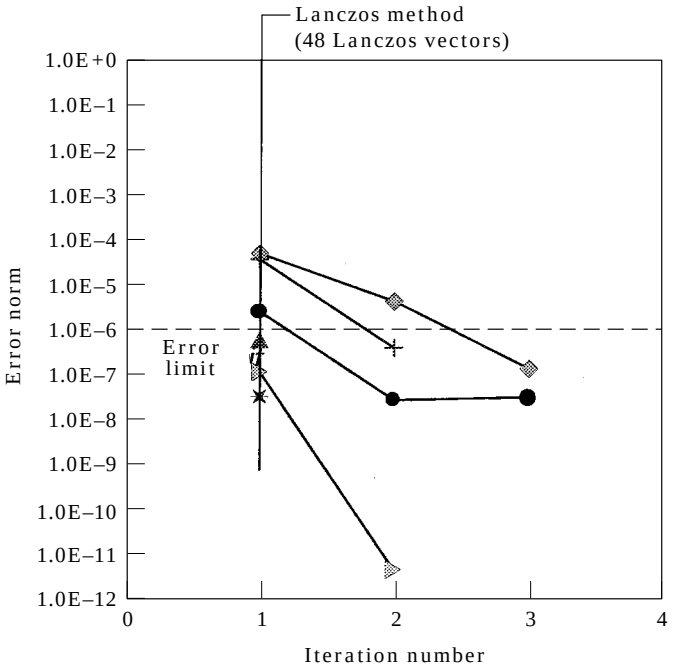


Figure 6. Variation of the error norm of the three-dimensional building by the proposed method. ×, 1st, 3rd eigenpairs; ▲, 2nd, 4th eigenpairs; ●, 5th, 6th eigenpairs; ◆, 7th, 8th eigenpairs; ▷, 9th, 11th eigenpairs; +, 10th, 12th eigenpairs 2.

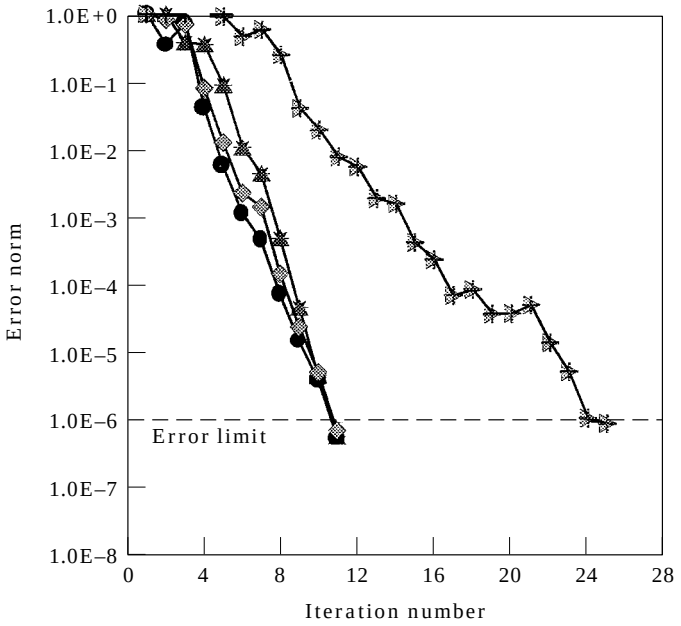


Figure 7. Variation of the error norm of the three-dimensional building by the subspace iteration method. Key as for Figure 6.

shown in Figure 1. The model is discretized in 200 beam elements resulting in the system of dynamic equations with a total of 590 degrees of freedom. Thus, the order of the associated eigenproblem is 1180. The consistent mass matrix is used

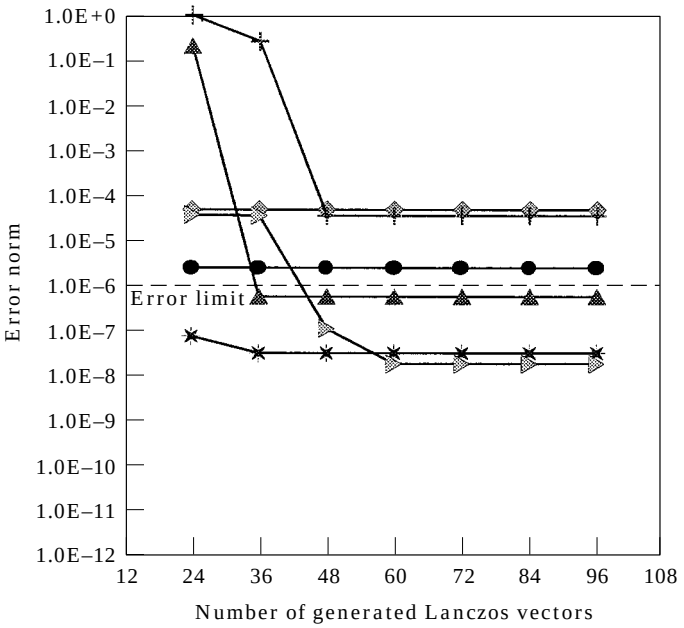


Figure 8. Variation of the error norm of the three-dimensional building by the Lanczos method. Key as for Figure 6.

TABLE 6

*CPU time spent for the first* 12 *eigenvalues of the three-dimensional building structure with concentrated dampers*

| Methods | CPU time in seconds (ratio) |
|---|---|
| Proposed method | 7641·94 (1·00) |
| Subspace iteration method | 8337·60 (1·09) |

for $M$. Its damping matrix is derived from the proportional damping expression given by $C = \alpha M + \beta K$ and concentrated dampers.

The eigenvalues of the model are shown in Table 2. All the eigenvalues of the model are multiple. The variations of the error norms to the iteration step are shown in Figures 2–4. The error norms of the initial values obtained by using the $4p(=48)$ Lanczos vectors are about 0·7 to $10^{-7}$. Using the results of the Lanczos method, the multiplicity of the desired eigenvalues can be checked. The number of iterations for the proposed method applied to the initial values that do not satisfy the error norm $10^{-6}$ is only one. The results in Figures 2 and 3 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The CPU time for the proposed method is compared with that of the subspace iteration method in Table 3. If we let the solution time for the proposed method be 1, it takes 3·55 times for the subspace iteration method. In Table 4, the CPU time for the Lanczos method is summarized. Because the method does not need complex operations, less solution time is required. However, the results of the Lanczos method, as shown in Figure 4, are not improved in spite of the increase of the number of Lanczos vectors.

### 3.2. THREE-DIMENSIONAL BUILDING STRUCTURE WITH CONCENTRATED DAMPERS

In this example a three-dimensional building structure with concentrated dampers is presented. The geometric configuration and material properties are shown in Figure 5. The model is divided into 436 beam elements and has 1128 degrees of freedom. The order of the associated eigenproblem is 2256. The

TABLE 7

*CPU time for the Lanczos method versus the number of generated Lanczos vectors of three-dimensional building structure*

| The number of generated Lanczos vectors | CPU time in seconds |
|---|---|
| 24 | 613·33 |
| 36 | 933·51 |
| 48 | 1246·60 |
| 60 | 1572·73 |
| 72 | 2000·39 |
| 84 | 2227·23 |
| 96 | 2582·77 |

consistent mass matrix is used to define $M$. The damping matrix consists of the Rayleigh damping and concentrated dampers.

The results of the proposed method are summarized in Table 5. The first and second eigenvalues are coincident, and also the ninth and tenth eigenvalues and their conjugate eigenvalues are coincident. The variations of the error norms to the iteration step are shown in Figures 6–8. The first step of the proposed method denotes the results of the Lanczos algorithm. The error norms of the initial values obtained by using the 48 Lanczos vectors are about $10^{-4}$ to $10^{-7}$. The number of iterations for the proposed method applied to the initial values that do not satisfy the error norm $10^{-6}$ is one or two. The results in Figures 6–8 indicate that the convergence of the proposed method is much better than that of the subspace iteration method. The CPU time for the proposed method is compared with the subspace iteration method in Table 6. If the solution time for the proposed method is 1, it takes 1·09 times for the subspace iteration method. In Table 7, the CPU time for the Lanczos method is summarized. Because the method does not need complex operations, less solution time is required. However, the results of the Lanczos method, as shown in Figure 8 are not improved in spite of the increase of the number of Lanczos vectors.

## 4. CONCLUSIONS

An efficient method for solving damped structural dynamic eigenproblems with multiple eigenvalues as well as distinct ones is presented. Characteristics of the proposed method identified by the numerical results from test problems are identified as follows: (1) Since the convergence rate of the proposed method is high, the proposed method is very effective for solving damped dynamic systems with a large number of degrees of freedom. (2) Non-singularity of the proposed method is always guaranteed, which is proved analytically. (3) The algorithm of the proposed method is simple.

## ACKNOWLEDGMENT

This research was supported by the Korea Science and Engineering Foundation (No. 961-1203-014-2). The support of the Korea Science and Engineering Foundation is greatly appreciated.

## REFERENCES

1. T. K. Caughey and M. E. J. O'Kelly 1965 *Transactions of the ASME*, *Journal of Applied Mechanics* **32**, 583–588. Classical normal modes in damped linear dynamic systems.
2. C. B. Moler and G. W. Stewart 1973 *SIAM Journal of Numerical Analysis* **10**, 241–256. An algorithm for generalized matrix eigenvalue problems.
3. L. Kaufman 1974 *SIAM Journal of Numerical Analysis* **11**, 997–1024. LZ algorithm to solve the generalized eigenvalue problem.
4. L. Veselic 1983 *Computer Methods in Applied Mechanics and Engineering* **38**, 128–143. A global Jacobi method for a symmetric indefinite problem $Sx = \lambda Tx$.

5. L. MEIROVITCH and G. RYLAND, II 1979 *Journal of Sound and Vibration* **67**, 1–19. Response of slightly damped gyroscopic systems.
6. D. L. CRONIN 1990 *Computers and Structures* **36**, 133–138. Eigenvalue and eigenvector determination for nonclassically damped dynamic systems.
7. M. K. KWAK 1993 *Journal of Sound and Vibration* **160**, 351–357. Perturbation method for the eigenvalue problem of lightly damped systems.
8. S. S. PERES-DA-SILVA, D. L. CRONIN and T. W. RANDOLPH 1995 *Computers and Structures* **57**, 883–891. Computation of eigenvalues and eigenvectors of nonclassically damped systems.
9. T. TANG and W. L. WANG 1995 *Journal of Sound and Vibration* **187**, 671–687. Perturbation method for determining eigensolutions of weakly damped systems.
10. K. K. GUPTA 1974 *International Journal of Numerical Methods in Engineering* **8**, 877–911. Eigenproblem solution of damped structural systems.
11. K. K. GUPTA 1981 *International Journal of Numerical Methods in Engineering* **17**, 187–198. Development of a unified numerical procedure for free vibration analysis of structures.
12. S. UTKU and J. L. M. CLEMENTE 1984 *Computers and Structures* **19**, 843–847. Computation of eigenpairs of $Ax = \lambda Bx$ for vibrations of spinning deformable bodies.
13. H. C. CHEN and R. L. TAYLOR 1986 *Report No. UCB/SEMM*-86/10, *University of California at Berkeley*. Properties and solutions of the eigensystem of non-proportionally damped linear dynamic systems.
14. A. Y. T. LEUNG 1995 *Journal of Sound and Vibration* **184**, 627–637. Subspace iteration method for complex symmetric eigenproblems.
15. C. LANCZOS 1950 *Journal of Research of the National Bureau of Standards* **45**, 255–282. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators.
16. C. C. PAIGE 1971 *Ph.D. Thesis*, *University of London*. The computation of eigenvalues and eigenvectors of very large sparse matrices.
17. C. C. PAIGE 1972 *Journal of the Institute of Mathematics and its Applications* **10**, 373–381. Computational variants of the Lanczos method for the eigenproblem.
18. C. C. PAIGE 1976 *Journal of the Institute of Mathematics and its Applications* **18**, 341–349. Error analysis of the Lanczos algorithm for tridiagonalizing a symmetric matrix.
19. B. N. PARLETT and D. S. SCOTT 1979 *Mathematics of Computation* **33**, 217–238. The Lanczos algorithm with selective orthogonalization.
20. H. D. SIMON 1984 *Mathematics of Computation* **42**, 115–142. The Lanczos algorithm with partial reorthogonalization.
21. B. N. PARLETT, D. R. TAYLOR and Z. A. LIU 1985 *Mathematics of Computation* **14**, 105–124. A look-ahead Lanczos algorithm for unsymmetric matrices.
22. H. M. KIM and R. R. CRAIG, JR. 1988 *International Journal of Numerical Methods in Engineering* **26**, 2305–2318. Structural dynamics analysis using an unsymmetric block Lanczos algorithm.
23. C. RAJAKUMAR and C. R. ROGERS 1991 *International Journal for Numerical Methods in Engineering* **32**, 1009–1026. The Lanczos algorithm applied to unsymmetric generalized eigenvalue problem.
24. C. RAJAKUMAR 1993 *International Journal for Numerical Methods in Engineering* **105**, 1–22. Lanczos algorithm for the quadratic eigenvalue problem in engineering applications.
25. H. C. CHEN and R. L. TAYLOR 1988 *Computers and Structures* **30**, 151–161. Solution of eigenproblems for damped structural systems by Lanczos algorithm.
26. H. C. CHEN 1994 *Computers and Structures* **53**, 1109–1114. Efficient vibration analysis of general structural systems.

27. Z. C. Zheng, G. X. Ren and W. J. Wang 1997 *Journal of Sound and Vibration* **199**, 253–268. A reduction method for large scale unsymmetric eigenvalue problems in structural dynamics.
28. I. W. Lee, M. C. Kim and A. R. Robinson 1997 *ASCE*, *Journal of Engineering Mechanics* **124**, 576–580. An efficient solution method of eigenproblems for damped structural systems using the modified Newton–Raphson technique.
29. I. W. Lee, M. C. Kim and A. R. Robinson 1997 *Computers and Structures* **63**, 61–68. Determination of the natural frequencies and mode shapes for large structures by accelerated Newton–Raphson method.
30. K. J. Bathe and S. Ramaswamy 1980 *Computers and Structures* **23**, 313–331. An accelerated subspace iteration method.