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FORCED RESPONSE OF A SEMI-INFINITE PLATE OF PARABOLICALLY VARYING THICKNESS

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Forced motion of a semi-infinite plate of parabolically varying thickness is analyzed by the eigenfunction method. Analysis is based on classical theory. An exact closed form solution is obtained for free vibration. Plates clamped at both the edges and cantilever plates subjected to constant and half sine pulse loads uniformly distributed over a portion of the plate are taken as example problems. Numerical results computed for the transverse deflection are plotted in graphs.

1. INTRODUCTION

The study of vibrations of plates of non-uniform thickness is of greater importance because of their applications in fields such as civil engineering, aerospace engineering, machine design and design of earthquake resistant structures etc. A few papers available on forced vibration/motion of beams of variable thickness are as follows:

Mayer Jr. [1] considered the vibration response of geometrically non-linear elastic beams subjected to pulse and impulse loading. Laura *et al*. [2] have analyzed the free and forced vibration of beams of non-uniform cross-section by using Rayleigh's optimization technique. Lee *et al*. [3] have analyzed the free and forced vibration of non-uniform beams by expressing the frequency equation and dynamic forced response in terms of the fundamental solution of the system. Takahashi and Yoshioka [4] have presented the analysis of vibration and stability of a non-uniform L-shaped beam subjected to a tangential follower force distributed over the centerline by use of the transfer matrix approach. Bapat and Bhutani [5] have considered an exact general approach to study the free and forced torsional vibrations of the a system with N stepped changes in its thickness. Gupta and Goyal [6] have analyzed the forced motion of a semi-infinite plate of linearly varying thickness by the eigenfunction method.

In the present paper, the motion of a semi-infinite plate of parabolically varying thickness is considered by classical theory. An exact closed form solution is obtained for free vibration. The forced motion of the plate is analyzed by the eigenfunction method. Plates clamped at both the edges and cantilever plates

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subjected to constant or half sine pulse load uniformly distributed over a portion of the plate are considered as example problems. The numerical results computed for transverse deflection for various values of the taper constant, time and space variables for loads uniformly distributed over the whole plate are plotted in the graphs. The variation in thickness is taken in such a way that the average thickness of the plate remains constant.

2. EQUATION OF MOTION

A plate of infinite length, finite breadth a and thickness h varying parabolically along the breadth is considered. The plate is referred to as cartesian co-ordinates by taking the y-axis along the infinite length, the middle plane of the plate in the plane $z=0$ and the two edges in the plane $x=0$ and $x=a$.

The non-dimensional equation of motion of the plate according to classical theory is taken as in reference [6]

$$H^3W_{,XXXX} + 6H^2H_{,X}W_{,XXX} + 3(2HH_{,X}^2 + H^2H_{,XX})W_{,XX} + 12HW_{,TT} = 12P(X, T), \tag{1}$$

where $X = x/a$, $W = w/a$, $P = p(1-\nu^2)/E$, $H = h/a$, $T = t\sqrt{E/(1-\nu^2)\rho a^2}$ and w, ρ, t, p, E and ν are the transverse deflection, density of the plate, time, load per unit area, Young's modulus and Possion's ratio respectively. The comma followed by the variable suffix denotes differentiation with respect to that variable.

The parabolically varying thickness of the plate is taken as

$$H = H_0(1+\beta X)^2, \tag{2}$$

where $H_0 = h_0/a$, h_0 is the thickness of the plate at $x=0$ and β is the taper constant.

3. FORMAL SOLUTION

3.1. FREE VIBRATION

For free vibration, one takes

$$W(X, T) = W_j(X)\, \mathrm{e}^{\mathrm{i}\Omega_j T}, \quad j = 1, 2, 3, \ldots, \tag{3}$$

where Ω_j and W_j are the circular frequency and mode shape function respectively in the jth normal mode of free vibration.

Substitution of equations (2) and (3) in equation (1) after putting $P=0$ yields

$$(1+\beta X)^4 W_{j,XXXX} + 12\beta(1+\beta X)^3 W_{j,XXX} + 30\beta^2(1+\beta X)^2 W_{j,XX} - \omega_j^2\beta^4 W_j = 0, \tag{4}$$

where $\omega_j^2 = 12\Omega_j^2/H_0^2\beta^4$.

The closed form solution of equation (4) comes out to be

$$W_j(X) = (1+\beta X)^{-1\cdot 5} S_j(X) D_j, \tag{5}$$

where

$$S_j(X) = [\cosh\{\lambda_{1j}\log(1+\beta X)\}\ \sinh\{(\lambda_{1j}\log(1+\beta X)\}$$
$$\times\ \cos\{\lambda_{2j}\log(1+\beta X)\}\ \sin\{\lambda_{2j}\log(1+\beta X)\}],$$
$$\lambda_{1j} = 0{\cdot}5[17+\{4+\omega_j\}^{1/2}]^{1/2}, \qquad \lambda_{2j} = 0{\cdot}5[-17+\{4+\omega_j\}^{1/2}]^{1/2},$$

$\mathbf{D}_j = [d_{1j}d_{2j}d_{3j}d_{4j}]'$ is the arbitrary constant vector and prime denotes the transpose of matrix.

The orthonormality condition for the normal modes of free vibration can be taken as

$$\int_0^1 HW_jW_k\,\mathrm{d}X = \delta_k^j, \tag{6}$$

where δ_k^j is the Kronecker delta.

3.2. FORCED MOTION

The solution of the force motion equation (1) is assumed in the form

$$W(X, T) = \sum W_j(X)g_j(T), \tag{7}$$

where summation over j is taken from 1 to ∞.

Substitution of equation (7) in equation (1) and use of equation (4) gives

$$\sum HW_j(g_{j,TT} + \Omega_j^2 g_j) = P(X, T). \tag{8}$$

Using orthonormality condition (6), one gets

$$g_{j,TT} + \Omega_j^2 g_j = G_j(T), \tag{9}$$

where

$$G_j(T) = \int_0^1 W_j(X)P(X, T)\,\mathrm{d}X. \tag{10}$$

The solution of equation (9) is

$$\Omega_j g_j(T) = \Omega_j g_j(0)\cos(\Omega_j T) + g_{j,T}(0)\sin(\Omega_j T) + \int_0^T G_j(\tau)\sin\{\Omega_j(T-\tau)\}\,\mathrm{d}\tau, \tag{11}$$

where

$$g_j(0) = \sum H\int_0^1 W(X, 0)W_j\,\mathrm{d}X, \qquad g_{j,T}(0) = \sum H\int_0^1 W_{,T}(X, 0)W_j\,\mathrm{d}X. \tag{12}$$

4. EXAMPLE PROBLEM

4.1. INITIAL CONDITIONS

The initial conditions are taken as $W(X, 0) = W_{,T}(X, 0) = 0$, which gives

$$g_j(0) = g_{j,T}(0) = 0. \tag{13}$$

4.2. LOADING CONDITIONS

The following two types of external loads uniformly distributed over a portion of the plate are taken:

4.2.1. *Constant load* (CL)

$$P(X, T) = [P_0/(X_2 - X_1)][U(X - X_1) - U(X - X_2)]U(T); \quad 0 \leqslant X_1 < X_2 \leqslant 1, \tag{14}$$

where P_0 is the total load on the plate and U denotes a unit step function.

4.2.2 *Half sine pulse load* (HL)

$$P(X, T) = \frac{P_0}{(X_2 - X_1)}[U(X - X_1) - U(X - X_2)]\{1 - U(T - t_1)\} \sin(\pi T/t_1);$$
$$0 \leqslant X_1 < X_2 \leqslant 1, \tag{15}$$

where t_1 is the duration of HL.

4.3. EDGE CONDITIONS

The plate is subjected to two types of edge conditions:

4.3.1. *Clamped at both the edges* (C–C)

For this condition $W = W_{,X} = 0$ at $X = 0$ and $X = 1$, which reduces to

$$W_j = W_{j,X} = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{16}$$

4.3.2 *Clamped at* $X = 0$ *and free at* $X = 1$ (C–F)

For this condition $W = W_{,X} = 0$ at $X = 0$ and $W_{,XX} = W_{,XXX} = 0$ at $X = 1$, which leads to

$$W_j = W_{j,X} = 0 \text{ at } X = 0 \text{ and } W_{j,XX} = W_{j,XXX} = 0 \text{ at } X = 1. \tag{17}$$

4.4. FREE VIBRATION ANALYSIS

For the sake of convenience the suffix j is suppressed in free vibration analysis.

4.4.1. *Frequency equation*

For a C–C plate, the edge conditions (16) give

$$d_1 + d_3 = 0, \qquad -1{\cdot}5d_1 + \lambda_1 d_2 - 1{\cdot}5d_3 + \lambda_2 d_4 = 0,$$
$$C_1 d_1 + S_1 d_2 + C_2 d_3 + S_2 d_4 = 0,$$

$$(\lambda_1 S_1 - 1{\cdot}5C_1)d_1 + (\lambda_1 C_1 - 1{\cdot}5S_1)d_2 - (\lambda_2 S_2 + 1{\cdot}5C_2)d_3 + (\lambda_2 C_2 - 1{\cdot}5S_2)d_4 = 0, \tag{18}$$

where

$$C_1 = \cosh\{\lambda_1 \log(1+\beta)\}, \qquad S_1 = \sinh\{\lambda_1 \log(1+\beta)\},$$
$$C_2 = \cos\{\lambda_2 \log(1+\beta)\}, \qquad S_2 = \sin\{\lambda_2 \log(1+\beta)\}. \tag{19}$$

The corresponding frequency equation is

$$\lambda_1\lambda_2(C_2 - C_1)^2 + (\lambda_1 S_2 - \lambda_2 S_1)(\lambda_2 S_2 + \lambda_1 S_1) = 0. \tag{20}$$

For a C–F plate, the edge conditions (17) gives

$$d_1 + d_3 = 0, \qquad -1{\cdot}5d_1 + \lambda_1 d_2 - 1{\cdot}5d_3 + \lambda_2 d_4 = 0,$$
$$(e_1 C_1 - e_3 S_1)d_1 + (e_1 S_1 - e_3 C_1)d_2 + (e_4 S_2 - e_2 C_2)d_3 + (-e_4 C_2 - e_2 S_2)d_4 = 0,$$
$$(f_1 S_1 - f_3 C_1)d_1 + (f_1 C_1 - f_3 S_1)d_2 + (f_4 C_2 - f_2 S_2)d_3 + (f_4 S_2 - f_2 C_2)d_4 = 0, \tag{21}$$

where

$$e_1 = \lambda_1^2 + 3{\cdot}75, \qquad e_2 = \lambda_2^2 - 3{\cdot}75, \qquad e_3 = 4\lambda_1, \qquad e_4 = 4\lambda_2,$$
$$f_1 = \lambda_1(\lambda_1^2 + 17{\cdot}25), \qquad f_2 = \lambda_2(\lambda_2^2 - 17{\cdot}25), \qquad f_3 = 7{\cdot}5\lambda_1^2 + 13{\cdot}125,$$
$$f_4 = 7{\cdot}5\lambda_2^2 - 13{\cdot}125.$$

The corresponding frequency equation is

$$a_1 b_2 - a_2 b_1 = 0, \tag{22}$$

where

$$a_1 = e_1 C_1 - e_3 S_1 + e_2 C_2 - e_4 S_2, \qquad b_1 = \lambda_2(e_1 S_1 - e_3 C_1) + \lambda_1(e_2 S_2 + e_4 C_2),$$
$$a_2 = f_1 S_1 - f_3 C_1 - f_2 S_2 - f_4 C_2, \qquad b_2 = \lambda_2(f_1 C_1 - f_3 S_1) + \lambda_1(f_2 C_2 - f_4 S_2).$$

The countably infinite roots of these frequency equations are the natural frequencies Ω_j for various normal modes of transverse vibration of the plate.

4.4.2. *Mode shape*

The mode normalization condition (6) is used for determining the unique solution for W. The solution is

$$W(X) = (1+\beta X)^{-1{\cdot}5} Z(X)[2\lambda_1\lambda_2\beta/H_0 f(\beta)]^{1/2}, \qquad Z(X) = S(X)[q \;\; r \;\; -q \;\; 1]' \tag{23}$$

$$\begin{aligned} f(\beta) = {} & \lambda_1\lambda_2(2q^2 - r^2 + 1)\log(1+\beta) + S_1\lambda_2\{(q^2 + r^2)C_1 + 2qrS_1\} \\ & + \lambda_1 S_2\{(q^2 - 1)C_2 - 2qS_2\} \\ & + \{-q^2(\lambda_1 S_1 C_2 + \lambda_2 C_1 S_2) + q(\lambda_1 S_1 S_2 - \lambda_2 C_1 C_2) \\ & - qr(\lambda_1 C_1 C_2 + \lambda_2 S_1 S_2) \\ & + r(\lambda_1 C_1 S_2 - \lambda_2 S_1 C_2)\}[4\lambda_1\lambda_2/(\lambda_1^2 + \lambda_2^2)], \end{aligned} \tag{24}$$

where

$$r = -\lambda_2/\lambda_1, \qquad q = \begin{Bmatrix} (\lambda_2 S_1 - \lambda_1 S_2)/\lambda_1(C_1 - C_2), & \text{for C–C case,} \\ b_1/\lambda_1 a_1, & \text{for C–F case,} \end{Bmatrix}.$$

4.5. FORCED MOTION ANALYSIS

The loading conditions, given by equation (14) or (15), and the mode shape function, given by equation (23) as the case may be, are substituted in equation (10). The $G_j(T)$ thus obtained are substituted in equation (11). The casewise results obtained are as follows:

4.5.1. C–C *or* C–F *plate subject to* CL

$$g_j(T) = P_j[1 - \cos(\Omega_j T)]/\Omega_j^2, \qquad P_j = P_0[\phi_j(X_2) - \phi_j(X_1)], \tag{25}$$

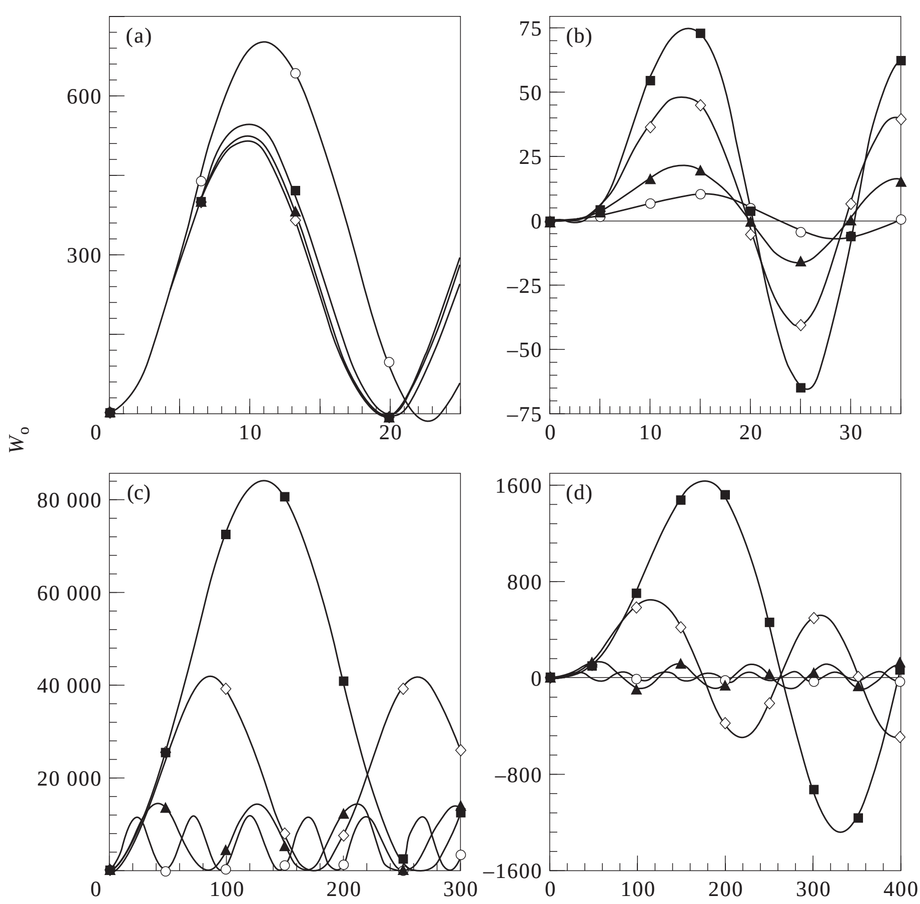

Figure 1. W_0 versus T for $H_a = 0{\cdot}05$: (*a*) C–C, CL ($X = 0{\cdot}5$); (b) C–C, HL ($X = 0{\cdot}5$); (c) C–F, CL ($X = 1{\cdot}0$) and (d) C–F, HL ($X = 1{\cdot}0$) for various value of β. Key: —○—, −0·7; —▲—, −0·3; —◇—, 0·3; —■—, 0·7.

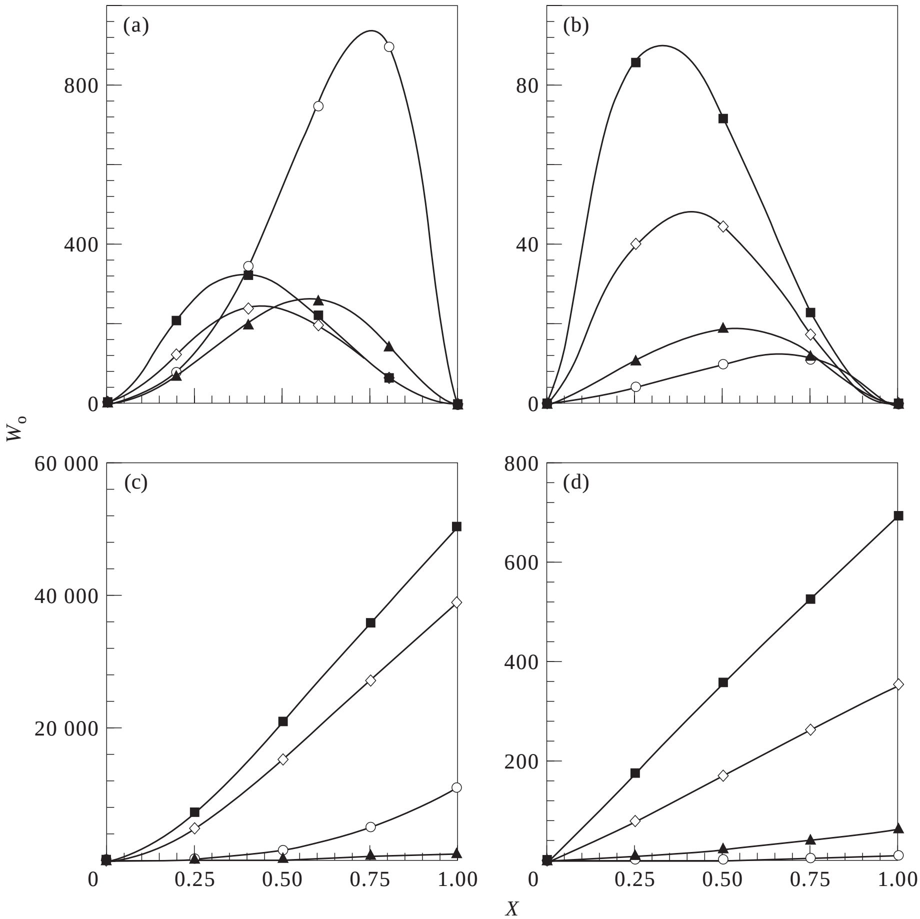

Figure 2. W_0 versus X for $H_a = 0{\cdot}05$: (a) C–C, CL ($T = 15$); (b) C–C, HL ($T = 15$); (c) C–F, CL ($T = 75$) and (d) C–F, HL ($T = 75$) for various value of β. Key as for Figure 1.

$$\begin{aligned}\phi_j(X) = (1+\beta X)^{-1/2}\{1/(0{\cdot}25-\lambda_{1j}^2)\}[(-0{\cdot}5q_j-\lambda_{1j}r_j)\cosh\{\lambda_{1j}\log(1+\beta X)\}\\ +(-\lambda_{1j}q_j-0{\cdot}5r_j)\sinh\{(\lambda_{1j}\log(1+\beta X))\}]\\ +\{1/(0{\cdot}25+\lambda_{2j}^2)\}[(0{\cdot}5q_j-\lambda_{2j})\cos\{(\lambda_{2j}\log(1+\beta X))\}\\ +(-0{\cdot}5-q_j\lambda_{2j})\sin\{\lambda_{2j}\log(1+\beta X)\}]][2\lambda_1\lambda_2\beta/H_0f(\beta)]^{1/2}.\end{aligned} \quad (26)$$

4.5.2. C–C *or* C–F *plate subjected to* HL

$$g_j(T) = \left\{\begin{array}{ll} P_j t_1[\pi\sin(\Omega_j T) - \Omega_j t_1 \sin(\pi T/t_1)]/[\Omega_j(\pi^2-\Omega_j^2 t_1^2)], & \text{when } T < t_1, \\ 2P_j\pi t_1[\sin\{\Omega_j(T-t_1/2)\}\cos(\Omega_j t_1/2)]/[\Omega_j(\pi^2-\Omega_j^2 t_1^2)], & \text{when } T \geqslant t_1, \end{array}\right\}. \quad (27)$$

The substitution of unique mode shape $W_j(X)$ given by equation (23) and $g_j(T)$ from equation (25) or (27), as the case may be, in equation (7) gives the transverse deflection $W(X, T)$ for forced motion.

5. RESULTS AND DISCUSSION

The variation in thickness is taken in such a way that the average thickness of the plate, h_a, remains constant by taking

$$\int_0^a h_0(1 + \beta x/a)^2 \, \mathrm{d}x = ah_a,$$

which leads to

$$H_0 = 2H_a\beta/[(1 + \beta)^3 - 1],$$

where $H_a = h_a/a$.

The frequencies Ω_j from equations (19) and (22) are computed by the bisection method up to an accuracy of five decimal places. The series of W (equation (7)) is summed up to 14 terms, which ensures an accuracy of four decimal places.

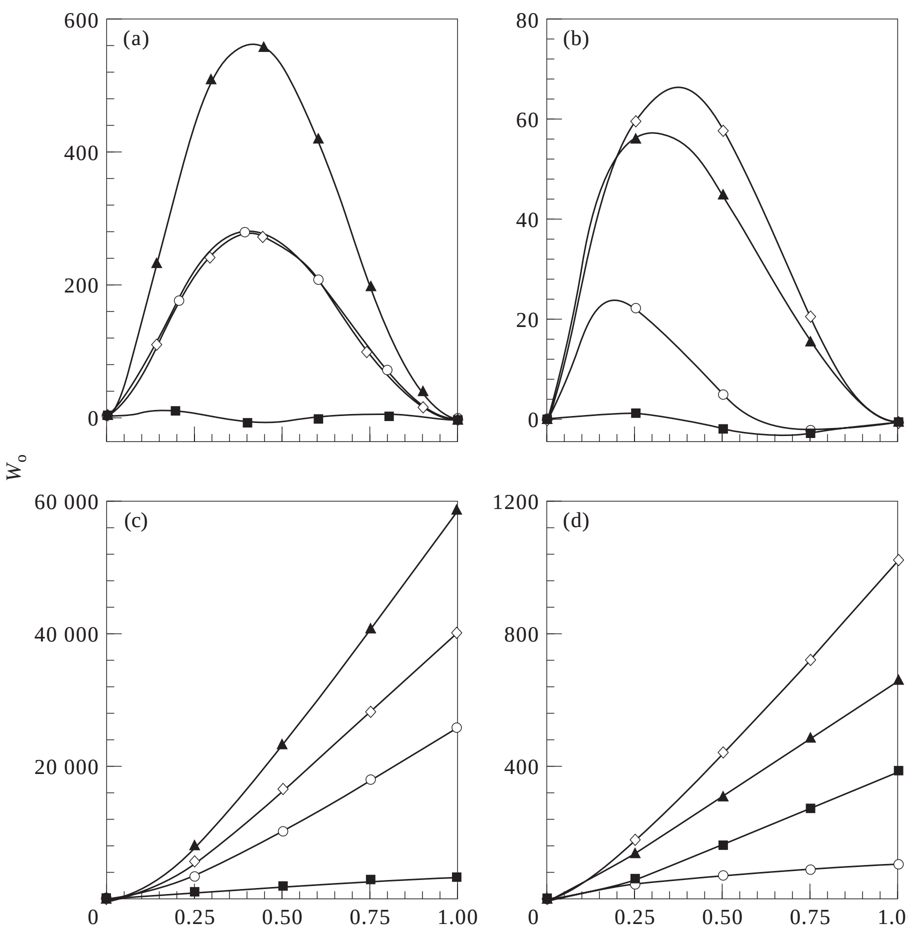

Figure 3. W_0 versus X for $H_a = 0{\cdot}05$ and $\beta = 0{\cdot}5$: (a) C–C, CL; (b) C–C, HL; (c) C–F, CL and (d) C–F, HL for various value of T. Key: —○—, 5(a), 10(b), 25(c), 35(d); —▲—, 10(a), 15(b), 35(c), 45(d); —◇—, 15(a), 20(b), 45(c), 55(d); —■—, 20(a), 25(b), 55(c), 65(d).

Numerical results are computed for $W_0 = W/P_0$ for various values of β, X, T by taking $\nu = 0{\cdot}3$, $H_a = 0{\cdot}05$ and $t_1 = 2\pi/\Omega_1$, where Ω_1 is the fundamental frequency.

The graphs for various values of β for W_0 at $X = 0{\cdot}5$ versus T for the C–C plate are plotted in Figures 1(a) and (b) and for W_0 at $X = 1{\cdot}0$ versus T for C–F plates are plotted in Figures 1(c) and (d) for CL and HL respectively. Figure 1(a) shows that with the increase in β the deflection first decreases and then increases. Figures 1(b–d) show that the deflection increase with the increase in β. The peaks are seen on one side of the plate for CL and alternatively on both sides of the plate of HL.

The graphs for various values of β for W_0 at $T = 15$ versus X for C–C plate are plotted in Figures 2(a) and (b) and for W_0 at $T = 75$ versus T for C–F plates are plotted in Figures 2(c) and (d) for CL and HL respectively. Figure 2(a) shows

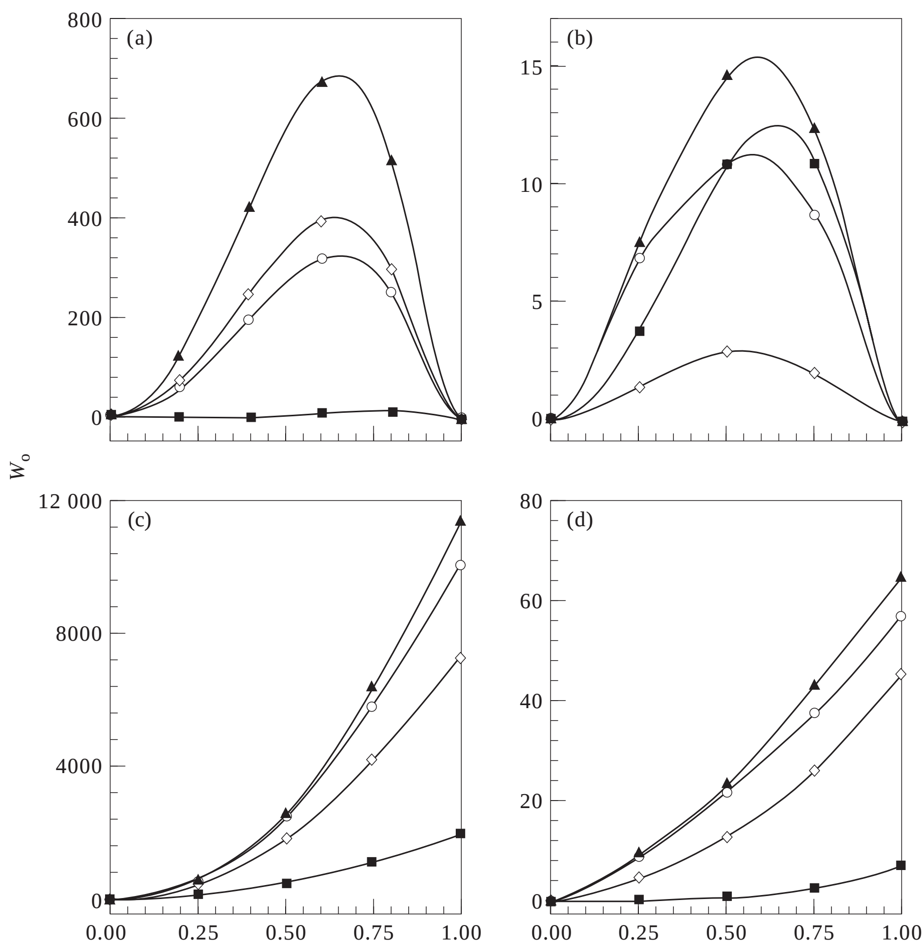

Figure 4. W_0 versus X for $H_a = 0{\cdot}5$ and $\beta = -0{\cdot}5$: (a) C–C, CL; (b) C–C, HL; (c) C–F, CL and (d) C–F, HL for various value of T. Key: —○—, 5(a,b), 50(c,d); —▲—, 10(a,b), 100(c,d); —◇—, 15(a,b), 150(c,d); —■—, 20(a,b), 200(c,d).

that the deflection first decreases and then increases with increase in β. Figures 2(b–d) show that the deflection increases with the increase in β.

The graphs of W_0 versus X for $\beta = 0{\cdot}5$ for various values of T for C–C plate are plotted in Figures 3(a) and (b) and for C–F plates in Figures 3(c) and (d) for CL and HL respectively. In all these figures the deflection first increases then decreases after attaining its maximum value.

The graphs of W_0 versus X for $\beta = -0{\cdot}5$ for various values of T for C–C plate are plotted in Figures 4(a) and (b) and for C–F plates in Figures 4(c) and (d) for CL and HL respectively. Similar variation as in Figure 3 is observed here.

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