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LETTERS TO THE EDITOR

DEGENERATED HOPF BIFURCATIONS IN A STOCHASTICALLY DISTURBED NON-LINEAR SYSTEM

C. W. S. To and D. M. Li

Department of Mechanical Engineering, University of Nebraska, 104*N Walter Scott Engineering Center, Lincoln, Nebraska* 68588-0656, *U.S.A.*

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1. INTRODUCTION

In an early paper a technique was presented for the analysis of Hopf bifurcations of a single-degree-of-freedom (sdof) non-linear system under stochastic disturbances [1] by employing the perturbation method [2], the stochastic averaging of Stratonovich [3] and some results from the singularity theory and group theory [4]. Loosely speaking, Hopf bifurcations are known as limit cycles in mechanics while in engineering they are called flutters. The unfolding of degenerated Hopf bifurcations can reveal some important and interesting global dynamic behaviors of the system. However, the issue of degenerated Hopf bifurcation in which the eigenvalue crossing condition is not satisfied has not been addressed. Therefore, in this note an approach is presented for the analysis and results of degenerated Hopf bifurcations of a system excited by stochastic disturbances are included. The approach is based on the perturbation method [2], the limit theorem of Khas'minskii [5] which is in essence similar to the stochastic averaging of Stratonovich [3] but is relatively more simple to use for second order approximations, and some results from the singularity theory and group theory [4]. It should be emphasized that the application of the results from the singularity theory and group theory enables us to examine the important issue of degenerated Hopf bifurcations of system excited by stochastic disturbances which cannot be dealt with using other existing established techniques, such as the multiple scale technique. A direct application of the currently presented analysis technique is in the random vibration control of structural systems which may be treated as sdof systems with random velocity feedback. An illustration of such an application is presented in section 5.

The main result presented in this note is that with a positive unfolding parameter the stochastically disturbed system bifurcates at a larger magnitude of the bifurcation point than the corresponding deterministic value. With a negative unfolding parameter the stochastically disturbed system can have bifurcation if a certain criterion is satisfied, otherwise, the system does not bifurcate similar to the corresponding deterministic system. It may be appropriate to mention that another purpose of this note is to serve as an example of applying advanced mathematical theories to difficult vibration problems which, otherwise, are impossible to analyse or solve.

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2. STATEMENT OF THE PROBLEM

In order to present the brief and yet tractable steps the minimal amount of algebra in this note is unavoidable and essential.

The equation of motion for the problem is

$$\dot{y} = \mathbf{F}(y, \lambda, \alpha) + \sqrt{\epsilon} f_i(t)\mathbf{A}_i y, \tag{1}$$

where $i = 1, 2, \ldots, n$ and $y \in R^2$; $\lambda, \alpha \in R^1$; $f_i(t)$ are independent stationary broad-band stochastic processes with zero mean values, $|\epsilon| \ll 1$ and $f_i(t)$ has arbitrary smoothly varying spectral density function. $\mathbf{F}(0, \lambda, \alpha) = 0$ and $\mathbf{A}_i$ are 2×2 matrices.

Let $\mathbf{A}(\lambda, \alpha) = D_y\mathbf{F}(0, \lambda, \alpha)$, then $\mathbf{A}(\lambda, \alpha)$ has eigenvalues $\sigma(\lambda, \alpha) \pm \mathrm{i}\omega(\lambda, \alpha)$. The generic assumptions that one adopted here are: there exists a point (λ_0, α_0) such that (i) $\sigma(\lambda_0, \alpha_0) = 0$, $\omega(\lambda_0, \alpha_0) = 1$, (*ii*) $\sigma_\lambda(\lambda_0, \alpha_0) = 0$, $\sigma_{\lambda\lambda}(\lambda_0, \alpha_0) \neq 0$, $\sigma_\alpha(\lambda_0, \alpha_0) \neq 0$, where the subscript denotes differentiation with respect to that subscript and the double subscripts designate differentiation twice with respect to the subscript; and

$$\mathbf{A}(\lambda, \alpha) = \begin{bmatrix} \sigma(\lambda, \alpha) & \omega(\lambda, \alpha) \\ -\omega(\lambda, \alpha) & \sigma(\lambda, \alpha) \end{bmatrix}.$$

The type of degenerated Hopf bifurcations of interest here is the one in which the eigenvalue crossing condition in (ii) has a zero crossing speed, that is, $\sigma_\lambda(\lambda_0, \alpha_0) = 0$.

To proceed further one expands the real part of the eigenvalue as

$$\begin{aligned} \sigma(\lambda, \alpha) &= \sigma(\lambda_0, \alpha_0) + \sigma_\lambda(\lambda_0, \alpha_0)(\lambda - \lambda_0) \\ &\quad + \sigma_\alpha(\lambda_0, \alpha_0)(\alpha - \alpha_0) + \tfrac{1}{2}\sigma_{\lambda\lambda}(\lambda_0, \alpha_0)(\lambda - \lambda_0)^2 + \cdots \\ &= \sigma_\alpha(\lambda_0, \alpha_0)(\alpha - \alpha_0) + \tfrac{1}{2}\sigma_{\lambda\lambda}(\lambda_0, \alpha_0)(\lambda - \lambda_0)^2 + \cdots. \end{aligned} \tag{2}$$

Let

$$\alpha - \alpha_0 = \epsilon\gamma \qquad \text{and} \qquad \lambda - \lambda_0 = \sqrt{\epsilon}\zeta, \tag{3}$$

then

$$\sigma(\lambda, \alpha) = \epsilon(\gamma\sigma_\alpha + (\zeta^2/2)\sigma_{\lambda\lambda}) + o(\epsilon) \tag{4}$$

and

$$\omega(\lambda, \alpha) = 1 + \sqrt{\epsilon}\omega_\lambda\zeta + \epsilon\omega_\alpha\gamma + (\epsilon/2)\omega_{\lambda\lambda}\zeta^2 + o(\epsilon). \tag{5}$$

Expanding $\mathbf{F}(y, \lambda, \alpha)$ of equation (1) at $y = 0$, one obtains

$$\mathbf{F}(y, \lambda, \alpha) = \mathbf{A}(\lambda, \alpha)y + H_2(y, \lambda, \alpha) + H_3(y, \lambda, \alpha) + \cdots,$$

where $H_2(y, \lambda, \alpha)$ and $H_3(y, \lambda, \alpha)$ are homogeneous polynomials in y with degree 2 and 3, respectively.

By rescaling $y = \sqrt{\epsilon}x$, then

$$H_2(y, \lambda, \alpha) = \epsilon H_2(x, \lambda_0, \alpha_0) + \epsilon^{3/2} H_2'(x, \lambda_0, \alpha_0) + o(\epsilon^{3/2}),$$

$$H_3(y, \lambda, \alpha) = \epsilon^{3/2} H_3(x, \lambda_0, \alpha_0) + o(\epsilon^{3/2}),$$

where the prime denotes partial differentiation with respect to λ. Thus, the above expression for $\mathbf{F}(y, \lambda, \alpha)$ becomes

$$\mathbf{F}(y, \lambda, \alpha) = \mathbf{A}(\lambda, \alpha)\sqrt{\epsilon}x + \epsilon H_2(x, \lambda_0, \alpha_0) + \epsilon^{3/2}[H_2'(x, \lambda_0, \alpha_0)\zeta + H_3(x, \lambda_0, \alpha_0)] + o(\epsilon^{3/2}),$$

where

$$\mathbf{A}(\lambda, \alpha)x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \mathbf{B}(x, \lambda, \alpha) + o(\epsilon),$$

in which

$$\mathbf{B}(x, \lambda, \alpha) = \begin{bmatrix} \epsilon(\gamma\sigma_\alpha + \frac{1}{2}\zeta^2\sigma_{\lambda\lambda}) & (\sqrt{\epsilon}\zeta\omega_\lambda + \epsilon\gamma\omega_\alpha + (\epsilon/2)\zeta^2\sigma_{\lambda\lambda}) \\ -(\sqrt{\epsilon}\zeta\omega_\lambda + \epsilon\gamma\omega_\alpha + (\epsilon/2)\zeta^2\sigma_{\lambda\lambda}) & \epsilon(\gamma\sigma_\alpha + \frac{1}{2}\zeta^2\sigma_{\lambda\lambda}) \end{bmatrix} x.$$

Applying the above results, equation (1) reduces to

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} \epsilon\sigma^* & \sqrt{\epsilon}\omega^* \\ -\sqrt{\epsilon}\omega^* & \epsilon\sigma^* \end{bmatrix} x + \sqrt{\epsilon} H_2(x, \lambda_0, \alpha_0) + \epsilon[H_2'\zeta + H_3(x, \lambda_0, \alpha_0)] + \sqrt{\epsilon}\mathbf{A}_i f_i(t) x + o(\epsilon), \tag{6}$$

where

$$\sigma^* = \gamma\sigma_\alpha + \tfrac{1}{2}\zeta^2\sigma_{\lambda\lambda}, \qquad \omega^* = \zeta\omega_\lambda + \sqrt{\epsilon}\gamma\omega_\alpha + (\sqrt{\epsilon}/2)\zeta^2\omega_{\lambda\lambda}; \tag{7a,b}$$

$$H_2(x, \lambda_0, \alpha_0) = \begin{Bmatrix} B_{rs}^{(1)} x_r x_s \\ B_{rs}^{(2)} x_r x_s \end{Bmatrix}, \quad r \leqslant s, \quad r, s = 1, 2, \tag{8a}$$

$$H_3(x, \lambda_0, \alpha_0) = \begin{Bmatrix} C_{rsv}^{(1)} x_r x_s x_v \\ C_{rsv}^{(2)} x_r x_s x_v \end{Bmatrix}, \quad r \leqslant s \leqslant v, \quad r, s, v = 1, 2, \tag{8b}$$

$$H_2'(x, \lambda_0, \alpha_0) = \begin{Bmatrix} D_{rs}^{(1)} x_r x_s \\ D_{rs}^{(2)} x_r x_s \end{Bmatrix}, \quad r \leqslant s, \quad r, s = 1, 2. \tag{8c}$$

Before proceeding further some results are introduced for degenerated Hopf bifurcations in the deterministic system [6] as a theorem below.

Theorem 1. When $\epsilon = 0$ in a system defined by equation (1) where $\mathbf{F}(y, \lambda, \alpha)$ satisfies the assumptions in section 2 above and that $r_z(0, \lambda_0, \alpha_0) \neq 0$ then the non-trivial periodic solutions with small amplitudes are Z_2-equivalent to

$$\delta_1 x^3 + \delta_2(\lambda - \lambda_0)^2 x + \mu x = 0, \tag{9}$$

where $\delta_1 = \operatorname{sgn} r_z(0, \lambda_0, \alpha_0)$, $\delta_2 = -\operatorname{sgn} \sigma_{\lambda\lambda}(\lambda_0, \alpha_0)$, and λ is the bifurcation parameter and μ is the unfolding parameter, that is, $\mu = \alpha - \alpha_0$.

In the language of discrete symmetry group theory, the Z_2 referred to here is the two-element "reflection" group. The bifurcation diagrams of the above deterministic system are given in reference [6] and $r_z(0, \lambda_0, \alpha_0)$ has been obtained and presented by the authors in reference [1] as

$$\begin{aligned} r_z(0, \lambda_0, \alpha_0) = -\tfrac{1}{8}(2B_{11}^{(1)}B_{11}^{(2)} - B_{11}^{(1)}B_{12}^{(1)} - B_{12}^{(1)}B_{22}^{(1)} - 2B_{22}^{(1)}B_{22}^{(2)} + B_{11}^{(2)}B_{12}^{(2)} \\ + B_{12}^{(2)}B_{22}^{(2)}) - \tfrac{1}{8}(3C_{111}^{(1)} + C_{112}^{(2)} + C_{122}^{(1)} + 3C_{222}^{(2)}). \end{aligned} \tag{10}$$

3. STANDARD FORM AND SIMPLIFIED EQUATIONS

In this section the standard form and simplified equations for equation (1) are considered. This is accomplished by making use of equation (6) and the transformations

$$x_1 = a \sin \Phi, \quad x_2 = a \cos \Phi, \quad \Phi = t + \phi \tag{11}$$

such that

$$\begin{aligned} a^2 = x_1^2 + x_2^2, \quad \Phi = \arctan(x_1/x_2), \qquad \dot{a} = \dot{x}_1 \sin \Phi + \dot{x}_2 \cos \Phi, \\ a\dot{\Phi} = \dot{x}_1 \cos \Phi - \dot{x}_2 \sin \Phi. \end{aligned} \tag{12}$$

By making use of equation (6) and (12), and after some lengthy algebraic manipulation one obtains

$$\begin{aligned} \dot{a} &= \sqrt{\epsilon}(a^2/4)X_1(\Phi) + \sqrt{\epsilon}aX_{2i}(\Phi)f_i(t) + \epsilon X_3(a, \Phi) + o(\epsilon), \\ \dot{\phi}_1 &= \sqrt{\epsilon}(a/4)Y_1(\Phi) + \sqrt{\epsilon}Y_{2i}(\Phi)f_i(t) + \epsilon[\gamma\omega_\alpha + \tfrac{1}{2}\zeta^2\omega_{\lambda\lambda} + X_4(a, \Phi)] + o(\epsilon), \end{aligned} \tag{13}$$

where

$$\begin{aligned} X_1(\Phi) &= (B_{11}^{(2)} + B_{12}^{(1)} + 3B_{22}^{(2)}) \cos \Phi + (3B_{11}^{(1)} + B_{12}^{(2)} + B_{22}^{(1)}) \sin \Phi \\ &\quad + (B_{22}^{(2)} - B_{12}^{(1)} - B_{11}^{(1)}) \cos 3\Phi + (B_{22}^{(1)} + B_{12}^{(2)} - B_{11}^{(1)}) \sin 3\Phi, \\ X_{2i}(\Phi) &= A_{i11} \sin^2 \Phi + \tfrac{1}{2}(A_{i12} + A_{i21}) \sin 2\Phi + A_{i22} \cos^2 \Phi, \\ X_3(a, \Phi) &= \sigma^* a + (a^3/8)(3C_{111}^{(1)} + 3C_{222}^{(2)} + C_{122}^{(1)} + C_{112}^{(2)}) + \text{o.t.}, \\ X_4(a, \Phi) &= (a^2/8)(-3C_{111}^{(2)} + 3C_{222}^{(1)} - C_{122}^{(2)} + C_{112}^{(1)}) + \text{o.t.}, \\ Y_1(\Phi) &= (B_{11}^{(1)} - B_{12}^{(2)} + 3B_{22}^{(1)}) \cos \Phi + (-3B_{11}^{(2)} + B_{12}^{(1)} - B_{22}^{(2)}) \sin \Phi \\ &\quad + (B_{22}^{(1)} + B_{12}^{(2)} - B_{11}^{(1)}) \cos 3\Phi + (-B_{22}^{(2)} + B_{12}^{(1)} + B_{11}^{(2)}) \sin 3\Phi, \\ Y_{2i}(\Phi) &= A_{i12} \cos^2 \Phi + \tfrac{1}{2}(A_{i11} - A_{i22}) \sin 2\Phi - A_{i21} \sin^2 \Phi, \\ \Phi &= (1 + \sqrt{\epsilon}\zeta\omega_\lambda)\mathrm{t} + \phi_1, \qquad \phi_1 = \phi - \sqrt{\epsilon}\zeta\omega_\lambda \mathrm{t}, \end{aligned}$$

and o.t. denotes oscillatory terms.

Equation (13) is the so-called equations in standard form. After all oscillatory terms such as sin 2Φ, sin 3Φ, cos 2Φ and cos 3Φ in equation (13) are eliminated, the resulting equations are known as simplified equations.

Note that in the foregoing $X_i(\Phi)$ and $Y_i(\Phi)$ are periodic in t with period

$$T = 2\pi/(1 + \sqrt{\epsilon}\zeta\omega_\lambda).$$

Clearly,

$$\int_0^{\mathrm{T}} X_1(\Phi)\,\mathrm{d}t = \int_0^{\mathrm{T}} Y_1(\Phi)\,\mathrm{d}t = 0.$$

By the limit theorem of Khas'minskii [5], as $\epsilon \to 0$, the solutions (a, ϕ_1) for equation (13) converge weakly to the Markov processes governed by the following Ito's equations:

$$\mathrm{d}a = \epsilon b_1(a, \phi_1)\,\mathrm{d}t + \sqrt{\epsilon}\eta_{1j}\,\mathrm{d}w_j, \qquad \mathrm{d}\phi_1 = \epsilon b_2(a, \phi_1)\,\mathrm{d}t + \sqrt{\epsilon}\eta_{2j}\,\mathrm{d}w_j, \quad j = 1, 2, \tag{14a,b}$$

where

$$\eta_{rs}\eta_{rs}^{\mathrm{T}} = a_{rs}, \qquad r, s = 1, 2,$$

$$a_{11} = (a^2/8)[\alpha_i S_i(0) + \beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)], \qquad a_{22} = \tfrac{1}{8}[\gamma_i S_i(0) + \beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)];$$

in this last equation

$$b_1 = \delta a - Ra^3,$$

$$\begin{aligned} b_1 = \sigma^* a &+ (a^3/8)(3C_{111}^{(1)} + 3C_{222}^{(2)} + C_{122}^{(1)} + C_{112}^{(2)}) \\ &+ (a/16)[\alpha_i S_i(0) + 3\beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)] \\ &- [a^3/16(1 + \sqrt{\epsilon}\zeta\omega_\lambda)](F_1G_1 + F_2G_2 + F_3G_3 + F_4G_4), \end{aligned}$$

$$a_{12} = a_{21} = (a/4)(A_{i11} + A_{i22})(A_{i12} - A_{i21})S_i(0),$$

$$\delta = \sigma^* + \tfrac{1}{16}[\alpha_i S_i(0) + 3\beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)],$$

$$R = -\tfrac{1}{8}(3C_{111}^{(1)} + 3C_{222}^{(2)} + C_{122}^{(1)} + C_{112}^{(2)}) + \sum_{r=1}^{4}(F_rG_r)[1/16(1 + \sqrt{\epsilon}\zeta\omega_\lambda)],$$

$$\begin{aligned} b_2 = \gamma\omega_\alpha &+ \tfrac{1}{2}\zeta^2\omega_{\lambda\lambda} + (a^2/8)(-3C_{111}^{(2)} + 3C_{222}^{(1)} - C_{122}^{(2)} + C_{112}^{(1)}) \\ &- \tfrac{1}{8}[\beta_i \Psi_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)] + [a^2/16(1 + \sqrt{\epsilon}\zeta\omega_\lambda)] \\ &\left[-\sum_{r=1}^{4} F_r^2 + (G_1F_2 - G_2F_1 + \tfrac{1}{3}G_3F_4 - \tfrac{1}{3}G_4F_3)\right], \end{aligned}$$

$$S_i(\omega) = 2\int_0^\infty R_i(\tau)\cos\omega\tau\,\mathrm{d}\tau, \qquad \Psi_i(\omega) = 2\int_0^\infty R_i(\tau)\sin\omega\tau\,\mathrm{d}\tau.$$

$$\alpha_i = 2(A_{i11} + A_{i22})^2, \qquad \gamma_i = 2(A_{i12} - A_{i21})^2,$$

$$\beta_i = (A_{i11} - A_{i22})^2 + (A_{i12} + A_{i21})^2.$$

Note that in the foregoing the superscript T designates the "transpose of" while the subscript i is understood to take values from $i = 1, 2, \ldots, n$. $R_i(\tau) = \langle f_i(t) f_i(t+\tau) \rangle$, here the angular brackets denotes the mathematical expectation, and

$$F_1 = B_{11}^{(1)} - B_{12}^{(2)} + 3B_{22}^{(1)}, \qquad F_2 = -3B_{11}^{(2)} + B_{12}^{(1)} - B_{22}^{(2)},$$

$$F_3 = B_{22}^{(1)} + B_{12}^{(2)} - B_{11}^{(1)}, \qquad F_4 = -B_{22}^{(2)} + B_{12}^{(1)} - B_{11}^{(2)},$$

$$G_1 = B_{11}^{(2)} + B_{12}^{(1)} + 3B_{22}^{(2)}, \qquad G_2 = 3B_{11}^{(1)} + B_{12}^{(2)} + B_{22}^{(1)},$$

$$G_3 = B_{22}^{(2)} - B_{12}^{(1)} - B_{11}^{(2)}, \qquad G_4 = B_{22}^{(1)} + B_{12}^{(2)} - B_{11}^{(1)}.$$

Note also that the above convergence is at the interval $t \in [0, \tau_0/\epsilon]$ for any positive $\tau_0 > 0$.

4. FOKKER-PLANCK EQUATION AND BIFURCATIONS

From the simplified equations of the last section it is clear that equation (14a) is independent of phase angle defined in equation (14b). Therefore, the Fokker–Planck equation for the amplitude a is

$$\partial p(a,t)/\partial t = -\epsilon \partial [b_1 p(a,t)]/\partial a + (\epsilon/2)\partial^2 [a_{11} p(a,t)]/\partial a^2.$$

By setting $u = \epsilon t$ the above Fokker–Planck equation becomes

$$\partial p(a,u)/\partial u = -\partial [b_1 p(a,u)]/\partial a + \tfrac{1}{2}\partial^2 [a_{11} p(a,u)]/\partial a^2. \tag{15}$$

Substituting for the drift coefficient from equation (14) and applying the lemma in reference [1] one has

$$\langle a^k(u) \rangle \to \text{constant as} \quad u \to \infty \quad \text{if} \quad \gamma = (\delta a^2 - \tfrac{1}{2} a_{11})/a_{11} > 0, \tag{16a}$$

where k now is the order of the statistical moment. Also,

$$\langle a^k(u) \rangle \to 0 \quad \text{as} \quad u \to \infty \quad \text{if} \quad \gamma < 0. \tag{16b}$$

Equation (16a) gives

$$\delta a^2 > \tfrac{1}{2} a_{11}. \tag{17}$$

Substituting for the symbols by using equations (14) and (7), equation (17) reduces to

$$\zeta^2 \sigma_{\lambda\lambda} > -2\gamma\sigma_\alpha - \tfrac{1}{4}\beta_i S_i (1 + \sqrt{\epsilon}\zeta\omega_\lambda). \tag{18}$$

This relation and equation (16) enable one to obtain the bifurcation diagrams.

To clarify this statement and to demonstrate the steps involved in such determination one proceeds further as follows by assuming

$$r_z > 0 \quad \text{and} \quad \sigma_{\lambda\lambda} < 0 \tag{19}$$

such that with the theorem in section 2 above $\delta_1 = \delta_2 = 1$ and the bifurcations in the deterministic system are Z_2 equivalent to the following equation

$$q^3 + \mu_1^2 q + \mu_2 q = 0, \tag{20}$$

where μ_1 and μ_2 are the bifurcation parameter and unfolding parameter, respectively.

Equation (20) is a universal unfolding of the normal form $q^3 + \mu_1^2 q$, which has Z_2 codimension one. The bifurcation diagrams in the deterministic case from reference [6] are presented in Figure 1.

Before making use of the above results and equation (18) for the determination of bifurcation diagrams one has to find the relationship between μ_1 and μ_2, and the parameters of our oscillator, λ and α. From Golubitsky and Langford [6], the bifurcation non-trivial solutions of the deterministic system are determined by $b(z, \lambda, \alpha) = 0$. Expanding $b(z, \lambda, \alpha)$ in a Taylor series at $(0, \lambda_0, \alpha_0)$,

$$b_{10} z + b_{02}(\lambda - \lambda_0)^2 + b_\alpha(\alpha - \alpha_0) + \cdots = 0. \tag{21}$$

Disregarding terms with order different from those in equation (20), one can write the last equations as

$$z + (b_\alpha / b_{10})(\alpha - \alpha_0) + (b_{02}/b_{10})(\lambda - \lambda_0)^2 = 0, \tag{22}$$

where

$$b_\alpha = \sigma_\alpha(\lambda_0, \alpha_0), \qquad b_{02} = -\tfrac{1}{2}\sigma_{\lambda\lambda}(\lambda_0, \alpha_0), \qquad b_{10} = r_z = r_z(0, \lambda_0, \alpha_0).$$

Comparing equations (22) and (20), one has

$$\mu_1^2 = (b_{02}/b_{10})(\lambda - \lambda_0)^2 = -\tfrac{1}{2}(\sigma_{\lambda\lambda}/r_z)(\lambda - \lambda_0)^2,$$
$$\mu_2 = (b_\alpha/b_{10})(\alpha - \alpha_0) = -(\sigma_\alpha / r_z)(\alpha - \alpha_0). \tag{23a,b}$$

In equation (23) the arguments (λ_o, α_o) for $\sigma_{\lambda\lambda}$ and σ_α are ignored for simplicity. To first order approximation $(\lambda - \lambda_o)$ and $(\alpha - \alpha_o)$ as defined in equation (3), equation (23) become

$$\mu_1^2 = -\epsilon\left(\frac{\zeta^2}{2}\right)\frac{\sigma_{\lambda\lambda}}{r_z}, \qquad \mu_2 = -\epsilon\gamma\frac{\sigma_\alpha}{r_z}. \tag{24a, b}$$

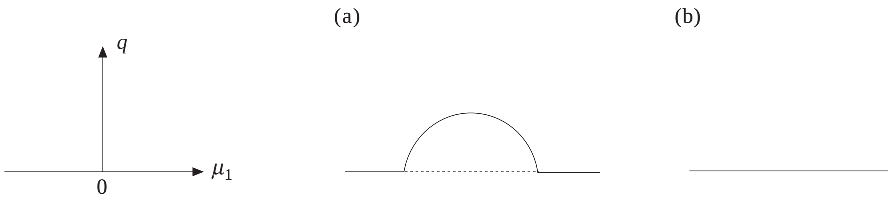

Figure 1. Bifurcations of a deterministic system: (a) $\mu_2 < 0$; (b) $\mu_2 > 0$.

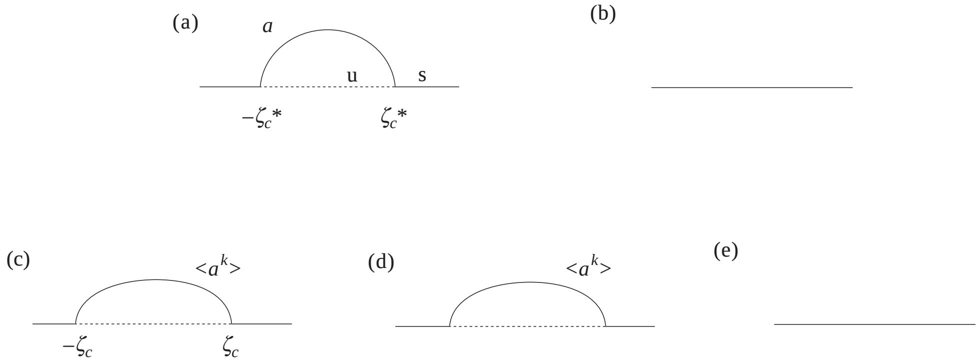

Figure 2. Degenerated Hopf bifurcations for $r_z > 0$ and $\sigma_{\lambda\lambda} < 0$; u = unstable, s = stable, bifurcations points $\zeta_c^* < \zeta_c$. Deterministic system: (a) $\gamma\sigma_\alpha > 0$, (b) $\gamma\sigma_\alpha < 0$; stochastically perturbed system: (c) $\gamma\sigma_\alpha > 0$, (d) $\gamma\sigma_\alpha < 0$ and $2\gamma\sigma_\alpha + \beta_i S_i/4 > 0$; (e) $\gamma\sigma_\alpha < 0$ and $2\gamma\sigma_\alpha + \beta_i S_i/4 < 0$.

Equation (23) establishes the relationship between μ_1^2 and λ as well as μ_2 and α, whereas equation (24) relates μ_1^2 to ζ^2 and μ_2 to γ. Thus, one can make use of equations (18) and (19) such that

$$\zeta^2 < [2\gamma\sigma_\alpha + \tfrac{1}{4}\beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)]/(-\sigma_{\lambda\lambda})$$

and $-\zeta_c < \zeta < \zeta_c$, where

$$\zeta_c = \sqrt{[2\gamma\sigma_\alpha + \tfrac{1}{4}\beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda)]/(-\sigma_{\lambda\lambda})}$$

if

$$2\gamma\sigma_\alpha + \tfrac{1}{4}\beta_i S_i(1 + \sqrt{\epsilon}\zeta\omega_\lambda) > 0. \tag{25}$$

By making use of Figure 1, and equations (24) and (25), one obtains the bifurcation diagrams presented in Figure 2. From the latter one can see that when the unfolding parameter $\gamma\sigma_\alpha < 0$, the deterministic system has no bifurcation, while the stochastic system may have bifurcation if the condition $2\gamma\sigma_\alpha + \beta_i S_i/4 > 0$ is satisfied.

To provide a closer inspection and understanding of the difference between degenerated Hopf bifurcations of the deterministic and stochastically perturbed

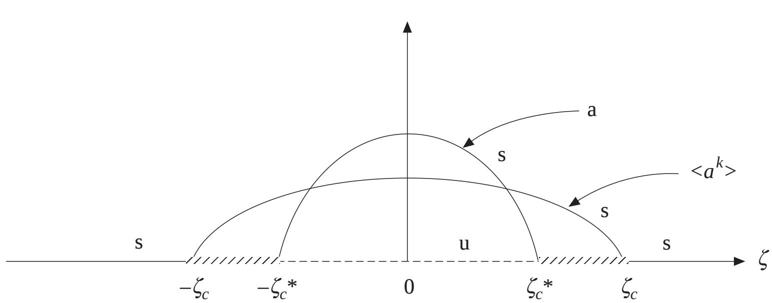

Figure 3. Degenerated Hopf bifurcations of stochastic and deterministic systems for $r_z < 0$, $\sigma_{\lambda\lambda} < 0$ and $\gamma\sigma_\alpha > 0$: ---, unstable for both stochastically perturbed and deterministic systems; ′′′, stable for deterministic system, unstable for stochastically perturbed system.

system and, in turn, reveal their implications, the solutions for the case $\gamma\sigma_\alpha > 0$ in Figure 2 are superimposed and amplified as in Figure 3.

5. DEGENERATED HOPF BIFURCATIONS IN VELOCITY FEEDBACK CONTROL SYSTEMS

It should be noted that the two-dimensional problem described by equation (1) has incorporated a wide class of practical non-linear systems.

As an illustration the velocity feedback control of a sdof nonlinear system under a parametric stationary random excitation process is considered. Without the velocity feedback control this is the Van der Pol oscillator under a parametric stationary random excitation process. Of course, the main purpose for employing feedback control here is to stabilize the system response. As such the question of existence of degenerated Hopf bifurcation is of interest.

Without loss of generality the non-linear system has the equation

$$\ddot{X} + \lambda^2\dot{X} + \beta X^2\dot{X} + X = f_f + \sqrt{\epsilon}f(t)X, \tag{26}$$

where X is the random displacement, f_f denotes the random control force derived from the force actuator, $f(t)$ is a stationary broad-band stochastic process with zero mean value, β and the remaining symbols have their usual meaning.

The random control force f_f is related to the actuator input p by the equation

$$f_f = G_a p, \tag{27}$$

where G_a designates the actuator gain.

To relate the velocity feedback to the system the output of the sensor q is expressed as

$$q = G_s\dot{X} \tag{28}$$

in which G_s is the weighting factor or sensor gain.

For the particular problem considered one assumes constant feedback gain G_f such that

$$p = -G_f q = -G_f G_s\dot{X}. \tag{29}$$

This type of control law is known as the output feedback. The objective is to choose G_f such that X has some desirable properties. Note that p and q here have different meanings from those in section 4.

Then equation (26) can be written as

$$\ddot{X} + (\lambda^2 + \alpha)\dot{X} + \beta X^2\dot{X} + X = \sqrt{\epsilon}f(t)X, \tag{30}$$

where $\alpha = G_a G_f G_s$.

Now, let $y_1 = X$, $y_2 = \mathrm{d}X/\mathrm{d}t$, and $y = [y_1 \;\; y_2]^{\mathrm{T}}$. Then equation (30) can be expressed in a similar manner to equation (1) as

$$\dot{y} = \mathbf{F}(y, \lambda, \alpha) + \sqrt{\epsilon}f(t)\mathbf{A}_1 y, \tag{31}$$

where

$$\mathbf{F}(y, \lambda, \alpha) = \begin{Bmatrix} y_2 \\ -y_1 - (\lambda^2 + \alpha)y_2 - \beta y_1^2 y_2 \end{Bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$\mathbf{A}(\lambda, \alpha) = \begin{bmatrix} 0 & 1 \\ -1 & -(\lambda^2 + \alpha) \end{bmatrix}, \quad \text{since} \quad \mathbf{A}(\lambda, \alpha) = D_y \mathbf{F}(0, \lambda, \alpha).$$

Thus, the matrix $\mathbf{A}(\lambda, \alpha)$ has two eigenvalues as $\sigma(\lambda, \alpha) \pm i\omega(\lambda, \alpha)$, where $\sigma(\lambda, \alpha) = -(\lambda^2 + \alpha)/2$ and $\omega(\lambda, \alpha) = [4 - (\lambda^2 + \alpha)^2]^{1/2}/2$. Clearly, $\sigma(0, 0) = 0$ and $\omega(0, 0) = 1$. Moreover, $\sigma_\lambda = -\lambda$, $\sigma_{\lambda\lambda} = -1$ and $\sigma_\alpha = -1/2$. In other words, conditions (i) and (ii) of section 2 are satisfied.

Following the remaining steps in section 2 and after some algebra, equation (10) gives

$$r_z(0, \lambda_0, \alpha_0) = r_z = \beta/8. \tag{32}$$

By making use of the results above, equation (9), and assuming that $\beta > 0$, one has $\delta_1 = 1$, $\delta_2 = 1$, $r_z > 0$, $\sigma_{\lambda\lambda} = -1/2 < 0$ such that the bifurcation solutions for this particular deterministic system are similar to those in Figure 1. Note that in the latter figure, $\mu_1 = \lambda = \sqrt{\epsilon\zeta}$ and $\mu_2 = \mu = \alpha$ of the oscillator. In the latter, if now the intensity, ϵ, of the stochastic parametric excitation in equation (30) is not equal to zero while the other system parameters are identical to those of the deterministic system considered above, then following the steps in sections 3 and 4 one also has the bifurcation solutions presented in Figures 2 and 3. From Figure 2 one notices that with a positive unfolding parameter, $\gamma\sigma_\alpha$ the stochastically excited oscillator bifurcates at a larger magnitude of the bifurcating point than the corresponding deterministic counterpart. On the other hand, with a negative unfolding parameter, the stochastic system can have bifurcation if a certain condition is satisfied.

6. CONCLUSION

In this note a method for degenerated Hopf bifurcation analysis of single-degree-of-freedom (sdof) systems disturbed by stationary random excitations has been presented. The method is based on the perturbation method, the limit theorem of Khas'minskii which in essence is similar to the stochastic averaging of Stratonovich but is relatively more simple to use for second order approximations, and some results from the singularity theory and group theory. The application of the results from the singularity theory and group theory enables us to examine the important issue of degenerated Hopf bifurcations of systems excited by stationary stochastic forces which cannot be dealt with using other existing established techniques.

The main results obtained by the above method and presented in the foregoing are summarized as

(1) the bifurcation points are independent of the order of statistical moments;
(2) with a positive unfolding parameter the stochastically disturbed system bifurcates at a larger magnitude of the bifurcation point than the corresponding deterministic value; and

(3) with a negative unfolding parameter, however, the stochastically disturbed system can have bifurcation if the criterion, $2\gamma\sigma_\alpha + \beta_i S_i/4 > 0$, is satisfied; otherwise, the system does not bifurcate. The latter is similar to the corresponding deterministic system.

Finally, an example of application of the presented technique has been made on the velocity feedback control of a sdof non-linear system under a parametric stationary random excitation.

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