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A SIMPLE APPROACH TO INVESTIGATE VIBRATORY BEHAVIOUR OF THERMALLY STRESSED LAMINATED STRUCTURES

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The aim of the present paper is to propose a simple method to investigate the vibratory behaviour of laminated composite structures subjected to large thermal loads (may be beyond critical temperature T_{cr}). von Karman type non-linear strain–displacement relationships are employed to derive non-linear finite element equations of motion. These finite element equations are based on secant stiffness rather than tangential stiffness. The secant stiffness matrix is separated into three parts, i.e., (i) linear stiffness matrix independent of field variables, (ii) non-linear stiffness matrix depending linearly on field variables and (iii) non-linear matrix depending quadratically on field variables. Linear thermal buckling and free vibration analyses are performed as a first step to compute the critical temperatures, natural frequency and corresponding mode shapes. Assuming the mode shape corresponding to fundamental frequency as the spatial distribution, large-order non-linear finite element equations are reduced to a single second-order ordinary non-linear differential equation. A direct numerical integration method is employed to compute the non-linear frequencies of thermally stressed structures. To demonstrate the method, vibratory behaviour of thermally stressed laminated beams is investigated. The proposed method is validated by comparing the non-linear frequencies of beams (not subjected to initial stress) obtained using the present method with those available in the literature. The influence of difference in buckling mode shape and vibration mode shape for certain boundary conditions on the non-linear behaviour is also studied. Some interesting observations regarding the finiteness of amplitude in the post-buckling regime are also made.

1. INTRODUCTION

The influence of thermal environment on aircraft/aerospace structures was realised much earlier (probably during World War II) with the advent of supersonic flights. Kinetic heating during supersonic flight results in temperature increase which in turn influences structural behaviour. This increase in temperature with restrained thermal expansion induces compressive stresses which could cause thermal buckling and alter natural frequencies significantly. In the last decade or so, the use of fibre-reinforced composites in the aerospace industry has increased

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significantly owing to their high specific strength and stiffness. This property and flexibility of tailoring the fibre orientation leads to low design thicknesses and thus make the structures more succeptable to buckling. But, thin composite structures are capable of carrying much larger loads beyond buckling loads without failure provided buckling stress is within the elastic range. Further, transverse shear flexibility is likely to play a significant role due to their low ratios of transverse shear moduli to in-plane modulus. Therefore, it is essential to understand the post-buckling behaviour and vibration characteristics of buckled composite structures, viz., beams, plates and shells when subjected to thermal loads.

The composite structures of high Mach number aerospace vehicles (re-entry/reusable) should be able to endure large thermal loads and still preserve their structural integrity. Therefore, thermal effects must be considered during the design of these structural elements. An excellent review article by Tauchert [1] provided an historical perspective of thermally induced flexure, buckling and vibration of plates. The problem of thermal buckling and post-buckling of laminated beams and plates has been investigated in references [2, 3]. Similarly, non-linear oscillatory behaviour of laminated beams and plates has been studied by several investigators [4–6].

The vibration behaviour of buckled isotropic plates has been investigated by Bisplinghoff and Pian [7]. They employed an extension of Marguerre's theory for plates subjected to longitudinal compression and obtained stress function. The modal equations of motion were derived from Lagrange's equation with the strain energy and kinetic energy of the buckled plate. Yang and Han [8] also investigated the Bisplinghoff and Pian's [7] plate problem using finite element method. They solved the problem in three steps, viz., (i) a simple relation was derived between uniform temperature change ΔT and in-plane compression P in the buckled region, $0{\cdot}5(\Delta T/\Delta T_{cr} + 1) = P/P_{cr}$, (ii) the post buckling state under P/P_{cr} was determined, and finally (iii) linear vibration about this state was obtained.

Zhou *et al.* [9] and Lee and Lee [10] investigated the vibrations of thermally buckled thin and thick composite plates by employing finite element method. In these studies, the authors determined the post-buckled state and then by using this tangent stiffness matrix corresponding to this state, frequencies are computed.

It is surprising to note that literature on vibrations of thermally buckled laminated beams is relatively scarce, thus becomes the subject of the present paper. In this study, a simple computational scheme is proposed to investigate the oscillatory behaviour of unsymmetrically laminated beams/plates subjected to large thermal loads.

2. THEORETICAL FORMULATION

In this section, von Karman non-linear plate theory, which is a subset of general non-linear theory of elasticity, is employed for the problem formulation. The derivation assumes large displacements, but the rotations and strains are assumed small compared to unity. It implies that the changes in the geometry in the definition of stresses and integration are neglected. Further, use is made of Mindlin's assumptions, i.e., planes normal to the undeformed middle surface remain plate but not necessarily normal. Under these assumptions the strain–displacement relationship can be written as:

$$\varepsilon_{xx} = u_{,x} + \tfrac{1}{2}w_{,x}^2 + z\Psi_{x,x} - \alpha_x \Delta T$$
$$\varepsilon_{yy} = v_{,y} + \tfrac{1}{2}w_{,y}^2 + z\Psi_{y,y} - \alpha_y \Delta T$$
$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} + z(\Psi_{x,y} + \Psi_{y,x}) - \alpha_{xy}\Delta T$$
$$\gamma_{xz} = \Psi_x + w_{,x}$$
$$\gamma_{yz} = \Psi_y + w_{,y} \tag{1}$$

where the transverse normal strain (ε_{zz}) is neglected and cartesian coordinates are used; with x–y being the plate mid-plane, u and v are the displacements of the mid-plane in the x–y direction, w is the transverse displacement in the z-direction and Ψ_x and Ψ_y are bending rotations in the x–z and y–z planes respectively. α_x, α_y and α_{xy} are the coefficients of thermal expansion and ΔT represents temperature difference. The stress–strain relations are given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{2}$$

where $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{16}$, $\bar{Q}_{22}$, $\bar{Q}_{26}$, $\bar{Q}_{66}$, $\bar{Q}_{44}$, $\bar{Q}_{45}$ and $\bar{Q}_{55}$ are reduced stiffnesses and are obtained by transforming the material properties from the principal material coordinate system to the plate coordinate system (x–y). The strain energy of the plate made up of linear elastic material reads

$$U = \frac{1}{2}\int_v (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{3}$$

where the quantities subjected to variation are the displacements (u, v and w) and rotations (Ψ_x and Ψ_y) after substituting equations (1) and (2) into equation (3). Now, defining the stress resultants as

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}\, \mathrm{d}z; \qquad N_y = \int_{-h/2}^{h/2} \sigma_{yy}\, \mathrm{d}z; \qquad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy}\, \mathrm{d}z$$

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} z\, \mathrm{d}z; \qquad M_y = \int_{-h/2}^{h/2} \sigma_{yy} z\, \mathrm{d}z; \qquad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z\, \mathrm{d}z$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, \mathrm{d}z; \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, \mathrm{d}z$$

where h is the plate thickness.

The constitutive equations can be rewritten as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{4}$$

where A_{ij}, B_{ij} and D_{ij} $(i, j = 1, 2, 6)$ are usual extensional, bending–extension and bending stiffness coefficients A_{ij} $(i, j = 4, 5)$ are transverse shear stiffness coefficients with

$$\varepsilon_{x0} = u_{,x} + \tfrac{1}{2} w_{,x}^2 - \alpha_x \Delta T; \qquad \varepsilon_{y0} = v_{,y} + \tfrac{1}{2} w_{,y}^2 - \alpha_y \Delta T$$

$$\gamma_{xy0} = u_{,y} + v_{,x} + w_{,x} w_{,y} - \alpha_{xy} \Delta T; \qquad \kappa_x = \Psi_{x,x}; \qquad \kappa_y = \Psi_{y,y}$$

$$\kappa_{xy} = \Psi_{x,y} + \Psi_{y,x}; \qquad \gamma_{xz} = \Psi_x + w_{,x} \quad \text{and} \quad \gamma_{yz} = \Psi_y + w_{,y}.$$

If the in-plane inertia of the plate is neglected, the kinetic energy expression reads:

$$T = \frac{1}{2} \int_v \rho(z) [(z\dot{\Psi}_x)^2 + (z\dot{\Psi}_y)^2 + (\dot{w})^2] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{5}$$

where ρ is the plate density and can vary from layer to layer, and $(\,\cdot\,)$ denotes differentiation with respect to time.

Hamilton's principle for the problem at hand can be written as,

$$\int_{t_0}^{t_1} \delta(T - U) \, \mathrm{d}t = 0 \tag{6}$$

where the functional (3) and (5) are to be used in equation (6) and the quantities subjected to variation are u, v, w, Ψ_x and Ψ_y. Applying the variational operation, equation (6) will furnish three equations of motion in w, Ψ_x and Ψ_y, and two equations of equilibrium in u and v together with the boundary conditions.

The governing equations of a laminated beam can be derived by using the same procedure. While doing so, one can consider the fact that $\sigma_{yy} = \tau_{xy} = 0$ and express ε_{yy} and γ_{xy} in terms of ε_{xx}. This information is used to modify the constitutive equation (2) and strain energy expression (3). Similarly, in kinetic energy expression $\dot{\Psi}_y$ vanishes if the motion is in the x–z plane. The modified kinetic potential (6) yields two equations of motion in w and Ψ_x and one equilibrium equation in u along with boundary conditions.

3. FINITE ELEMENT FORMULATION

Let the domain of the structure be divided into sub-domains called finite elements. The quantities that are subjected to variation, i.e., field variables u, v, w, Ψ_x and Ψ_y are expressed in terms of nodal quantities as:

$$u = \sum_{i=1}^{NN} N_i u_i; \qquad v = \sum_{i=1}^{NN} N_i v_i; \qquad w = \sum_{i=1}^{NN} N_i w_i$$

$$\Psi_x = \sum_{i=1}^{NN} N_i \Psi_{xi}; \qquad \Psi_y = \sum_{i=1}^{NN} N_i \Psi_{yi} \tag{7}$$

where u_i, v_i, w_i, Ψ_{xi} and Ψ_{yi} are nodal variables and NN is the number of nodes in an element. N_i denotes shape functions. These shape functions can be the same or different for each field variable. Substituting equation (7) in the strain–displacement relation (1), the strains can be expressed in terms of nodal quantities as:

$$\{\varepsilon\} = \{[B_L] + \tfrac{1}{2}[B_{NL}]\}\{\delta\} - \{\varepsilon_T\} \tag{8}$$

$[B_L]$ and $[B_{NL}]$ are matrices relating the linear and non-linear components of strain vector to the nodal quantities $\{\delta\}$. $\{\varepsilon_T\}$ denotes thermal strain vector. The strain energy expression (3) after substituting constitutive relation (2) and strain–displacement relation (8) becomes:

$$U^{(e)} = \frac{1}{2}\int_v \{\varepsilon\}^{\mathrm{T}}[\bar{Q}]\{\varepsilon\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{9}$$

where e denotes element. Using equations (5) and (7), the kinetic energy expression reads:

$$T^{(e)} = \frac{1}{2}\int_v \rho(z)[z\dot{\Psi}_{xi}^{\mathrm{T}}N_i^{\mathrm{T}}N_i\dot{\Psi}_{xi} + z\dot{\Psi}_{yi}^{\mathrm{T}}N_i^{\mathrm{T}}N_i\dot{\Psi}_{yi} + \dot{w}_i^{\mathrm{T}}N_i^{\mathrm{T}}N_i\dot{w}_i]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{10}$$

The variation of the kinetic potential (T–U) gives the following non-linear finite element equation;

$$[K_0 + n_1(\delta) + n_2(\delta)]\{\delta\} + \lambda[K_G]\{\delta\} + [m]\{\ddot{\delta}\} = \{f_t\} \tag{11}$$

where

$$K_0 = \int_v B_L^{\mathrm{T}}\bar{Q}B_L\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$n_1 = \int_v (B_{NL}^{\mathrm{T}}\bar{Q}B_L + \tfrac{1}{2}B_L^{\mathrm{T}}\bar{Q}B_{NL})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$n_2 = \int_v \frac{1}{2} B_{NL}^{\mathrm{T}} \bar{Q} B_{NL} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

and

$$\{f_t\} = \int_v [B_L]^{\mathrm{T}} [\bar{Q}]^{\mathrm{T}} \{\varepsilon_T\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$[n_1]$ and $[n_2]$ are first-order unsymmetric and second-order symmetric stiffness matrices respectively. That is, $[n_1]$ depends linearly and $[n_2]$ depends quadratically on the elemental degrees of freedom due to bending, viz., w, Ψ_x and Ψ_y. This dependence of non-linear stiffness matrices on bending degrees of freedom is due to the consideration of von Karman non-linear plate theory. If Green–Lagrange non-linear strain vector is employed then $[n_1]$ and $[n_2]$ will depend on all the element degrees of freedom.

The geometric stiffness matrix $[K_G]$ and mass matrix $[m]$ are computed using the standard procedure [11] as follows:

$$[K_G]\{\delta\} = \int_v B_{NL}^{\mathrm{T}} \sigma \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where stress vector $\sigma = \lfloor \sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{xz} \;\; \tau_{yz} \rfloor$ denotes pre-buckling stress state in the structure and can be obtained by solving

$$[K_0]\{\delta\} = \{f_t\}. \tag{12}$$

Now the pre-buckling stress state $\{\sigma\}$ can be computed as:

$$\{\sigma\} = [\bar{Q}]\{[B_L]\{\delta\} - \{\varepsilon_T\}\}$$

with $\{\varepsilon_T\}^{\mathrm{T}} = \{\alpha_x \Delta T \;\; \alpha_y \Delta T \;\; \alpha_{xy} \Delta T\}$ where α_x, α_y and α_{xy} are the coefficients of thermal expansion and ΔT is the temperature difference. In the present study ΔT is assumed constant and

$$[m] = \int_v \rho(z) [c]^{\mathrm{T}} [c] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where $[c]$ is the matrix relating displacement anywhere in the element to nodal variables as:

$$[c] = \begin{bmatrix} 0 & 0 & 0 & zN_i & 0 \\ 0 & 0 & 0 & 0 & zN_i \\ 0 & 0 & N_i & 0 & 0 \end{bmatrix}.$$

It may be noted that in the computation of mass matrix in-plane/axial inertia is neglected. The element matrices in equation (11) are assembled to obtain the global finite element equations. While assembling the in-plane/axial nodal quantities are kept together so that partitioning is possible. These assembled finite element equations read;

$$\begin{bmatrix} K_{uu} & K_{uw} \\ K_{wu} & K_{ww} \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \begin{bmatrix} 0 & n_{uw1} \\ n_{wu1} & n_{ww1} \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & n_{ww2} \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix}$$

$$+\lambda \begin{bmatrix} 0 & 0 \\ 0 & K_G \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m_{ww} \end{bmatrix} \begin{Bmatrix} \ddot{q}_u \\ \ddot{q}_w \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_w \end{Bmatrix} \tag{13}$$

where the subscript u refers to in-plane/axial nodal quantities (u_i, v_i) and subscript w refers to nodal quantities due to bending (w_i, Ψ_{xi} and Ψ_{yi}). For the case of symmetrically laminated, orthotropic and isotropic plates/beams, $[K_{uw}] = [K_{wu}] = 0$, due to the vanishing of bending-extension coupling. The submatrix $[n_{ww1}]$ is also found to vanish for these cases. As mentioned earlier submatrices $[n_{uw1}]$, $[n_{wu1}]$ and $[n_{ww1}]$ are linear functions and submatrices $[n_{ww2}]$ are quadratic functions of $\{q_w\}$ only.

4. METHOD OF SOLUTION

To begin with, linear vibration and buckling analyses are performed by dropping the non-linear stiffness matrices $[n_1]$ and $[n_2]$ as:

$$\begin{bmatrix} K_{uu} & K_{uw} \\ K_{wu} & K_{ww} \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m_{ww} \end{bmatrix} \begin{Bmatrix} \ddot{q}_u \\ \ddot{q}_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

and

$$\begin{bmatrix} K_{uu} & K_{uw} \\ K_{wu} & K_{ww} \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & K_G \end{bmatrix} \begin{Bmatrix} q_u \\ q_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{15}$$

Linear buckling analysis is performed to estimate the level of initial stress compared to critical load. The linear mode corresponding to the fundamental frequency is assumed as the spatial distribution for equation (13), i.e.,

$$\{q_w\} = \{\bar{q}_w\} A(t) \tag{16}$$

where $\{\bar{q}_w\}$ is a normalised eigenvector obtained from linear vibration analysis such that the maximum transverse displacement due to bending is unity and $A(t)$ is a time function (represents temporal variable) and will be written as A in the following. Substituting equation (16) into equation (13), the in-plane/axial displacement vector can be expressed as:

$$\{q_u\} = -[k_{uu}]^{-1}[k_{uw}]\{\bar{q}_w\}A - [k_{uu}]^{-1}[n_{uw1}]\{\bar{q}_w\}A^2. \tag{17}$$

By substituting equation (17) in equation (13), the equation of motion for transverse displacements becomes:

$$[k_{ww} - k_{wu}k_{uu}^{-1}k_{uw}]\{\bar{q}_w\}A + [\bar{n}_{ww1} - k_{wu}k_{uu}^{-1}\bar{n}_{uw1} - \bar{n}_{wu1}k_{uu}^{-1}k_{uw}]\{\bar{q}_w\}A^2$$
$$+[\bar{n}_{ww2} - \bar{n}_{wu1}k_{uu}^{-1}\bar{n}_{uw1}]\{\bar{q}_w\}A^3 + \lambda[K_G]\{\bar{q}_w\}A + [m_{ww}]\{\bar{q}_w\}\ddot{A} = \{0\}. \tag{18}$$

The bars over non-linear stiffness submatrices indicate that these are computed using the normalised mode $\{\bar{q}_w\}$. Pre-multiplying equation (18) with $\{\bar{q}_w\}^{\mathrm{T}}$ and dividing throughout with $\bar{q}_w^{\mathrm{T}}m_{ww}\bar{q}_w$ gives an equation of the form

$$\ddot{A} + \alpha A + \beta A^2 + \gamma A^3 + gA = 0$$

or

$$\ddot{A} + (\alpha + g)A + \beta A^2 + \gamma A^3 = 0. \tag{19}$$

5. DIRECT INTEGRATION METHOD

In the direct integration method, equation of motion (19) is transformed into energy balance equation by multiplying it with $\dot{A}$ and integrating with respect to time. The resulting energy balance equation reads,

$$\dot{A}^2 + (\alpha + g)A^2 + \tfrac{2}{3}\beta A^3 + \tfrac{1}{2}\gamma A^4 = C. \tag{20}$$

Using the condition that $\dot{A} = 0$ when $A = A_{\max}$, equation (20) becomes:

$$\dot{A}^2 - (\alpha + g)(A_{\max}^2 - A^2) - \tfrac{2}{3}\beta(A_{\max}^3 - A^3) - \tfrac{1}{2}\gamma(A_{\max}^4 - A^4) = 0. \tag{21}$$

Equation (21) for $\dot{A} = 0$ will result in at least two real roots. In the absence of β, these are $\pm A_{\max}$. This indicates that structure executes oscillations with same amplitude in positive and negative deflection cycle, i.e., $\pm A_{\max}$. However, in the presence of β, the other real root will not be $-A_{\max}$ indicating that asymmetrically laminated structures oscillate with different amplitude in positive and negative deflection cycles. Equation (21) is rearranged to give

$$T = 2\int_0^{A_{\max}} \frac{\mathrm{d}A}{\sqrt{(\alpha + g)(A_{\max}^2 - A^2) + \frac{2}{3}\beta(A_{\max}^3 - A^3) + \frac{1}{2}\gamma(A_{\max}^4 - A^4)}}$$
$$+2\int_0^{B_{\max}} \frac{\mathrm{d}A}{\sqrt{(\alpha + g)(B_{\max}^2 - A^2) + \frac{2}{3}\beta(B_{\max}^3 - A^3) + \frac{1}{2}\gamma(B_{\max}^4 - A^4)}}. \tag{22}$$

The singularity at $A = A_{\max}$ or $A = B_{\max}$ in equation (22) is removed by substituting $A = A_{\max} \sin\theta$ and $B = B_{\max} \sin\theta$ and appropriately changing the limits. The modified integral takes the form,

$$T = \frac{2\pi}{\omega} = 2\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{(\alpha + g) + \frac{2}{3}\beta F_1 A_{\max} + \frac{1}{2}\gamma F_2 A_{\max}^2}} + 2\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{(\alpha + g) + \frac{2}{3}\beta F_1 B_{\max} + \frac{1}{2}\gamma F_2 B_{\max}^2}} \tag{23}$$

where

$$F_1 = (1 + \sin\theta + \sin^2\theta)/(1 + \sin\theta)$$

$$F_2 = (1 + \sin^2\theta).$$

The integrals involved in equation (23) can be computed numerically. In the present study five point Gauss quadrature formulae are employed to compute these integrals and thus nonlinear frequency.

6. NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the method proposed in the preceeding section, dynamic behaviour of thermally stressed composite beams is studied in detail. The constitutive equation (4) is modified by setting the stress resultants N_y, N_{xy} and stress couples M_y, M_{xy} to zero. The resulting constitutive relationship becomes:

$$\begin{Bmatrix} N_x \\ M_x \end{Bmatrix} = \left\{ \begin{bmatrix} A_{11} & B_{11} \\ \mathrm{B}_{11} & D_{11} \end{bmatrix} - \begin{bmatrix} A_{12} & A_{16} & B_{12} & B_{16} \\ B_{12} & B_{16} & D_{12} & D_{16} \end{bmatrix} \times \begin{bmatrix} A_{22} & A_{26} & B_{22} & B_{26} \\ A_{26} & A_{66} & B_{26} & B_{66} \\ B_{22} & B_{26} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} A_{12} & B_{12} \\ A_{16} & B_{16} \\ B_{12} & D_{12} \\ B_{16} & D_{16} \end{bmatrix} \right\} \begin{Bmatrix} \varepsilon_{x0} \\ k_x \end{Bmatrix} \tag{24}$$

or

$$\begin{Bmatrix} N_x \\ M_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \kappa_x \end{Bmatrix}$$

and

$$Q_x = \tfrac{5}{6} A_{44}\gamma_{xz} \tag{25}$$

where constant 5/6 is the shear correction factor.

It may be noted that $N_y = N_{xy} = M_y = M_{xy} = 0$ does not imply $\sigma_y = \tau_{xy} = 0$ in individual layers. However, the fact that the above mentioned stress resultants and stress couples vanish in case of even laminated beams is utilised in avoiding the pre-buckling stress analysis. The only stress resultant that affects the geometric stiffness matrix is computed as:

$$N_x = -\sum (\bar{Q}_{11}\alpha_x + \bar{Q}_{12}\alpha_y + \bar{Q}_{16}\alpha_{xy})^k \Delta T h_k \tag{26}$$

where superscript k denotes the kth layer and h_k refers to its thickness.

In the finite element formulation, the interpolation functions employed for extensional variable u and flexural variables w and Ψ_x are linear and cubic respectively. Thus the 2 node beam element developed herein has five degrees of freedom namely u, w, w', Ψ_x and Ψ_x' per node.

The element is found to give a fairly accurate estimate of buckling loads and frequencies even with highly coarse mesh sizes. However, in the present study the beam is discretised into 32 elements so as to obtain the spatial variation of mode more accurately. During the coarse of study it was realised that finer meshes result in better and converged estimate of non-linear stiffness coefficients, i.e., β and γ.

The mechanical properties considered in this study are:

(I) Isotropic: $v = 0{\cdot}3$; $\alpha_L = \alpha_T$

(II) Composite: $E_L/E_T = 25$; $G_{LT} = G_{LZ} = 0{\cdot}5E_T$; $v = 0{\cdot}25$; $\alpha_T/\alpha_L = 10$.

The end conditions considered are:

Both ends Simply-Supported (SS):

$$\text{At } x = 0, L; \qquad u = w = 0$$

Both ends Clamped (CC):

$$\text{At } x = 0, L; \qquad u = w = \Psi_x = 0$$

One end Clamped and other end Simply-Supported (CS):

$$\text{At } x = 0; \qquad u = w = 0$$

$$\text{At } x = L\ ; \qquad u = w = \Psi_x = 0.$$

The results in this section are presented in the form of following non-dimensional parameters;

$$\lambda_{\omega_0} = \frac{m\omega_0^2 L^4}{\bar{D}_{11}}; \qquad \lambda_\omega = \frac{\omega}{\omega_0}; \qquad W_c = \frac{A_{\max}}{\zeta}; \qquad \lambda_{cr} = \frac{100\alpha_T T_{cr} L^2}{h^2}$$

where ω_0 is fundamental frequency, ω is non-linear frequency, $A_{\max}$ is the maximum amplitude in a deflection cycle and $\zeta = \sqrt{I/A}$.

In order to validate the method proposed herein, non-linear frequencies of simply-supported slender ($L/\zeta = 100$) and short ($L/\zeta = 25$) isotropic beams are compared with the exact solution due to Woinowski-Krieger [6] and Singh [12] in Tables 1 and 2. In this study the beam is not subjected to initial stress. The comparison of non-linear frequencies at various amplitudes indicates that the

TABLE 1

Comparison of non-linear frequency ratios of simply supported slender and short isotropic beams

	ω/ω_0			
	$L/\zeta = 25$, $\lambda_{\omega_0} = 91{\cdot}522$		$L/\zeta = 100$, $\lambda_{\omega_0} = 97{\cdot}015$	
W_c	Present	Ref. [12]	Present	Ref. [6]
0·2	1·00393	1·00392	1·00376	1·0037
0·4	1·01562	1·01556	1·01494	1·0148
0·6	1·03477	1·03464	1·03327	1·0331
0·8	1·06091	1·06069	1·05831	1·0581
1·0	1·09348	1·09316	1·08956	1·0892
2·0	1·33178	1·33077	1·31907	1·3178

present results are in excellent agreement with those due to Woinowski-Krieger.

The influence of the end conditions on the non-linear frequencies of slender and short isotropic beams are studied in Figure 1. The results indicate that the influence of non-linearity is least for beams with clamped ends and maximum for beams with simply-supported ends. Further, the frequency ratios for short beams are higher than the corresponding slender beams, implying short beams exhibit higher non-linearity.

The influence of initial thermal stress on the non-linear frequency of simply-supported, isotropic and 2-layered angle-ply ($45°/-45°$) composite beams of slenderness ratio 25 is presented in Figures 2 and 3. As expected the compressive stress developed due to thermal loading causes softening and thereby decrease in the frequency. The frequency of such beams drops to zero when the temperature difference responsible for compressive stress state reaches its critical value.

TABLE 2

Comparison of non-linear frequency ratios of isotropic slender beam ($L/\zeta = 100$) with CC and CS end conditions

	ω/ω_0			
	CS, $\lambda_{\omega_0} = 235{\cdot}467$		CC, $\lambda_{\omega_0} = 492{\cdot}458$	
W_c	Present	Ref. [12]	Present	Ref. [12]
0·2	1·00185	1·0019	1·00091	1·0009
0·4	1·00739	1·0077	1·00364	1·0036
0·6	1·01654	1·0172	1·00816	1·0080
0·8	1·02920	1·0304	1·01446	1·0142
1·0	1·04521	1·0471	1·02249	1·0221
2·0	1·16917	1·1758	1·08679	1·0854

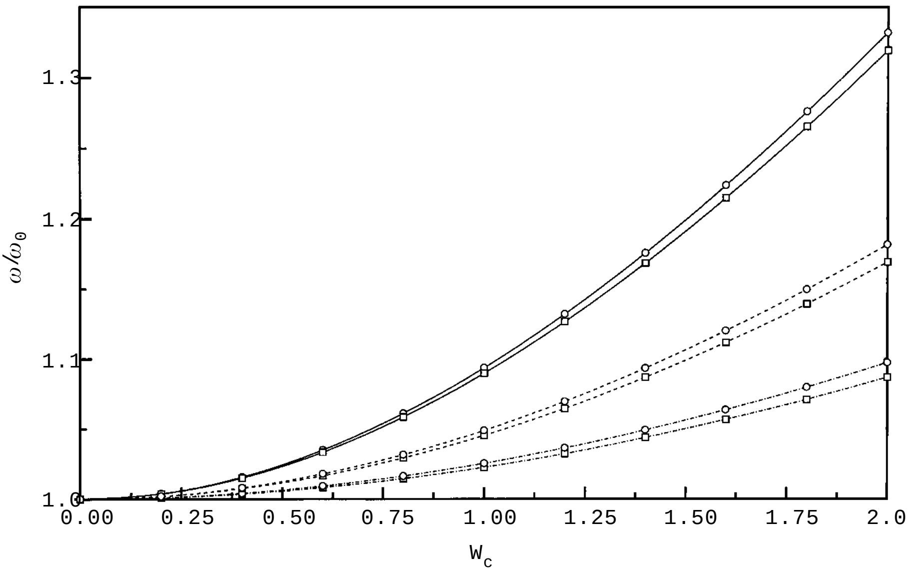

Figure 1. Variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for slender ($L/\zeta = 100$) and short ($L/\zeta = 25$) isotropic beams. ○——○, SS: $L/\zeta = 25$, $\lambda_{\omega_0} = 91{\cdot}5520$; □——□, SS: $L/\zeta = 100$, $\lambda_{\omega_0} = 97{\cdot}0150$; ○- - -○, CS: $L/\zeta = 25$, $\lambda_{\omega_0} = 206{\cdot}373$; □- - -□, CS: $L/\zeta = 100$, $\lambda_{\omega_0} = 235{\cdot}467$; ○–·–○, CC: $L/\zeta = 25$, $\lambda_{\omega_0} = 397{\cdot}053$; □–·–□, $L/\zeta = 100$, $\lambda_{\omega_0} = 492{\cdot}458$.

Further, the beams subjected to large temperature loads (beyond critical temperature) can oscillate with a finite amplitude. In other words, the beams subjected to subcritical temperatures can oscillate with infinitesimal amplitude and finite frequency (which is lower than the natural frequency of the beam) and the

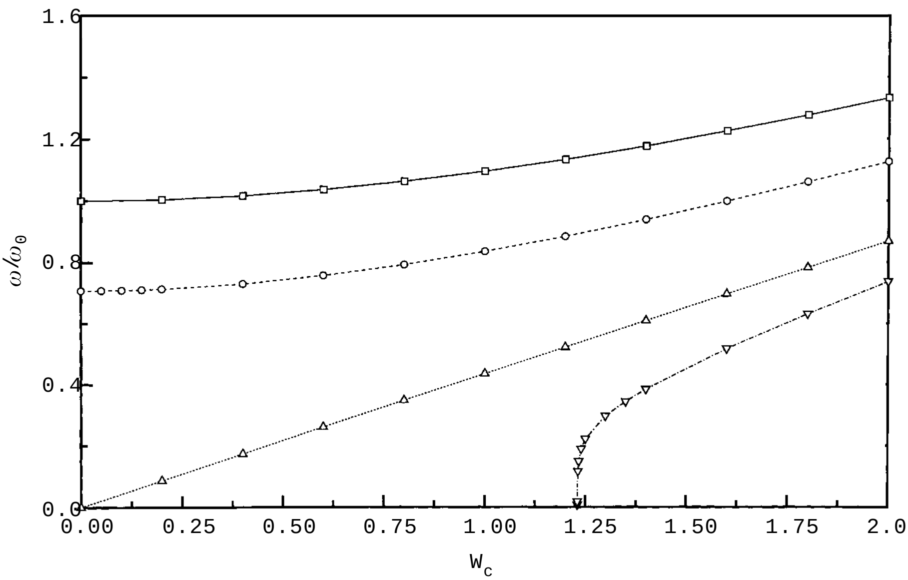

Figure 2. Influence of initial thermal stress on the variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for simply-supported short ($L/\zeta = 25$) isotropic beams. □, $F = 0$; ○, $F = 0{\cdot}5{*}\lambda_{cr}$; △, $F = 1{\cdot}0{*}\lambda_{cr}$; ▽, $F = 1{\cdot}2{*}\lambda_{cr}$; $\lambda_{\omega_0} = 91{\cdot}522$; $\lambda_{cr} = 54{\cdot}869$.

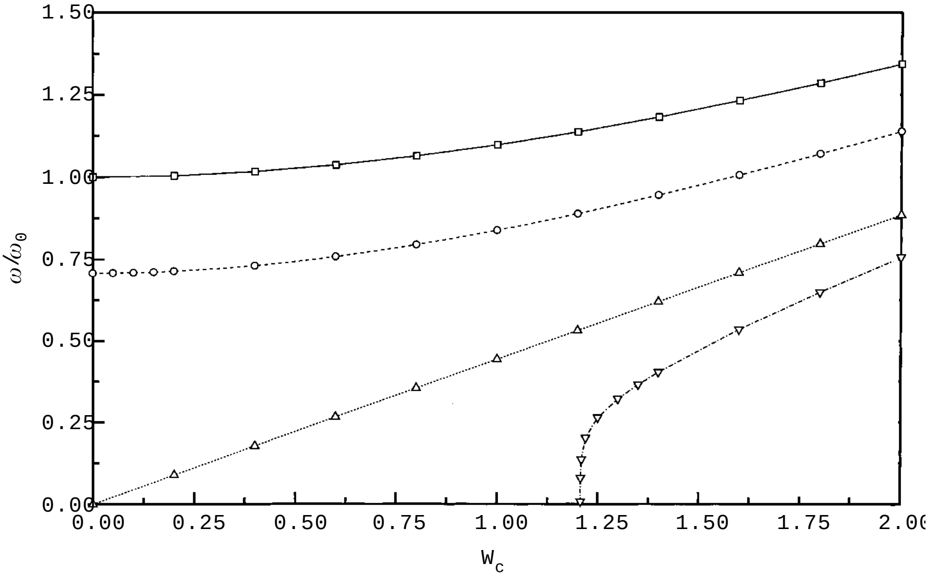

Figure 3. Influence of initial thermal stress on the variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for simply-supported two layered angle-ply short ($L/\zeta = 25$) beams. Symbols as in Figure 2. $\lambda_{\omega_0} = 88{\cdot}225$; $\lambda_{cr} = 16{\cdot}153$.

ones subjected to temperatures higher than their critical value can oscillate only with a finite amplitude.

Figures 4 and 5 show the effect of temperature loading on the variation of frequency ratio with amplitude ratio for clamped–clamped, isotropic and 2-layered

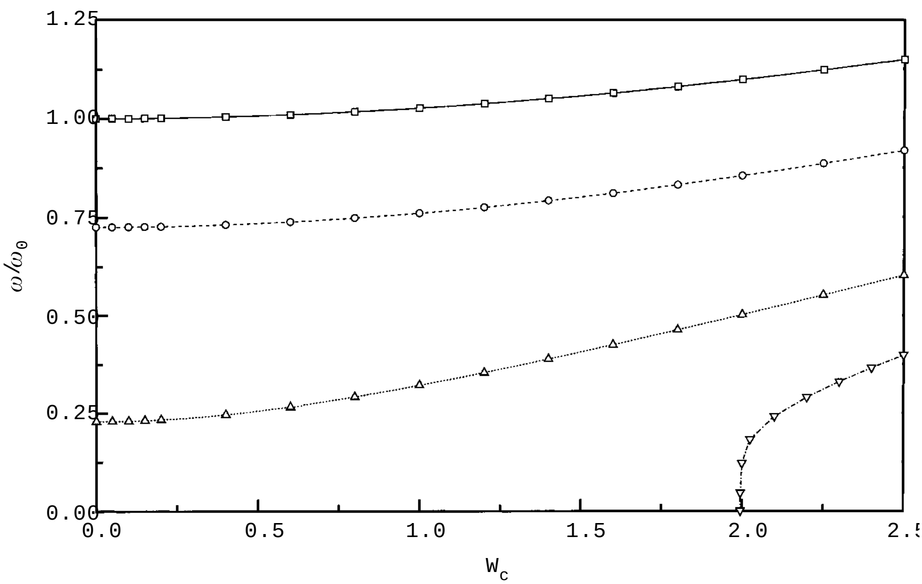

Figure 4. Influence of initial thermal stress on the variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for clamped–clamped short ($L/\zeta = 25$) isotropic beams. Symbols as in Figure 2. $\lambda_{\omega_0} = 397{\cdot}053$; $\lambda_{cr} = 192{\cdot}378$.

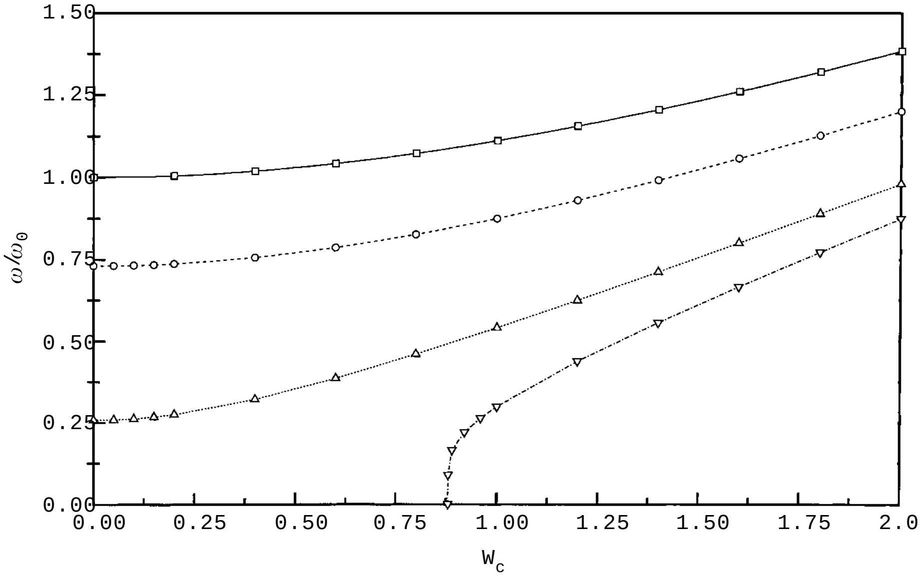

Figure 5. Influence of initial thermal stress on the variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for clamped–clamped two layered cross-ply short ($L/\zeta = 25$) beams. Symbols as in Figure 2. $\lambda_{\omega_0} = 79{\cdot}363$; $\lambda_{cr} = 60{\cdot}210$.

cross-ply (0°/90°) composite short ($L/\zeta = 25$) beams. It is interesting to note that the frequency of such beams does not become zero even when temperature has reached its critical value. It is mainly because the buckling and vibration mode

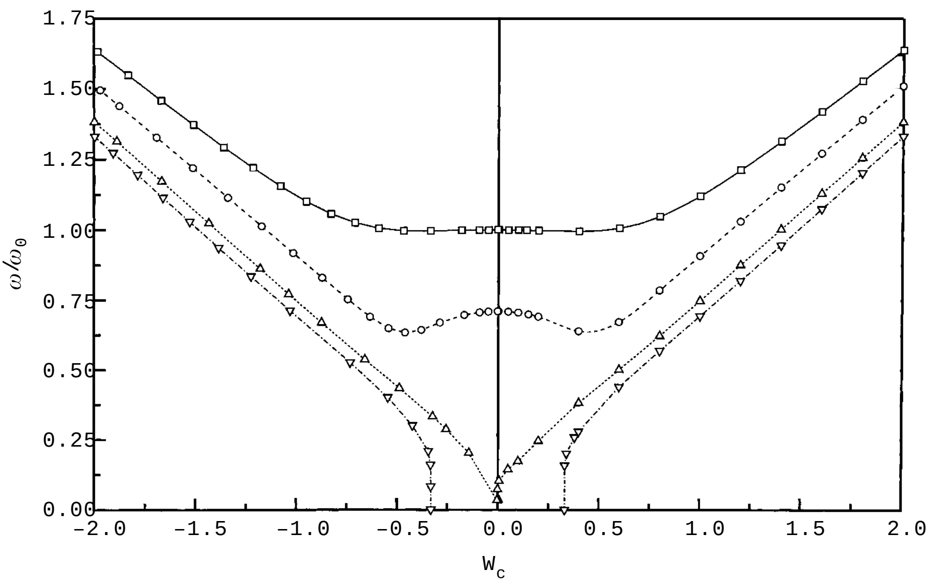

Figure 6. Influence of initial thermal stress on the variation of frequency ratio (ω/ω_0) with amplitude ratio (W_c) for simply-supported two layered cross-ply short ($L/\zeta = 25$) beams. Symbols as in Figure 2. $\lambda_{\omega_0} = 47{\cdot}466$; $\lambda_{cr} = 40{\cdot}657$.

shapes of clamped beams are different unlike those of simply-supported beams. Thus, dissimilar vibration and buckling mode shapes tend to stabilise the structure.

In all the cases investigated so far, i.e., Figures 1–5, the non-linear stiffness coefficient β is zero. It is so even for 2-layered cross-ply beams with clamped ends, bending-extension coupling is non-zero. It is because the stress couples caused by asymmetry of lay-up react to clamped end conditions. Therefore, the amplitude of positive and negative deflection half cycles throughout these studies is same. However, the non-linear stiffness coefficient β is non-zero for 2-layered cross-ply simply-supported beams. The influence of initial thermal stress on the oscillatory behaviour of such beams is investigated in Figure 6. This study indicates that asymmetrically laminated beams with simply-supported ends oscillate with different amplitude in positive and negative deflection half cycles. It is also noticed that frequency decreases initially and then increases with an increase in amplitude.

7. CONCLUSIONS

A simple method is proposed to study the vibratory behaviour of unsymmetrically laminated structures subjected to large thermal loads. The comparison of the results indicates that the proposed method is shown to yield accurate prediction of non-linear frequencies in the absence of initial stress. It is shown that unsymmetrical laminates oscillate with different amplitudes in positive and negative cycles. It is also shown that buckled beams oscillate with a finite amplitude. The boundary conditions have significant influence on the vibratory behaviour of beams. It is more so, when the buckling mode and vibration modes are dissimilar. The fixed-fixed end conditions, where vibration and buckling modes are dissimilar, are found to have a stabilising effect, viz., fundamental frequency remains finite even when initial stress corresponds to the buckling load.

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