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MODEL REDUCTION OF VISCOELASTIC FINITE ELEMENT MODELS

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This paper examines a method of adding viscoelastic properties to finite element models by using additional co-ordinates to account for the frequency dependence usually associated with such damping materials. Several such methods exist and all suffer from an increase in order of the final finite model which is undesirable in many applications. Here we propose to combine one of these methods, the GHM (Golla–Hughes–McTavish) method, with model reduction techniques to remove the objection of increased model order. The result of combining several methods is an ability to add the effects of viscoelastic components to finite element or other analytical models without increasing the order of the system. The procedure is illustrated by a numerical example. The method proposed here results in a viscoelastic finite element of a structure without increasing the order of the original model.

1. INTRODUCTION

Viscoelastic damping is exhibited in many polymeric and glassy materials, and such materials are often used to add damping to a structure to reduce vibration and noise. However, accurate mathematical modelling of structures with viscoelastic materials is difficult because the measured dynamic properties of viscoelastic material are frequency and temperature dependent and can depend on the type of deformation and amplitude. Several researches have presented successful methods of modelling the effects of viscoelastic damping mechanisms which introduce hysterisis or frequency dependence. One effective and popular approach is to introduce additional co-ordinates to account for the frequency dependent and hysteretic behavior. Motivated by a need to produce finite element models (FEM) that are capable of predicting the dynamic response of a structure or component with viscoelastic materials, Hughes and his coworkers [1–3] and Lesieutre and his coworkers [4–7] developed independent means of augmenting a FEM with new co-ordinates containing damping properties found from material loss factor curves. The GHM uses a second order physical co-ordinate system and the Lesieutre approach uses a first order state space method called the Augmenting

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Thermodynamic Fields (ATF) method and Anelastic Displacement Fields (ADF) which is augmented from ATF by considering the effects of multiple relaxation processes. Each of these methods are superior to the Modal Strain Energy method (MSE, Johnson *et al*. [8]) because they allow for the existence of complex modes and transient response calculation. While MSE is substantially easier to use, both ADF and GHM are able to account for damping effects over a range of frequencies, complex mode behavior, transient responses and both time and frequency domain modelling. Inman [9] applied the GHM approach to simple beams and Banks and Inman [10] used the alternate time domain method for modelling hysteresis.

Unfortunately both the GHM and ADF methods increase the order of the finite element model as they add co-ordinates to the system to compensate for frequency dependence. This is a distinct disadvantage when compared to MSE. Hence, the goal of this paper is to apply model reduction methods along with the GHM method to remove this objection of increased size. Here we examine the GHM finite element modelling method which represents a FEM model of viscoelastic structures by introducing internal co-ordinates and modelling the hysteresis with a transfer function.

If the damping is modelled using the GHM method, the structural degrees of freedom are at least doubled. This increases the computation time using a finite element model to predict the time response. Thus we are motivated to consider the effects of model reduction techniques on the GHM model. In particular we examine Guyan reduction and Internal Balancing reduction methods. Guyan reduction [11, 12] removes some of the insignificant physical co-ordinates, thereby producing a model that has smaller mass and stiffness matrices, but is not generally applicable to systems with damping as it is based on static considerations. However, the Guyan reduction produces a reduced order model with co-ordinates that are a subset of the original co-ordinate system. On the other hand, the internal balancing method [13] is derived in a state–space form and takes into consideration the dynamic response of the system. Internal balancing methods are applicable to damped systems (in fact asymptotic stability is required). On the other hand, the balanced reduction schemes do not produce co-ordinates which are a subset of the original finite element co-ordinates. The Guyan reduction method has the advantage of allowing the user to retain specific co-ordinates of interest, while the internal balancing method yields non-physical co-ordinates. Yae and Inman [14] produced a version of the internal balancing method which combines the desirable features of both methods by retaining dynamic fidelity and producing a reduced model in terms of a subset of the original states.

2. FORMULATION OF A VISCOELASTIC MATERIAL

The stress–strain constitutive relationship for viscoelastic material is

$$\sigma(x, t) = E\varepsilon(x, t) + \int_0^t g(t - s)\frac{\mathrm{d}\varepsilon(x, s)}{\mathrm{d}s}\,\mathrm{d}s, \tag{1}$$

where $\sigma(x, t)$ is the stress, $x \in (0, l)$ is the distance along the beam, $t > 0$ is the time, $\varepsilon(x, t)$ is the strain, E is the elastic modulus, and the kernel $g(t - s)$ describes the hysteresis as developed by Christensen [15], for example. This gives rise to a hysteretic or complex modulus description for the vibration of a structure with a viscoelastic material component.

The Golla–Hughes–McTavish method requires the representation of the material modulus function as a series of (damped) "mini-oscillator" terms or internal variables. This method was developed for direct incorporation into the finite element method. The material complex modulus can be written in the Laplace domain in the form

$$E^*(s) = E_0(1 + h(s)) = E_0\left(1 + \sum_{n=1}^{k} \hat{\alpha}_n \frac{s^2 + 2\hat{\zeta}_n\hat{\omega}_n s}{s^2 + 2\hat{\zeta}_n\hat{\omega}_n s + \hat{\omega}_n^2}\right), \tag{2}$$

where E_0 is the equilibrium value of the modulus, and s is the Laplace operator. The hatted terms are free variables for curve fitting to the complex data for a particular material at a given temperature. Also, the number of expansion terms, k, may be modified to represent the high or low frequency dependence of the complex terms. The expansion of $h(s)$ represents the material modulus as a series of the mini oscillator (second order equation) terms [2, 3]. The other methods of representing the hysteresis term, $h(s)$, in equation (2) are summarized in Table 1. The real and imaginary parts of Young's modulus for DYAD-606 (SOUNDCOAT) at temperature 25°C are plotted in Figure 1. These are compared to the corresponding curve fit values, indicated by (—), using two mini oscillator terms [$k = 2$ in equation (2)]. Different viscoelastic material will have different frequency dependence and have a different number of terms, k, of the GHM fit.

TABLE 1

Summary of methods for modeling viscoelastic effects using internal variables

$h(s) = \sum \frac{a_i s}{s + b_i}$	Boit (1955)
$h(s) = \frac{E_0 + E_1 s^\beta}{1 + b s^\beta}$	Bagly and Torvik (1981)
$h(s) = \sum_n \hat{\alpha}_n \frac{s^2 + 2\hat{\zeta}_n\hat{\omega}_n s}{s^2 + 2\hat{\zeta}_n\hat{\omega}_n s + \hat{\omega}_n^2}$	Hughes *et al.* (1985)
$h(s) = 1 + \sum_i \frac{\Delta_i s}{s + \beta_i}$	Lesieutre (1990)
$h(s) = 1 + \sum_i \frac{\alpha_i \tau_i s}{\tau_i s + 1}$	Yiu (1993)

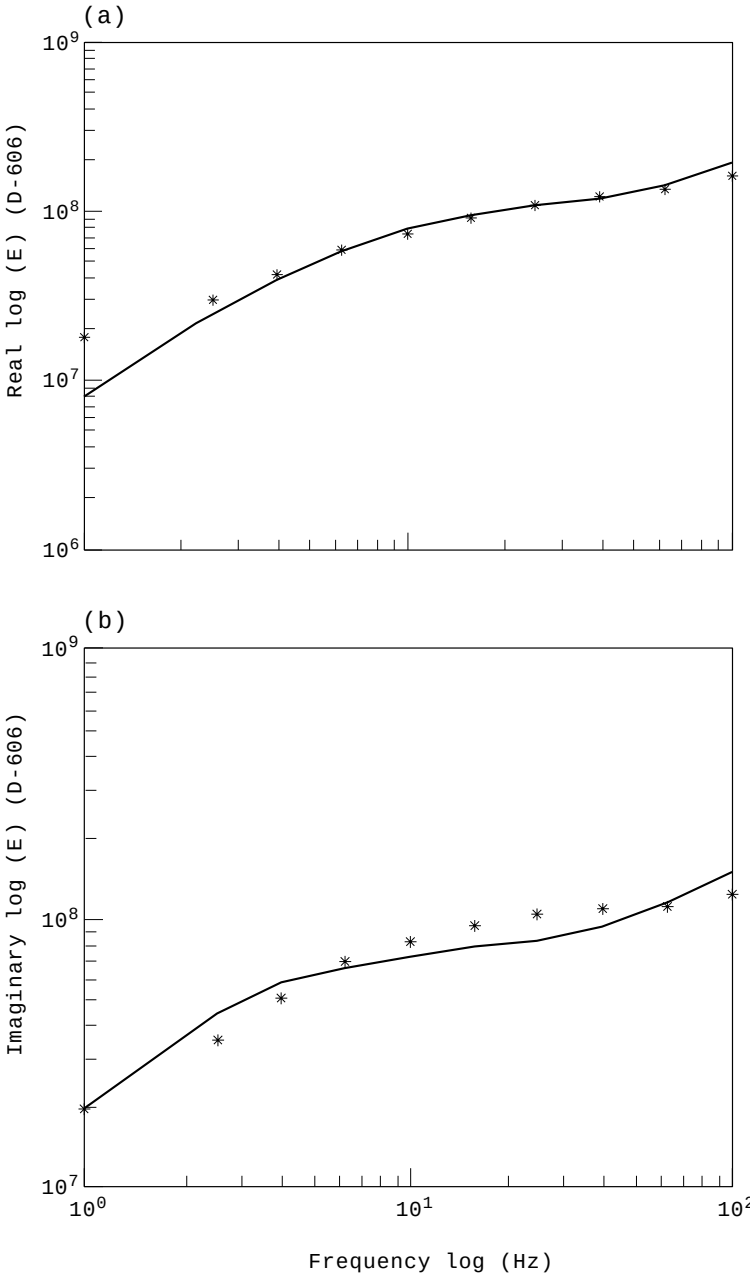

Figure 1. Two-term GHM modulus function (*, true value, —, curve-fitting): (a) real value of DYAD-606, (b) imaginary value of DYAD-606.

The constrained optimization algorithm used MATLAB's "constr" command, which is a Sequential Quadratic Programming (SQP) method, which finds the best choice of hatted mini-oscillator terms. This routine computes the constrained minimum of an objective function of the hatted terms α, ζ and ω, starting at initial estimates. That is, the material complex modulus in equation (2) is minimized with

satisfying the constraints to find the optimum hatted terms of α, ζ, ω. This is mathematically stated as "minimize $E^*(s)$ subject to the constraints $g_i(s) \leqslant 0$." The constraints are the hatted terms α_i in this case. This gives more accurate results since the terms are bounded by some limits. A method of solving optimization problems with inequality constraints is to use the Hessian of the Lagrangian function,

$$L(x, \lambda) = E^*(s) + \sum_{i=1}^{m} \lambda_i g_i(x). \tag{3}$$

This method uses a vector of Lagrange multipliers, λ_i, to add the constraints directly to the objective function. The new cost function, $L(x, \lambda)$, is then minimized through iteration using a quasi-Newton updating method. This is then used to generate the Quadratic Programming (QP) sub-problem based on a quadratic approximation. The solution of the QP sub-problem is used to form a search direction for a line search procedure. The minimum along the line formed from this search direction is generally approximated using this search procedure or by a polynomial method involving interpolation or extrapolation. The problem is to find a new iterate x_{k+1} of the form, $x_{k+1} = x_k + \alpha^* d$, where x_k denotes the current iterate, d the search direction obtained by an appropriate method and α^* is a scalar step length parameter which is the distance to the minimum. When the objective function is minimized after some iterations, the approximate values of the hatted terms of α, ζ and ω are compared with the real values, giving the tolerance between them. When the tolerance is satisfied at a limit (some small value), the optimum hatted two mini-oscillator terms are $E_0 = 1{\cdot}18\text{E}6$, $\hat{\alpha} = [87{\cdot}5\ \ 263{\cdot}13]$, $\hat{\zeta} = [1344{\cdot}6\ \ 129{\cdot}6]$, $\hat{\omega} = [1494{\cdot}5\ \ 39999{\cdot}9]$. The important effect of frequency is that the Young's modulus always increases with increasing frequency, as shown Figure 1.

3. GHM FINITE ELEMENT MODEL

The equation of motion for a finite element in the Laplace domain is

$$M(s^2\mathbf{x}(s) - s\mathbf{x}_0 - s\dot{\mathbf{x}}_0) + K(s)\mathbf{x}(s) = \mathbf{f}(s), \tag{4}$$

where M is the mass matrix, $\mathbf{x}(s)$ is the displacement vector, $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$ are the initial displacement and initial velocity vectors, respectively, $\mathbf{f}(s)$ is the forcing function and

$$K(s) = (E^{*1}(s)\overline{K}^1 + E^{*2}(s)\overline{K}^2 + \cdots + E^{*n}(s)\overline{K}^n). \tag{5}$$

Here the variable $E^{*n}(s)$ represents the nth complex modulus in the Laplace domain and $\overline{K}^n$ is the contribution of the nth modulus to the stiffness matrix.

Considering a single modulus model with a single expansion term for simplicity, and neglecting initial conditions, the GHM method represents equation (4) as

$$Ms^2\mathbf{x}(s) + E_0\left(1 + \hat{\alpha}\frac{s^2 + 2\hat{\zeta}\hat{\omega}s}{s^2 + 2\hat{\zeta}\hat{\omega}s + \hat{\omega}^2}\right)\bar{K}\mathbf{x}(s) = \mathbf{f}(s). \tag{6}$$

Introducing a column of dissipation co-ordinates $\hat{\mathbf{z}}$ such that

$$\hat{\mathbf{z}}(s) = \left(\frac{\hat{\omega}^2}{s^2 + 2\hat{\zeta}\hat{\omega}s + \hat{\omega}^2}\right)\mathbf{x}(s), \tag{7}$$

then the following Laplace domain element equation of motion is equivalent to equation (6) with the material modulus function $E^{*1}(s)$:

$$\begin{bmatrix} M & 0 \\ 0 & \dfrac{\hat{\alpha}}{\hat{\omega}^2}E_0\bar{K} \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \hat{\mathbf{z}}(s) \end{bmatrix}s^2 + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{2\hat{\alpha}\hat{\zeta}}{\hat{\omega}}E_0\bar{K} \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \hat{\mathbf{z}}(s) \end{bmatrix}s$$

$$+\begin{bmatrix} \bar{K}E_0(1+\hat{\alpha}) & -\hat{\alpha}E_0\bar{K} \\ -\hat{\alpha}E_0\bar{K} & \hat{\alpha}E_0\bar{K} \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \hat{\mathbf{z}}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(s) \\ \mathbf{0} \end{bmatrix}. \tag{8}$$

Here it is assumed that there are no rigid body modes as discussed by Slater *et al*. [16]. Using an elastic diagonal matrix of the non-zero eigenvalues Λ_e and corresponding eigenvectors R_e of the modulus-factored stiffness matrix and substituting

$$\mathbf{z}(s) = R_e^T\hat{\mathbf{z}}(s), \tag{9}$$

one arrives at

$$\begin{bmatrix} M & 0 \\ 0 & \dfrac{\hat{\alpha}}{\hat{\omega}^2}E_0\Lambda_e \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix}s^2 + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{2\hat{\alpha}\zeta}{\hat{\omega}}E_0\Lambda_e \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix}s$$

$$+\begin{bmatrix} \bar{K}E_0(1+\hat{\alpha}) & -\hat{\alpha}E_0R_e\Lambda_e \\ -\hat{\alpha}E_0\Lambda_eR_e^T & \hat{\alpha}E_0\Lambda_e \end{bmatrix}\begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(s) \\ \mathbf{0} \end{bmatrix}. \tag{10}$$

This is the final form of the GHM model of the element viscoelastic equation of motion, and the submatrices Λ_e and R_e in GHM viscoelastic element mass,

damping and stiffness matrices are found through spectral decomposition of the elastic component matrices:

$$\Lambda_e = \begin{bmatrix} \dfrac{2I}{l} & 0 \\ 0 & \dfrac{6I(4+l^2)}{l^3} \end{bmatrix}, \qquad R_e = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2+l^2/2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{l/2}{\sqrt{2+l^2/2}} \\ 0 & \dfrac{-1}{\sqrt{2+l^2/2}} \\ -\dfrac{1}{\sqrt{2}} & \dfrac{l/2}{\sqrt{2+l^2/2}} \end{bmatrix}. \tag{11}$$

The size of the additional co-ordinates $\mathbf{z}(s)$ depends on the nature of the material and on how many terms are needed to fit the particular material's loss modulus data. In addition, the size of $\mathbf{z}(s)$ greatly increases the order of the analytical model, rendering this approach somewhat undesirable. Reducing the order of equation (10) forms the focus of the result presented here.

4. REDUCTION METHODS

Model reduction methods are briefly introduced here as they have been developed in two different disciplines: finite element analysis and control theory. In the condensation process or static reduction, such as the Guyan reduction commonly done in finite element analysis, some of the insignificant physical co-ordinates are removed such as rotational degrees of freedom. This condensation method provides a reduced order model with co-ordinates that are a subset of the original co-ordinate system. However, static reduction does not necessarily apply to a damped system. Furthermore, this static reduction method does not reflect the system's dynamic behavior.

There are many additional methods on model reduction in structural dynamics. Some methods common to control design theory are applicable to a damped system and present the same dynamic response. For control design the model is converted to a state–space form and reduced by one of the reduction methods [13, 17–24]. These methods approximate a larger, dynamic system with a fewer number of the state variables while minimally changing the structural dynamic response. The problem with these reduction methods is that the reduced model is expressed in a co-ordinate different from a physically meaningful co-ordinate. Consequently, it is difficult to recognize any connection between the states of the reduced model and those of the original model.

One of these basic methods, called the internal balancing method, is used here. The drawback of these control methods is that it is not directly possible to express the reduced model in terms of a subset of the original states. Yae and Inman [14] found it is possible to express the reduced model in terms of a subset of the original states if an additional co-ordinate transformation is properly applied.

4.1. GUYAN REDUCTION METHOD

The eigenvalue problem of an undamped system can be partitioned as follows:

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \lambda \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{12}$$

where the generalized co-ordinate $\mathbf{q}_2$ is to be retained if it is critical (excited externally) for the performance of the system and the generalized co-ordinate $\mathbf{q}_1$ is to be removed by condensation (usually the rotational degree of freedom). Thus, in state reduction all mass terms except M_{22} are ignored in order to obtain a relation between $\mathbf{q}_1$ and $\mathbf{q}_2$. From the upper partition of equation (12),

$$\mathbf{q}_1 = -K_{11}^{-1} K_{12} \mathbf{q}_2 = \bar{T} \mathbf{q}_2, \tag{13}$$

where $\bar{T}$ defined as $\bar{T} = -K_{11}^{-1} K_{12}$; the transformation to reduce the original model is given by

$$\begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} = \begin{bmatrix} \bar{T} \\ I \end{bmatrix} \mathbf{q}_2 = T \mathbf{q}_2, \tag{14}$$

where I is the identity matrix of appropriate size. Substituting equation (14) into equation (12) and premultiplication by T^T yields the condensed or reduced eigenvalue problem

$$(\bar{K} - \lambda \bar{M}) \mathbf{q}_2 = \mathbf{0}. \tag{15}$$

Here the reduced matrices are given by

$$\bar{K} = T^T K T \qquad \text{and} \qquad \bar{M} = T^T M T. \tag{16}$$

While this static reduction formulation was developed for undamped systems, a straightforward application of the above transformations to a damped system with an external force of $\mathbf{f}$ yields that the condensed damping matrix and external loads are given by

$$\bar{D} = T^T D T \qquad \text{and} \qquad \bar{\mathbf{F}} = T^T \mathbf{f}. \tag{17}$$

Equations (16) and (17) yield a reduced order model for a damped system. It has the advantage of retaining the desired co-ordinate $\mathbf{q}_2$ as a subset of the original states, but the distinct disadvantage of not reflecting the system dynamics so that the response of the reduced system is a poor representation of the actual response.

4.2. INTERNAL BALANCING METHOD

The internal balancing method of model reduction forms the standard for most model reduction methodologies in control theory. Internal balancing works with damped systems and is very accurate. However, it does not provide a connection to the physical co-ordinates. Here the original equations of motion are taken to be

$$M \ddot{\mathbf{q}} + D \dot{\mathbf{q}} + K \mathbf{q} = \mathbf{f}, \tag{18}$$

where M, D, and K are the $n \times n$ real, symmetric, positive definite matrices. The $n \times 1$ vector $\mathbf{q}$ is the displacement vector. The overdots denote differentiation with respect to time. The $n \times 1$ vector $\mathbf{f}$ represents the external forces applied to the structure. Equation (18) is converted into the state space form such that

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{y}(t) = C\mathbf{x}(t), \tag{19}$$

where

$$A = \begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} M^{-1}B_1 \\ 0 \end{bmatrix}, \qquad C = [C_1 \quad C_2],$$

and $\mathbf{y}(t)$ is a vector consisting of those states that are to be measured. Since the B and C are directly related to the locations of applied force and measurement, they influence the degree of controllability and observability. It should be assumed that the system (A, B, C) is controllable, observable and asymptotically stable for a valid application of this method. The balanced reduction allows those states that minimally affect the response to be removed from the model. A useful measure of this is provided for asymptotically stable systems by defining the *controllability and observability grammians*, denoted by W_c and W_o, respectively and defined by

$$W_c = \int_0^\infty \mathrm{e}^{At} B B^T \mathrm{e}^{A^T t}\, \mathrm{d}t, \qquad W_o = \int_0^\infty \mathrm{e}^{A^T t} C^T C \mathrm{e}^{At}\, \mathrm{d}t, \tag{20}$$

where e^{At} is the state transition matrix of the open-loop system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$. W_c and W_o are the unique symmetric positive definite matrices which satisfy the Lyapunov matrix equations,

$$AW_c + W_c A^T = -BB^T, \qquad A^T W_o + W_o A = -C^T C, \tag{21}$$

for asymptotically stable systems. Moore [13] has shown that there exists a co-ordinate system in which these two grammians are equal and diagonal. Such a system is then called *balanced*. These two balanced systems are

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\mathbf{u}(t), \qquad \hat{\mathbf{y}}(t) = \hat{C}\hat{\mathbf{x}}(t), \tag{22}$$

where

$$\hat{\mathbf{x}} = P^{-1}\mathbf{x}, \qquad \hat{A} = P^{-1}AP, \qquad \hat{B} = P^{-1}B, \qquad \hat{C} = CP, \tag{23}$$

with P denoting a linear transformation of the system into the balanced co-ordinate system. In addition, the two grammians are equal in this co-ordinate system:

$$\hat{W}_c = \hat{W}_o = \mathrm{diag}\,[\sigma_1, \sigma_2, \ldots, \sigma_{2n}], \tag{24}$$

where $\hat{W}_c = P^{-1}W_c P$, $\hat{W}_o = P^{-1}W_o P$ and the σ_is denote the singular values of the grammians. Applying the idea of singular values as a measure of rank deficiency to the controllability and observability grammians yields a systematic model

reduction method. The matrix P that transforms the original system (A, B, C) into a balanced system $(\hat{A}, \hat{B}, \hat{C})$ can be obtained using the following algorithm:

(1) The reduced order model can be calculated by first calculating an intermediate transformation matrix P_1 based on the controllability grammians. Solve for W_c and find eigenvalues Λ_c and eigenvectors V_c such that $V_c^T W_c V_c = \Lambda_c$. Then define $P_1 = V_c \Lambda_c^{-1/2}$.

(2) The co-ordinate transformation $\mathbf{x} = P_1 \tilde{\mathbf{x}}$ yields an intermediate system $(\tilde{A}, \tilde{B}, \tilde{C})$ calculated by $\tilde{A} = P_1^{-1} A P_1$, $\tilde{B} = P_1^{-1} B$, $\tilde{C} = C P_1$.

(3) To complete the balancing algorithm, these intermediate equations are balanced with respect to $\tilde{W}_o$. Solve for $\tilde{W}_o$ and find eigenvalues $\tilde{\Lambda}_o$ and eigenvectors $\tilde{V}_o$ such that $\tilde{V}_o^T \tilde{W}_o \tilde{V}_o = \tilde{\Lambda}_o$. Let $P_2 = \tilde{V}_o \tilde{\Lambda}_o^{-1/4}$.

(4) Another co-ordinate transformation $\tilde{\mathbf{x}} = P_2 \hat{\mathbf{x}}$ yields the desired balanced system $(\hat{A}, \hat{B}, \hat{C})$:

$$\hat{A} = P_2^{-1} \tilde{A} P_2 = P_2^{-1} (P_1^{-1} A P_1) P_2, \qquad \hat{B} = P_2^{-1} \tilde{B} = P_2^{-1} P_1^{-1} B,$$

$$\hat{C} = \tilde{C} P_2 = C P_1 P_2.$$

The transformation P is given by P_1 and P_2 as $P = P_1 P_2$. Using the above equations, the balanced system $(\hat{A}, \hat{B}, \hat{C})$ can be partitioned as

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}_r \\ \dot{\hat{\mathbf{x}}}_d \end{bmatrix} = \begin{bmatrix} \hat{A}_r & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_r \\ \hat{\mathbf{x}}_d \end{bmatrix} + \begin{bmatrix} \hat{B}_r \\ \hat{B}_d \end{bmatrix} \mathbf{u}, \qquad \mathbf{y} = [\hat{C}_r \quad \hat{C}_d] \begin{bmatrix} \hat{\mathbf{x}}_r \\ \hat{\mathbf{x}}_d \end{bmatrix}. \tag{25}$$

Deleting the k least controllable and observable states, i.e., setting $\hat{\mathbf{x}}_d = 0$, provides

$$\dot{\hat{\mathbf{x}}}_r(t) = \hat{A}_r \hat{\mathbf{x}}_r(t) + \hat{B}_r \mathbf{u}(t), \qquad \hat{\mathbf{y}}_r(t) = \hat{C}_r \hat{\mathbf{x}}_r(t), \tag{26}$$

a reduced model of order $(n - k)$. This produces the balanced system which is reduced by looking at the singular values of the balanced system and throwing away those co-ordinates which have relatively small singular values, indicating that those co-ordinates do not have much effect on the system response. This leaves a smaller order system with essentially the same dynamics as the full order system.

4.3. MODIFIED INTERNAL BALANCING METHOD

Unfortunately the co-ordinates left after a balanced reduction are not a subset of the finite element nodal co-ordinates. For example, if one applies standard balancing to a system with GHM elements, one may not get that $\dot{\hat{\mathbf{x}}}_r(t) = \mathbf{z}(t)$, which is what is desired. The reduced model is not simple to relate back to the original finite element model as is the case in the Guyan reduction. This problem is solved by introducing an additional co-ordinate transformation [14] to produce a reduced order model in a co-ordinate system consisting of a subset of the original finite element co-ordinate system. For finite element and measurement applications, it is desirable to provide a clear, physical relationship between the original vector $\mathbf{q}$ and the reduced state vector $\hat{\mathbf{x}}_r$. Such a relationship is found by

using the fact that the balanced states are linear combinations of the original states. Symbolically this is written as

$$\hat{x}_1 = \sum_{j=1}^{2n} c_{1j}x_j, \qquad \ldots, \hat{x}_{2n-k} = \sum_{j=1}^{2n} c_{(2n-k)j}x_j,$$

$$\hat{x}_{2n-(k-1)} = \sum_{j=1}^{2n} c_{(2n-k+1)j}x_j \to 0, \qquad \ldots, \hat{x}_{2n} = \sum_{j=1}^{2n} c_{2nj}x_j \to 0, \tag{27}$$

where the c_{ij}'s are the coefficients in the linear combinations of $\{x_1, x_2, \ldots, x_{2n}\}$. Here the last k states are set to zero because they represent the least significant states in the balanced system. That is, the last k states are those that yield the least response. Setting each of these summations equal to zero is equivalent to imposing k constraints on the original $2n$ states, which means that the modal reduction imposes dependencies on k number of the original states. In other words, one can construct a reduced order model by selecting $(2n-k)$ states out of the original $2n$ states. If the $(2n-k)$ selected states from the original system are denoted by $\mathbf{x}_r = [x_{j1}\ x_{j2}\ \cdots\ x_{j2n-k}]^T$ and the $(2n-k)$ states of the balanced system by $\hat{\mathbf{x}}_r = [\hat{x}_1\ \hat{x}_2\ \cdots\ \hat{x}_{2n-k}]^T$, then the states in $\hat{\mathbf{x}}_r$ are linear combinations of the states in $\mathbf{x}_r$. Thus, there exists a new transformation matrix P_r of order $(2n-k)\times(2n-k)$ such that $\mathbf{x}_r = P_r\hat{\mathbf{x}}_r$. The above constraints and the resulting transformation allow the user to specify which nodal co-ordinates of the original model are to be retained in the model reduction. In the following it is shown that the matrix P_r consists of certain rows and columns of the original transformation matrix P, and that there is a systematic way of constructing P_r from P.

(1) Select the state variables to be retained from $\{x_1, x_2, \ldots, x_{2n-k}\}$. Let the indices of those selected be $\{j_1, \ldots, j_{2n-k}\}$ rows from P.
(2) The transformation matrix P_r can be obtained by selecting first $2n-k$ columns and $\{j_1, \ldots, j_{2n-k}\}$ rows from P.
(3) The reduced order system (A_r, B_r, C_r),

$$\dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r\mathbf{u}(t), \qquad \mathbf{y}_r(t) = C_r\mathbf{x}_r(t), \tag{28}$$

is now expressed in terms of a subset $\mathbf{x}_r$ of the original state vector $\mathbf{x}$, where

$$A_r = P_r\hat{A}_rP_r^{-1}, \qquad B_r = P_r\hat{B}_r, \qquad C_r = \hat{C}_rP_r^{-1}. \tag{29}$$

Thus, a scheme has been provided that has the best features of the Guyan reduction method and of the balanced reduction method. It is possible to both

TABLE 2

Physical and geometrical properties of the viscoelastic beam

Length (m)	Thickness (m)	Width (m)	Density (kg/m^3)	Poisson's ratio (v)
0·1	5·08E-5	0·01	1105	0·49

specify which co-ordinates to keep while providing a dynamically based reduction scheme. This proposed method allows the removal of the internal co-ordinates, $\mathbf{z}(s)$, added to the system to build a damping matrix, and thus, to incorporate complex mode behavior in the original finite element co-ordinates of the original order.

5. NUMERICAL EXAMPLE

A numerical example is presented in order to demonstrate the use of viscoelastic element matrices in the finite element analysis of a viscoelastic beam (Figure 2) through the three reduction methods as described above. All the calculations are performed on a PC using MATLAB for windows by The Math works, Inc. The physical and geometrical parameters of the DYAD-606 are as shown in Table 2. The viscoelastic beam is equally divided into four elements so that it has four active node points. Each node point has six degrees of freedom, that is one translational displacement, one rotational displacement, and four additional viscoelastic auxiliary degrees of freedom. Hence, the viscoelastic beam has 24 degrees of freedom in total.

The equation of motion of the transverse vibration of a viscoelastic beam may be presented by a GHM finite element form with two mini-oscillator terms; two terms are chosen because of the curve slope given in Figure 1:

$$M_e\ddot{\mathbf{q}} + D_e\dot{\mathbf{q}} + K_e\mathbf{q} = \mathbf{f}_e, \tag{30}$$

where the finite element matrices are given by

$$M_e = \begin{bmatrix} \bar{M} & 0 & 0 \\ 0 & (\hat{\alpha}_1/\hat{\omega}_1^2)E_0\Lambda_e & 0 \\ 0 & 0 & (\hat{\alpha}_2/\hat{\omega}_2^2)E_0\Lambda_e \end{bmatrix}, \tag{31}$$

$$D_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (2\hat{\alpha}_1\hat{\zeta}_1/\hat{\omega}_1)E_0\Lambda_e & 0 \\ 0 & 0 & (2\hat{\alpha}_2\hat{\zeta}_2/\hat{\omega}_2)E_0\Lambda_e \end{bmatrix}, \tag{32}$$

$$K_e = \begin{bmatrix} \bar{K}E_0(1+\hat{\alpha}_1+\hat{\alpha}_2) & -\hat{\alpha}_1E_0R_e\Lambda_e & -\hat{\alpha}_2E_0R_e\Lambda_e \\ -\hat{\alpha}_1E_0\Lambda_eR_e^T & \hat{\alpha}_1E_0\Lambda_e & 0 \\ -\hat{\alpha}_2E_0\Lambda_eR_e^T & 0 & \hat{\alpha}_2E_0\Lambda_e \end{bmatrix}, \tag{33}$$

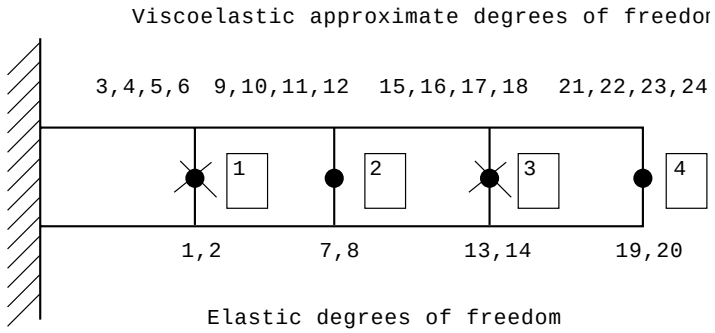

Figure 2. A cantilever beam with four elements.

and the element force $\mathbf{f}_e = \text{col}\,\{\mathbf{f}, 0, 0\}$ and the co-ordinate vector $\mathbf{q} = \text{col}\,\{\mathbf{x}, z_1, z_2\}$ with an element displacement vector $\mathbf{x} = \text{col}\,\{w_1, \theta_1, w_2, \theta_2\}$. The elastic mass and stiffness matrices for a beam element in a plane are given by [25]

$$\bar{M} = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix},$$

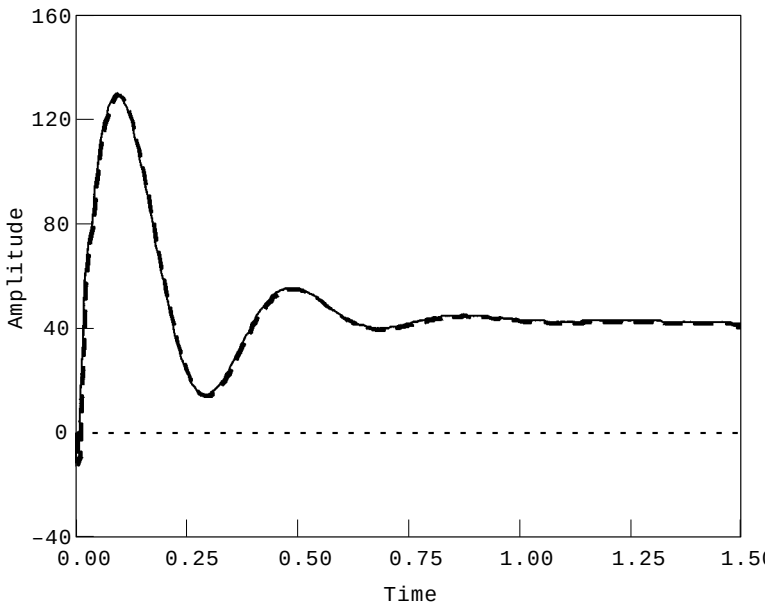

Figure 3. Time response of output in original (—), internal balancing (- - -), and difference (· · ·) between the two models.

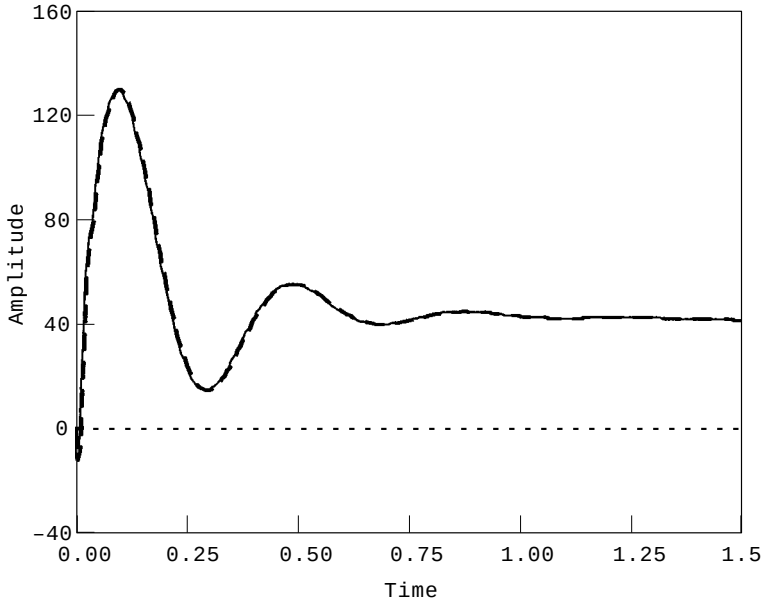

Figure 4. Time response of output in original (—), modified internal balancing (- - -), and difference (· · ·) between the two models.

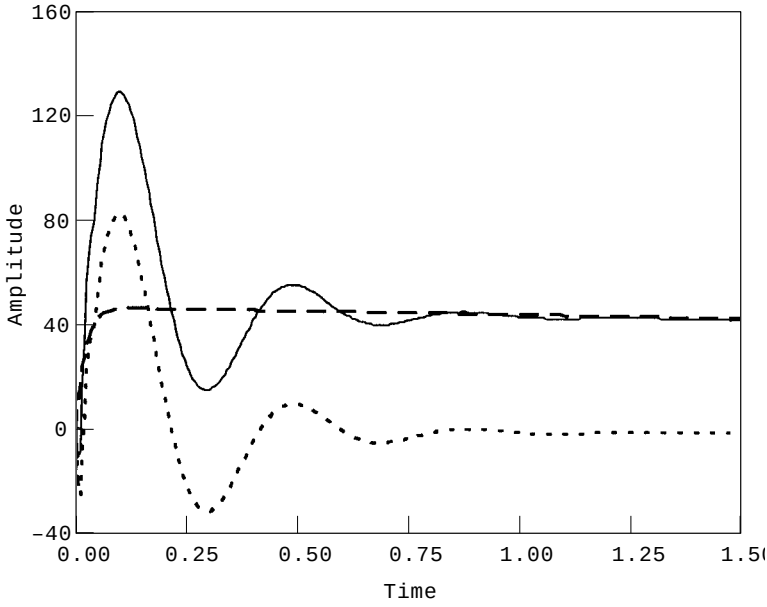

Figure 5. Time response of output in original (—), Guyan reduction (- - -), and difference (· · ·) between the two models.

TABLE 3

Modal frequency (rad/s) *and damping ratio*

	Original mode		Guyan		Internal balancing		Mod. Int. balancing	
	Frequency	Damping	Frequency	Damping	Frequency	Damping	Frequency	Damping
1	16·94	0·308	16·94	0·308	16·94	0·308	16·94	0·308
2	186·48	0·297	187·85	0·299	186·61	0·297	186·90	0·297
3	555·03	0·104	671·40	0·123	577·84	0·117	577·84	0·117
4	1100·49	0·052	1963·07	0·069	1100·69	0·053	1100·69	0·053

$$\bar{K} = \frac{I}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}. \tag{34}$$

The values of the hatted constants α, ζ and ω were obtained from curve fitting, as shown in section 2.

All three procedures discussed in section 3, Guyan, internal balancing, and modified internal balancing reduction methods, are applied to this GHM finite element model of a cantilever viscoelastic beam (Figure 2). For the purpose of demonstration, the impulse input is placed on the node 2 and the displacement

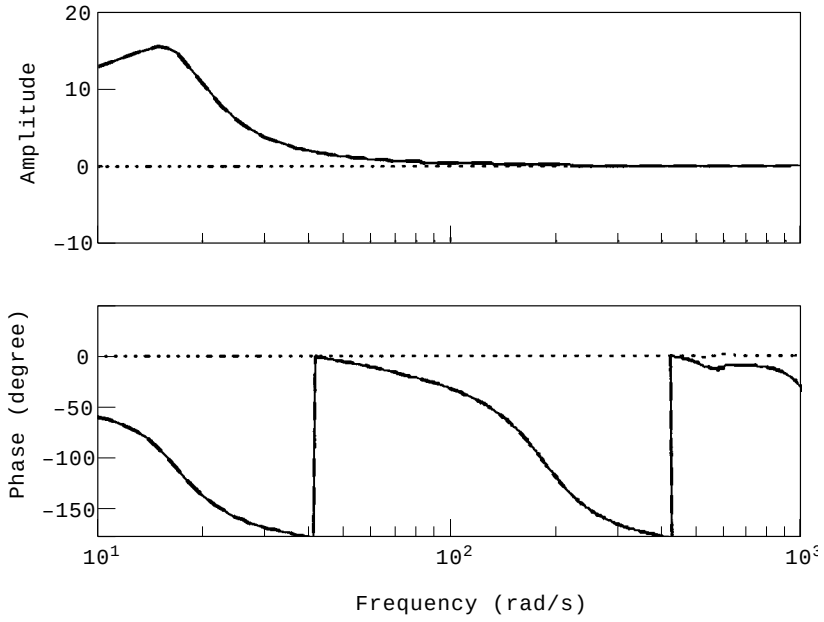

Figure 6. Frequency response function of original (—), internal balancing model (- - -), and difference (· · ·) between the two models.

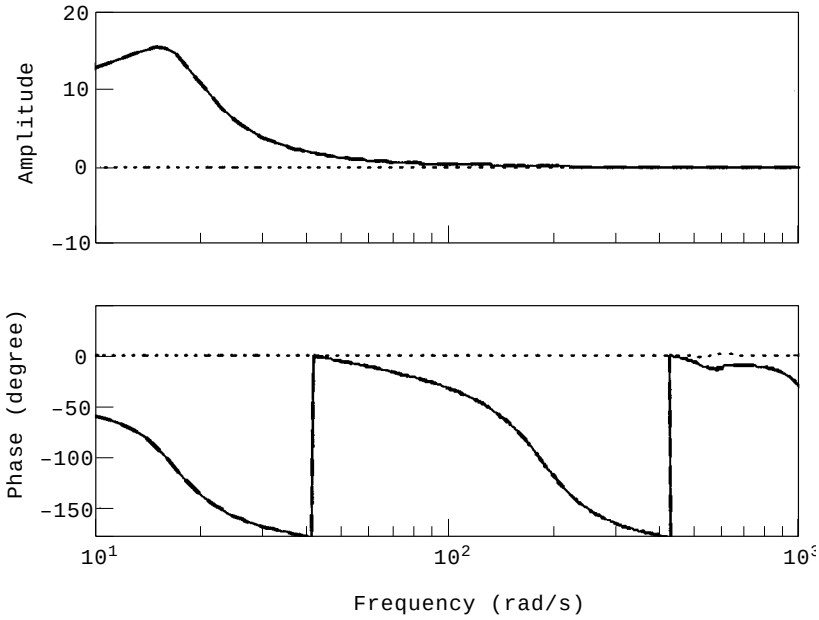

Figure 7. Frequency response function of original (—), modified internal balancing model (- - -), and difference (· · ·) between the two models.

of the tip (node 4) is measured. Thus, nodes 2 and 4 become important co-ordinates that should be retained in the final reduced model. In Figure 3, the time response curves of the original model and the model reduced by internal balancing are plotted. Here one is able to delete the viscoelastic states, that is, GHM internal variables and maintain the elastic states. The difference between the responses of the output amplitude in the original and the reduced model shown by the dashed line of Figure 3 is almost zero. In this case, the difference between the full and reduced system response is $\pm 0{\cdot}1$. In Figure 4 the responses of the original states and those of the modified internal balanced states are plotted. Here all viscoelsatic states and the elastic states at nodes 1 and 3 have been removed. Again both responses and their difference are plotted. The modified internally balanced, reduced order model is expressed by the remaining eight states. The time response curves of the original model and reduced model by the Guyan reduction method are plotted in Figure 5. One thing to note here is that the Guyan reduction method does not remove the viscoelastic states properly, as it does not take into account system dynamics as shown in Figure 5. In Figures 3 and 4, it is shown that despite some non-zero differences detected in the transient region, the differences are nearly zero in comparison to the response of the original state.

In order do obtain the natural frequencies and damping ratios, the GHM model is converted into state space form, the model reduction is applied to remove the internal variables and the resulting eigenvalue problem is computed. The damping ratio for each mode for each of the reduction methods is calculated by computing the ratio of the negative of the real part of the eigenvalue to the absolute value of the eigenvalue. Table 3 shows that the natural frequencies and damping ratios of the third and fourth modes acquired from the Guyan reduction method do not

agree with the original ones, while the values obtained by the internal balancing and the modified internal balancing methods, agree with the original value.

Figures 6 and 7 show the phase and magnitude frequency responses for the reduced order models using the balanced and modified balanced methods to remove the internal variables associated with the GHM method. The original Bode plot is shown as a solid line and the Bode plots of the internal balancing and the modified balancing method are shown as dashed lines. For the Bode plots, the impulse input is placed on node 2 and the displacement of the tip (node 4) is measured. As can be seen, both magnitude and phase relationship agree well with the full order model. Figures 6 and 7 also show that a single mode dominates the dynamic response because it is a VEM beam. The reduced models using the internal balancing and the modified internal balancing method preserve the phase relationships inherent in the complex original models.

A relative error [13] is obtained to secure the accuracy due to modal reduction. It provides a quantitative measure of error introduced by the reduction. As mentioned earlier, the reduced model $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ of order $(n-k)$ can be obtained from the balanced representation by deleting k number of the least controllable and observable states. One error measure is

$$\text{Relative error} = \left[\sum_{i=n-k+1}^{n} \sigma_i^2\right]^{1/2} \Big/ \left[\sum_{i=1}^{n-k} \sigma_i^2\right]^{1/2}, \tag{35}$$

where the numerator is composed of singular values of deleted states and the denominator is composed of those remaining states. The relative errors are 1·094E-17 and 1·786E-20 when using internal balancing and the modified internal balancing method, respectively, indicating that the reduced models are indeed a respectable realization of the original system.

6. CONCLUSION

This paper introduces the use of model reduction methods to remove internal variables (mini oscillators) used to account for viscoelastic properties in finite element modelling. These internal variable methods (Hughes, Lesieutre) have become effective ways to model viscoelastic material by finite element analysis. Unfortunately, they do so at the expense of increasing the model order. The method proposed here eliminates the need to increase the order over that of the original model. First, a model reduction method, based on internal balancing, is developed to represent the reduced model by converting it into state space form and removing the viscoelastic states that do not have much effect on the system response. As a result, the reduced model represents the original model with fewer states than the original model requires. Next, the model is expressed by a subset of the original states through another transformation that is derived from the deleted states in reduction. The method thereby provides a clear, physical relationship between the states in the reduced model and those in the original model. Finally, the Guyan reduction method does not remove the viscoelastic states (internal variables) without introducing substantial error. To check the

accuracy between the original model and the reduced models, the natural frequencies, damping ratios, Bode plots and relative error criterion are given. There are good agreements between the full model and the reduced models derived from the internal balancing and the modified internal balancing method. In particular, the modified internal balancing produces a reduced model that is an excellent representation of the viscoelastic system. The approach presented here renders modelling methods of the previous literature more useful to those situations where it is desired to keep the order of the finite element model as low as possible, yet accounts for viscoelastic effects.

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